Opuscula Math. 42, no. 2 (2022), 305–335
https://doi.org/10.7494/OpMath.2022.42.2.305

ON SOME INVERSE PROBLEM FOR BI-PARABOLIC EQUATION WITH OBSERVED DATA IN L^p SPACES

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Communicated by Ruzhang Xu

Abstract. The bi-parabolic equation has many practical significance in the field of heat transfer. The objective of the paper is to provide a regularized problem for bi-parabolic equation when the observed data are obtained in L^p. We are interested in looking at three types of inverse problems. Regularization results in the L^2 space appears in many related papers, but the survey results are rare in L^p, $p \neq 2$. The first problem related to the inverse source problem when the source function has split form. For this problem, we introduce the error between the Fourier regularized solution and the exact solution in L^p spaces. For the inverse initial problem for both linear and nonlinear cases, we applied the Fourier series truncation method. Under the terminal input data observed in L^p, we obtain the approximated solution also in the space L^p. Under some reasonable smoothness assumptions of the exact solution, the error between the the regularized solution and the exact solution are derived in the space L^p. This paper seems to generalize to previous results for bi-parabolic equation on this direction.

Keywords: bi-parabolic equations, Fourier truncation method, inverse source parabolic, inverse initial problem, regularization, Sobolev embeddings.

Mathematics Subject Classification: 35A05, 35A08.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we are interested to study the following biparabolic equation

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = F(x,t,u(x,t)), & \text{in } \Omega \times (0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where F is the source function and u describe the distribution of the temperature at position x and time t.

The main objective of this paper is to investigate the stability and approximate for two inverse problems for bi-parabolic equation (1.1) as follows.

Inverse source problem for (1.1). Let us assume that the source term in (1.1) in the linear case. The inverse source problem is stated as follows. Let the distribution of u at the final time

$$u(x,T) = g(x)$$

and assume further that the additional condition $u(x,0) = u_t(x,0) = 0$ for $x \in \Omega$. We need to recover the source function F where F has a split form $F(x,t) = \varphi(t)f(x)$. More specifically, let us given the function φ. Our task here is to recovering the function f. In [19], the input data is noisy by the observation data in L^2 space. The inverse source problem will be more difficult if the data is noisy in L^p space with $p \neq 2$. To the best of our knowledge, there are very few results related to the observation data in L^p space with $p \neq 2$. The first part of this paper deal with the stated topic.

Inverse initial value problem for (1.1). Let us consider the inverse initial value problem (1.1) with the following terminal observation

$$u_t(x,T) = 0, \quad u(x,T) = g(x) \quad \text{in } \Omega. \tag{1.2}$$

If g is disturbed by observed data in L^2 space, then we have some known studies, for example Lakhdari–Boussetila [16], Besma–Nadjib–Rebbani [3], Tuan–Kirane–Nam–Au [28], Tuan–Caraballo–Thach [25], and [2, 7, 8, 14, 17, 18, 22, 29]. This problem is one of the branches of the back-in-time problem for PDEs, that has been explored in many interesting papers, some of which can be listed as [1, 24, 27]. However, in the mentioned papers, the observed data is only in L^2. The result for the problem (1.1)–(1.2) when the final data g is disturbed by the observed data in L^p, $p \neq 2$, is still an open problem.

Before stating the main results, we would like to explain why the bi-parabolic equation is of interest. Bi-parabolic equation describes heat conduction and has many applications for thermal processes [5, 10, 21, 30] and it has been used to describe the special phenomena of process dynamics filter merge [4, 6, 15]. In the interesting work of Fushchich–Galitsyn–Polubinskii [10], the authors suggested that the classical quadratic parabolic equations do not completely describe heat and mass accurately. transfer processes and there have been some well-known paradoxes. To overcome some possible paradoxes, mathematicians have devised a new parabolic model, which is to replace the second-order operator by the fourth-order operator. To put it more clearly, they find that bi-parabolic equation by substituting the second-order operator$\mathscr{Q}u = \left(\frac{\partial}{\partial t} + \mathcal{A}\right) u$ by new fourth-order operator

$$\mathscr{Q}^2 u = \mathscr{Q}(\mathscr{Q}u) = \frac{\partial^2}{\partial t^2} u + 2\frac{\partial}{\partial t}\mathcal{A}u + \mathcal{A}^2 u.$$

This is also the reason why we have the equation (1.1) which is of the hyperbolic form. The (1.1) problem is a form of the higher-order PDE equation: it is quadratic in terms of time variables and quadratic in terms of space variables, which attracts many mathematicians to study and interest [9, 11, 12, 20, 23, 26, 31]. Following closely the work of Greer–Bertozzi–Sapiro [13], quadratic PDEs are modeled in a number of natural phenomena such as including ice formation, fluid on the lungs, warping of the brain, and the design of distinctive curves on the surface.

In the following, we would like to briefly discuss the contributions and novelties of this paper. As we stated earlier, this paper is the first result for the inverse problem for the biprabolic equation when the observed data is in the L^p space with $p \neq 2$. One big difficulty is that we cannot use Parseval equality directly because the data is not in L^2. We overcome these difficulties by using embedding between L^p and Hilbert scales spaces $\mathbb{H}^s(\Omega)$.

We detail the main contribution in three parts.

The first part deals with the inverse problem of determining the source function. We achieve the results of regularization result when the observed data $(\varphi_\varepsilon, g_\varepsilon) \in L^p(0,T) \times L^p(\Omega)$. Using the Fourier series truncation method, we obtain the error between the regularized solution and the exact solution. These results are clearly shown in the Theorem (3.1). The main analytical technique is to use some embeddings and some evaluations using Hölder inequality.

The second part of the paper deals with the final value problem for Problem (1.1) with a linear source function. The new feature of this part is the appearance of observed data, namely $(g_\varepsilon, F_\varepsilon) \in L^p(\Omega) \times L^\infty(0,T; L^p(\Omega))$. We also investigate the case that F has the split form $F(x,t) = \Phi(t)f(x)$, where both functions Φ, f are perturbed by $\Phi_\varepsilon, f_\varepsilon$ in $L^p(0,T)$ and $L^p(\Omega)$ respectively. These results are well described in Theorem (4.1) and Theorem (4.2).

The final part of the paper investigates the inverse initial problem for Problem (1.1) with nonlinear source. The new feature of this section is that we use the observation data $g_\varepsilon \in L^p(\Omega)$. The first one concerns the existence and uniqueness of a regularized solution in the space $L^\infty(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))$ for θ is suitable chosen. The problem stated here is illustrated in the theorem (5.1). The second one provides the upper bound of the error between the regularized solution and the exact solution in $L^{\frac{2N}{N-4\theta}}(\Omega)$. To overcome the difficulties arising from nonlinear components, we have learned and used the techniques in the interesting paper by Tuan–Caraballo [24].

This article is organized as follows. Section 2 deals with some function spaces and embeddings. Section 3 deals with regularized solution for the inverse source problem. The backward problem for linear bi-parabolic equation is investigated in Section 4 with observed data in L^p. Section 5 examines the backward problem for nonlinear bi-parabolic equation. The Fourier series truncation method was used for both sections. The main technique of this paper is to use embeddings and Fourier series.

2. PRELIMINARY RESULTS

In this section, we introduce the notation and the functional setting which shall be used in our paper. Recall that the spectral problem

$$\begin{cases} \Delta\psi_n(x) = -\lambda_n\psi_n(x), & x \in \Omega, \\ \psi_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are $\psi_n \in H_0^1(\Omega)$.

Definition 2.1 (Hilbert scale space)**.** We recall the Hilbert scale space, which is given as follows

$$\mathbb{H}^s(\Omega) = \left\{ f \in L^2(\Omega), \quad \sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_{\Omega} f(x)\psi_n(x)dx \Big)^2 < \infty \right\},$$

for any $s \ge 0$. It is well-known that $\mathbb{H}^s(\Omega)$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_{\Omega} f(x)\psi_n(x)dx \Big)^2 \right)^{1/2}, \quad f \in \mathbb{H}^s(\Omega).$$

Lemma 2.2 ([24])**.** *The following statement are true:*

$$\begin{aligned} &L^p(\Omega) \hookrightarrow \mathbb{H}^\mu(\Omega), \quad \text{if} \quad -\frac{N}{4} < \mu \le 0, \quad p \ge \frac{2N}{N-4\mu}, \\ &\mathbb{H}^s(\Omega) \hookrightarrow L^p(\Omega), \quad \text{if} \quad 0 \le s < \frac{N}{4}, \quad p \le \frac{2N}{N-4s}. \end{aligned} \tag{2.1}$$

3. INVERSE SOURCE PROBLEM

3.1. REPRESENTATION OF SOURCE FUNCTION

Let us give the explicit formula of the mild solution. First, taking the inner product of both sides of (1.1) with $\psi_n(x)$, we find that

$$\begin{aligned} &\frac{d^2}{dt^2}\Big(\int_{\Omega} u(x,t)\psi_n(x)dx \Big) + 2\lambda_n \Big(\int_{\Omega} u(x,t)\psi_n(x)dx \Big) + \lambda_n^2 \Big(\int_{\Omega} u(x,t)\psi_n(x)dx \Big) \\ &= \int_{\Omega} F(x,t)\psi_n(x)dx. \end{aligned}$$

It is easy to see that the latter problem has a solution given by

$$\begin{aligned} \int_{\Omega} u(x,t)\psi_n(x)dx = e^{-t\lambda_n} (1+t\lambda_n) \int_{\Omega} u(x,0)\psi_n(x)dx \\ + \int_0^t (t-\theta)e^{-(t-\theta)\lambda_n} \Big(\int_{\Omega} F(x,\theta)\psi_n(x)dx \Big) d\theta. \end{aligned}$$

Let $t = T$ and from $u(x,0) = 0$ and $u(x,T) = g(x)$, we find that

$$\begin{aligned}\int_\Omega g(x)\psi_n(x)dx &= \int_0^T (t-\theta)e^{-(t-\theta)\lambda_n}\Big(\int_\Omega F(x,\theta)\psi_n(x)dx\Big)d\theta \\ &= \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta\Big)\Big(\int_\Omega f(x)\psi_n(x)dx\Big),\end{aligned}$$

where we have used the fact that $F(x,\theta) = \varphi(\theta)f(x)$. This equality implies that

$$\Big(\int_\Omega f(x)\psi_n(x)dx\Big) = \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta\Big)^{-1}\Big(\int_\Omega g(x)\psi_n(x)dx\Big).$$

3.2. REGULARIZATION OF INVERSE SOURCE PROBLEM IN L^p SPACES

In this subsection, we provide a regularization result concerning on the observed data in L^p spaces. Fourier truncation method is applied to establish approximate solution.

Theorem 3.1. *Let us take $(\varphi_\varepsilon, g_\varepsilon) \in L^p(0,T) \times L^p(\Omega)$ such that $\varphi_\varepsilon(t) > C^* > 0$ for any $0 \le t \le T$, $p \in (1,2)$ and*

$$\|\varphi_\varepsilon - \varphi\|_{L^p(0,T)} + \|g_\varepsilon - g\|_{L^p(\Omega)} \le \varepsilon.$$

Let us assume that $f \in \mathbb{H}^{s+\delta}(\Omega)$ for $0 < \delta < 1$ and $0 < s < \frac{N}{4}$. We construct a regularized solution as follows

$$f_\varepsilon(x) = \sum_{n=1}^{n \le M_\varepsilon} \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi_\varepsilon(\theta)d\theta\Big)^{-1}\Big(\int_\Omega g_\varepsilon(x)\psi_n(x)dx\Big)\psi_n(x).$$

Set $\gamma = \max\left(\frac{1}{p} - \frac{\delta}{p}, \frac{2s+4}{N} - \frac{1}{2} + \frac{1}{p}\right)$ and choose $M_\varepsilon = \varepsilon^{\frac{k-1}{\gamma}}$ for any $0 < k < 1$,

$$\|f_\varepsilon - f\|_{L^{\frac{2N}{N-4s}}(\Omega)} \lesssim \varepsilon^{\frac{(1-k)\delta}{\gamma}}\|f\|_{\mathbb{H}^{\delta+s}(\Omega)} + \varepsilon^k\|f\|_{\mathbb{H}^{\delta+s}(\Omega)} + \varepsilon^k, \tag{3.1}$$

where the hidden constant depends on N, s, δ, p, T.

Proof. Since $1 < p < 2$, we can choose $\mu = \frac{Np-2N}{4p}$. It is easy to verify that $-\frac{N}{4} < \mu \le 0$ and by applying Lemma (2.1), we know that $\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega) \hookrightarrow L^p(\Omega)$. Therefore, we get that the following bound

$$\|g_\varepsilon - g\|_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)} \le C(N,p)\|g_\varepsilon - g\|_{L^p(\Omega)} \le C(N,p)\varepsilon. \tag{3.2}$$

In order to provide the upper bound of the error $\|f_\varepsilon - f\|_{\mathbb{H}^s(\Omega)}$, we need to establish two new following functions

$$\overline{f}_\varepsilon(x) = \sum_{n=1}^{n\le M_\varepsilon} \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi_\varepsilon(\theta)d\theta\Big)^{-1}\Big(\int_\Omega g(x)\psi_n(x)dx\Big)\psi_n(x)$$

and

$$\widetilde{f}_\varepsilon(x) = \sum_{n=1}^{n\le M_\varepsilon} \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta\Big)^{-1}\Big(\int_\Omega g(x)\psi_n(x)dx\Big)\psi_n(x).$$

By using triangle inequality, we obtain that

$$\begin{aligned}\|f_\varepsilon - f\|_{\mathbb{H}^s(\Omega)} &\le \|\widetilde{f}_\varepsilon - f\|_{\mathbb{H}^s(\Omega)} + \|\widetilde{f}_\varepsilon - \overline{f}_\varepsilon\|_{\mathbb{H}^s(\Omega)} + \|f_\varepsilon - \overline{f}_\varepsilon\|_{\mathbb{H}^s(\Omega)} \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{3.3}$$

Our next aim is to derive the upper bound for three terms J_1, J_2 and J_3. Let us divide into three steps as follows.

Step 1. Estimate of $J_1 = \|\widetilde{f}_\varepsilon - f\|_{\mathbb{H}^s(\Omega)}$.

Let us estimate the term $J_1 = \|\widetilde{f}_\varepsilon - f\|_{\mathbb{H}^s(\Omega)}$ for $0 \le s < \frac{N}{4}$. Using Parseval's equality and noting that f belongs to the space $\mathbb{H}^{\delta+s}(\Omega)$ for $0<\delta<1$, we infer the following bound

$$\begin{aligned}|J_1|^2 &= \|\widetilde{f}_\varepsilon - f\|^2_{\mathbb{H}^s(\Omega)} \\ &= \sum_{n=1}^{n>M_\varepsilon} \lambda_n^{2s}\Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi_\varepsilon(\theta)d\theta\Big)^{-2}\Big(\int_\Omega g(x)\psi_n(x)dx\Big)^2 \\ &= \sum_{n=1}^{n>M_\varepsilon} \lambda_n^{2s}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2 = \sum_{n=1}^{n>M_\varepsilon} \lambda_n^{-2\delta}\lambda_n^{2s+2\delta}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2 \\ &\le (M_\varepsilon)^{-2\delta}\sum_{n=1}^{n>M_\varepsilon} \lambda_n^{2s+2\delta}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2 \le (M_\varepsilon)^{-2\delta}\|f\|^2_{\mathbb{H}^{\delta+s}(\Omega)},\end{aligned}$$

for any $\delta > 0$. Thus, we deduce that the following bound holds for the first term on the right hand side of (3.3)

$$J_1 \le (M_\varepsilon)^{-\delta}\|f\|_{\mathbb{H}^{\delta+s}(\Omega)}. \tag{3.4}$$

Step 2. Estimate of $J_2 = \|\widetilde{f}_\varepsilon - \overline{f}_\varepsilon\|_{\mathbb{H}^s(\Omega)}$.

Using Parseval's equality and noting that

$$\int_\Omega g(x)\psi_n(x)dx = \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta\Big)\int_\Omega f(x)\psi_n(x)dx$$

we arrive at

$$\|\widetilde{f}_\varepsilon - \overline{f}_\varepsilon\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{n\le M_\varepsilon} \Bigg[\Big(\int_0^T (T-\theta) e^{-(T-\theta)\lambda_n} \varphi(\theta) d\theta \Big)^{-1} - \Big(\int_0^T (T-\theta) e^{-(T-\theta)\lambda_n} \varphi_\varepsilon(\theta) d\theta \Big)^{-1} \Bigg]^2 \lambda_n^{2s} \Big(\int_\Omega g(x)\psi_n(x) dx \Big)^2$$

$$= \sum_{n=1}^{n\le M_\varepsilon} \underbrace{\left[\frac{\int_0^T (T-\theta) e^{-(T-\theta)\lambda_n} \left(\varphi_\varepsilon(\theta) - \varphi\right) d\theta}{\int_0^T (T-\theta) e^{-(T-\theta)\lambda_n} \varphi_\varepsilon(\theta) d\theta} \right]^2}_{J_{2,1}} \lambda_n^{2s} \Big(\int_\Omega f(x)\psi_n(x) dx \Big)^2. \tag{3.5}$$

Let us consider the term $J_{2,1}$. Using Hölder's inequality, we obtain that

$$\begin{aligned} &\Big| \int_0^T (T-\theta) e^{-(T-\theta)\lambda_n} \left(\varphi_\varepsilon(\theta) - \varphi\right) d\theta \Big| \\ &\le \Big(\int_0^T |\varphi_\varepsilon(\theta) - \varphi|^p \, dt \Big)^{1/p} \Big(\int_0^T (T-\theta)^{p^*} e^{-(T-\theta)p^*\lambda_n} dt \Big)^{1/p^*} \\ &\le \|\varphi_\varepsilon - \varphi\|_{L^p(0,T)} \Big(\int_0^T (T-\theta)^{p^*} e^{-(T-\theta)p^*\lambda_n} dt \Big)^{1/p^*}. \end{aligned} \tag{3.6}$$

By in view of the inequality $e^{-z} \le C_\beta z^{-\beta}$ for any $\beta > 0$, we get that

$$\begin{aligned} \int_0^T (T-\theta)^{p^*} e^{-(T-\theta)p^*\lambda_n} dt &\le C_\beta \int_0^T (T-\theta)^{p^*} \left((T-\theta)p^*\lambda_n\right)^{-\beta} dt \\ &= C_\beta (p^*)^{-\beta} \lambda_n^{-\beta} \int_0^T (T-\theta)^{p^*-\beta} dt \\ &= \frac{C_\beta (p^*)^{-\beta} T^{p^*-\beta+1}}{p^*-\beta+1} \lambda_n^{-\beta}, \end{aligned}$$

where we note that $p^* + 1 > \beta$, i.e, $\beta < \frac{2p-1}{p-1}$. Therefore, we find that

$$\Big(\int_0^T (T-\theta)^{p^*} e^{-(T-\theta)p^*\lambda_n} dt \Big)^{1/p^*} \le C(\beta, p^*, T) \lambda_n^{-\frac{\beta(p-1)}{p}}. \tag{3.7}$$

Combining (3.6) and (3.7) and noting that $\|\varphi_\varepsilon - \varphi\|_{L^p(0,T)} \le \varepsilon$, we arrive at the following estimate

$$\Big|\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\left(\varphi_\varepsilon(\theta)-\varphi\right)d\theta\Big| \le C(\beta,p^*,T)\lambda_n^{-\frac{\beta(p-1)}{p}}\varepsilon.$$

Since $\varphi_\varepsilon > C^*$, we find that

$$\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi_\varepsilon(\theta)d\theta > C^*\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}d\theta = C^*\frac{1-(1+T\lambda_n)e^{-\lambda_n T}}{\lambda_n^2},$$

where we easily to check that

$$\int_0^T e^{z(s-T)}(T-s)ds = \frac{1-(1+Tz)e^{-zT}}{z^2}.$$

Combining two above results, we find that

$$J_{2,1} \le C(\beta,p^*,T)\frac{\lambda_n^{2-\frac{\beta(p-1)}{p}}\varepsilon}{1-(1+T\lambda_n)e^{-\lambda_n T}}.$$

This together with (3.5) implies that

$$\begin{aligned}\|\widetilde{f}_\varepsilon - \overline{f}_\varepsilon\|^2_{\mathbb{H}^s(\Omega)} &\le C(\beta,p^*,T)\sum_{n=1}^{n\le M_\varepsilon}\left(\frac{\lambda_n^{2-\frac{\beta(p-1)}{p}}\varepsilon}{1-(1+T\lambda_n)e^{-\lambda_n T}}\right)^2\lambda_n^{2s}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2\\ &\le \sum_{n=1}^{n\le M_\varepsilon}\frac{C(\beta,p^*,T)\varepsilon^2}{\left(1-(1+T\lambda_n)e^{-\lambda_n T}\right)^2}\lambda_n^{4+2s-\frac{\beta(2p-2)}{p}}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2.\end{aligned} \tag{3.8}$$

Let $h(z) = (1+Tz)e^{-Tz}$ for $z>0$. Then its derivative of it is $h'(z) = -T^2ze^{-Tz} < 0$. Hence, it is an decreasing function on $(0,\infty)$. It implies that $h(\lambda_n) \le h(\lambda_1)$. Therefore, we know immediately that

$$1-(1+T\lambda_n)e^{-\lambda_n T} \ge 1-(1+T\lambda_1)e^{-\lambda_1 T}. \tag{3.9}$$

Let us choose $\delta = \frac{2p-1}{p-1} - \beta$, we follows from (3.8) and (3.9) that

$$|J_2|^2 = \|\widetilde{f}_\varepsilon - \overline{f}_\varepsilon\|^2_{\mathbb{H}^s(\Omega)} \le \frac{C(\beta,p^*,T)\varepsilon^2}{1-(1+T\lambda_1)e^{-\lambda_1 T}}\sum_{n=1}^{n\le M_\varepsilon}\lambda_n^{2s+2\delta+\frac{2}{p}-\frac{2\delta}{p}}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2.$$

Since $\delta \in (0,1)$, we can verify that for $n \le M_\varepsilon$

$$\lambda_n^{2s+2\delta+\frac{2}{p}-\frac{2\delta}{p}} \le \lambda_n^{2s+2\delta}\left(M_\varepsilon\right)^{\frac{2}{p}-\frac{2\delta}{p}},$$

which allows us to provide that

$$
\begin{aligned}
|J_2|^2 &\le C_2^2 \left(M_\varepsilon\right)^{\frac{2}{p}-\frac{2\delta}{p}} \varepsilon^2 \sum_{n=1}^{n\le M_\varepsilon} \lambda_n^{2s+2\delta} \Big(\int_\Omega f(x)\psi_n(x)dx \Big)^2 \\
&= C_2^2 \left(M_\varepsilon\right)^{\frac{2}{p}-\frac{2\delta}{p}} \varepsilon^2 \|f\|^2_{\mathbb{H}^{\delta+s}(\Omega)}
\end{aligned}
$$

where C_2 depends on δ, λ_1, T. Thus, we deduce that the following bound for the second term on the right hand side of (3.3)

$$
J_2 \le C_2 \left(M_\varepsilon\right)^{\frac{1}{p}-\frac{\delta}{p}} \varepsilon \|f\|_{\mathbb{H}^{\delta+s}(\Omega)}. \tag{3.10}
$$

Step 3. Estimate of $J_3 = \|f_\varepsilon - \overline{f}_\varepsilon\|_{\mathbb{H}^s(\Omega)}$.
Using Parseval's equality, we find that

$$
\begin{aligned}
|J_3|^2 &= \sum_{n=1}^{n\le M_\varepsilon} \lambda_n^{2s} \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta \Big)^{-2} \Big(\int_\Omega \Big(g_\varepsilon(x)-g(x)\Big)\psi_n(x)dx \Big)^2 \\
&\le \sum_{n=1}^{n\le M_\varepsilon} \lambda_n^{2s-\frac{Np-2N}{2p}} \lambda_n^{\frac{Np-2N}{2p}} \Big(\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta \Big)^{-2} \\
&\quad \times \Big(\int_\Omega \Big(g_\varepsilon(x)-g(x)\Big)\psi_n(x)dx \Big)^2.
\end{aligned} \tag{3.11}
$$

Due to the condition $\varphi > C$, we get that the following estimate

$$
\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}\varphi(\theta)d\theta > C\int_0^T (T-\theta)e^{-(T-\theta)\lambda_n}d\theta = C\frac{1-(1+T\lambda_n)e^{-\lambda_n T}}{\lambda_n^2}.
$$

This inequality together with (3.9) and (3.11) yields

$$
\begin{aligned}
|J_3|^2 &\le \frac{1}{C\left(1-(1+T\lambda_1)e^{-\lambda_1 T}\right)^2} \\
&\quad \times \sum_{n=1}^{n\le M_\varepsilon} \lambda_n^{2s+4-\frac{Np-2N}{2p}} \lambda_n^{\frac{Np-2N}{2p}} \Big(\int_\Omega \Big(g_\varepsilon(x)-g(x)\Big)\psi_n(x)dx \Big)^2.
\end{aligned} \tag{3.12}
$$

From the fact that $\lambda_n \le \overline{C}n^{2/N}$, we can verify that for $n \le M_\varepsilon$

$$
\lambda_n^{2s+4-\frac{Np-2N}{2p}} \le \mathbf{C}n^{\frac{2}{N}\left(2s+4-\frac{Np-2N}{2p}\right)} \le \mathbf{C}\left(M_\varepsilon\right)^{\frac{4s+8}{N}-1+\frac{2}{p}}. \tag{3.13}
$$

By combining (3.12) and (3.13), we arrive at

$$|J_3|^2 \leq \frac{\mathbf{C}\left(M_\varepsilon\right)^{\frac{4s+8}{N}-1+\frac{2}{p}}}{C\left(1-(1+T\lambda_1)e^{-\lambda_1 T}\right)^2}\sum_{n=1}^{\infty}\lambda_n^{\frac{Np-2N}{2p}}\Big(\int_\Omega \left(g_\varepsilon(x)-g(x)\right)\psi_n(x)dx\Big)^2$$
$$\leq C(\lambda_1,T)\left(M_\varepsilon\right)^{\frac{4s+8}{N}-1+\frac{2}{p}}\|g_\varepsilon-g\|^2_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}.$$

Since the estimate (3.2), we follows from the latter observation that

$$|J_3|^2 \leq C_3^2\left(M_\varepsilon\right)^{\frac{4s+8}{N}-1+\frac{2}{p}}\varepsilon^2.$$

Here C_3 depends on N, p, λ_1, T. Thus, we obtain the following bound for the third term on the right hand side of (3.3)

$$J_3 \leq C_3\left(M_\varepsilon\right)^{\frac{2s+4}{N}-\frac{1}{2}+\frac{1}{p}}\varepsilon. \tag{3.14}$$

Combining (3.3), (3.4), (3.10) and (3.14), one has the following bound

$$\begin{aligned}\|f_\varepsilon-f\|_{\mathbb{H}^s(\Omega)} &\leq \left(M_\varepsilon\right)^{-\delta}\|f\|_{\mathbb{H}^{\delta+s}(\Omega)}+C_2\left(M_\varepsilon\right)^{\frac{1}{p}-\frac{\delta}{p}}\varepsilon\|f\|_{\mathbb{H}^{\delta+s}(\Omega)}\\ &\quad+C_3\left(M_\varepsilon\right)^{\frac{2s+4}{N}-\frac{1}{2}+\frac{1}{p}}\varepsilon.\end{aligned} \tag{3.15}$$

This follows from (3.15) that

$$\|f_\varepsilon-f\|_{\mathbb{H}^s(\Omega)} \leq \varepsilon^{\frac{(1-k)\delta}{\gamma}}\|f\|_{\mathbb{H}^{\delta+s}(\Omega)}+C_2\varepsilon^k\|f\|_{\mathbb{H}^{\delta+s}(\Omega)}+C_3\varepsilon^k.$$

By using Lemma 2.2, we know that $\mathbb{H}^s(\Omega) \hookrightarrow L^{\frac{2N}{N-4s}}(\Omega)$, which yields to the desired result (3.1). □

4. INVERSE INITIAL PROBLEM FOR BIPARABOLIC EQUATION

In this section, we consider the following inverse initial problem

$$\begin{cases} u_{tt}(x,t)+2\Delta u_t(x,t)+\Delta^2 u(x,t)=F(x,t), & \text{in } \Omega\times(0,T],\\ u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0, & \text{in } \Omega,\end{cases} \tag{4.1}$$

with the following final observation

$$u_t(x,T)=0, \quad u(x,T)=g(x) \quad \text{in } \Omega. \tag{4.2}$$

The main objective of this section is to provide a normalized solution to the inverse problem (4.1)–(4.2) when the input data is noisy in the L^p space.

Let us first give the explicit formula of the mild solution to Problem (4.1)–(4.2). First, taking the inner product of both sides of (1.1) with $\psi_n(x)$, we find that

$$\frac{d^2}{dt^2}\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big) + 2\lambda_n\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big) + \lambda_n^2\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big)$$
$$= \int_\Omega F(x,t)\psi_n(x)dx,$$

and from final observation as in (4.2), we have immediately that

$$\int_\Omega u(x,T)\psi_n(x)dx = \int_\Omega g(x)\psi_n(x)dx, \quad \frac{d}{dt}\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big)|_{t=T} = 0.$$

By solving the above system, we can obtain that

$$\int_\Omega u(x,t)\psi_n(x)dx = e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g(x)\psi_n(x)dx$$
$$- \int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Big(\int_\Omega F(x,\theta)\psi_n(x)dx\Big)d\theta,$$

which allows us to deduce that the mild solution to Problem (4.1)–(4.2) in the following

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g(x)\psi_n(x)dx\right]\psi_n(x) \\ &- \sum_{n=1}^{\infty}\left[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Big(\int_\Omega F(x,\theta)\psi_n(x)dx\Big)d\theta\right]\psi_n(x). \end{aligned} \tag{4.3}$$

By applying Fourier truncation method, we provide a regularized solution as follows

$$\begin{aligned} u_\varepsilon(x,t) = &\sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g_\varepsilon(x)\psi_n(x)dx\right]\psi_n(x) \\ &- \sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\left[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Big(\int_\Omega F_\varepsilon(x,\theta)\psi_n(x)dx\Big)d\theta\right]\psi_n(x), \end{aligned} \tag{4.4}$$

where $\mathcal{N}_\varepsilon$ is a parameter regularization which will be defined later.

Theorem 4.1. *Let us take the noisy data* $(g_\varepsilon, F_\varepsilon) \in L^p(\Omega) \times L^\infty(0,T;L^p(\Omega))$ *such that*

$$\|g_\varepsilon - g\|_{L^p(\Omega)} + \|F_\varepsilon - F\|_{L^\infty(0,T;L^p(\Omega))} \le \varepsilon.$$

Let us assume that $u \in L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))$ *for any* $\beta > 0$. *If we choose*

$$\mathcal{N}_\varepsilon = \left(\frac{1-\theta}{T\overline{C}} \log\left(\frac{1}{\varepsilon}\right)\right)^{N/2}$$

for any $0<\theta<1$, *then* $\|u(\cdot,t) - u_\varepsilon(\cdot,t)\|^2_{L^{\frac{2N}{N-4s}}(\Omega)}$ *is of order*

$$\max\left(\varepsilon^{2\theta} \left(\frac{1-\theta}{T\overline{C}} \log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{4s-N+\frac{2N}{p}}{2}}, \left(\frac{1-\theta}{T\overline{C}} \log\left(\frac{1}{\varepsilon}\right)\right)^{-2\beta}\right).$$

Proof. By the Sobolev embedding $L^p(\Omega) \hookrightarrow \mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)$, we have the following statement

$$\begin{aligned} &\|g_\varepsilon - g\|_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)} + \|F_\varepsilon - F\|_{L^\infty(0,T;\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega))} \\ &\le C(N,p)\|g_\varepsilon - g\|_{L^p(\Omega)} + C(N,p)\|F_\varepsilon - F\|_{L^\infty(0,T;L^p(\Omega))} \le C(N,p)\varepsilon. \end{aligned} \tag{4.5}$$

Our next goal is to provide the upper bound of the term $\|u - u_\varepsilon\|_{\mathbb{H}^s(\Omega)}$. To facilitate this investigation, we introduce a new function

$$\begin{aligned} \mathcal{U}_\varepsilon(x,t) = &\sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right) \int_\Omega g(x)\psi_n(x)dx \right]\psi_n(x) \\ &- \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \left[\int_t^T (\theta - t)e^{(\theta-t)\lambda_n} \Big(\int_\Omega F(x,\theta)\psi_n(x)dx\Big) d\theta \right]\psi_n(x). \end{aligned} \tag{4.6}$$

Step 1. Estimate of the term $\|\mathcal{U}_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$.

From the definition of two functions $\mathcal{U}_\varepsilon$ and u_ε as in (4.6) and (4.4) respectively, we find that

$$\begin{aligned} &\mathcal{U}_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t) \\ &= \underbrace{\sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right) \int_\Omega \Big(g(x) - g_\varepsilon(x)\Big)\psi_n(x)dx \right]\psi_n(x)}_{\mathscr{B}_1(x,t)} \\ &\underbrace{- \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \left[\int_t^T (\theta - t)e^{(\theta-t)\lambda_n} \Big(\int_\Omega \Big(F(x,\theta) - F_\varepsilon(x,\theta)\Big)\psi_n(x)dx\Big) d\theta \right]\psi_n(x)}_{\mathscr{B}_2(x,t)} \\ &= \mathscr{B}_1(x,t) + \mathscr{B}_2(x,t). \end{aligned} \tag{4.7}$$

In view of Parseval's equality, we estimate the first term on the right hand side of (4.7) as follows

$$\|\mathscr{B}_1\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s-\frac{Np-2N}{2p}} \lambda_n^{\frac{Np-2N}{2p}} e^{2(T-t)\lambda_n} \left(1-(T-t)\lambda_n\right)^2 \times \left[\int_\Omega \Big(g(x)-g_\varepsilon(x)\Big)\psi_n(x)dx\right]^2.$$

Using the inequality $(a+b)^2 \le 2a^2+2b^2$, one has $\left(1-(T-t)\lambda_n\right)^2 \le 2+2T^2\lambda_n^2$. This implies that

$$\|\mathscr{B}_1\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s-\frac{Np-2N}{2p}} \Big(2+2T^2\lambda_n^2\Big) e^{2T\lambda_n} \lambda_n^{\frac{Np-2N}{2p}} \times \left[\int_\Omega \Big(g(x)-g_\varepsilon(x)\Big)\psi_n(x)dx\right]^2. \tag{4.8}$$

From the fact that $\lambda_n \le \overline{C} n^{2/N}$ and noting that $s \ge \frac{N}{4} - \frac{N}{2p}$, we can verify that for $n \le M_\varepsilon$

$$\begin{aligned}
&\lambda_n^{2s-\frac{Np-2N}{2p}} \Big(2+2T^2\lambda_n^2\Big) e^{2T\lambda_n} \\
&= 2e^{2T\lambda_n} \lambda_n^{2s-\frac{Np-2N}{2p}} + 2T^2 e^{2T\lambda_n} \lambda_n^{2s-\frac{Np-2N}{2p}+2} \\
&\le 2C_1 e^{2T\overline{C}n^{2/N}} n^{\frac{4s}{N}+\frac{2}{p}-1} + 2C_2T^2 e^{2T\overline{C}n^{2/N}} n^{\frac{4s+4}{N}+\frac{2}{p}-1} \\
&\le 2C_1 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1} + 2C_2T^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1} \\
&\le C_3 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1} \Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big).
\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we know that

$$\begin{aligned}
\|\mathscr{B}_1\|^2_{\mathbb{H}^s(\Omega)} &\le C_3 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1} \Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big) \\
&\quad \times \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{\frac{Np-2N}{2p}} \left[\int_\Omega \Big(g(x)-g_\varepsilon(x)\Big)\psi_n(x)dx\right]^2 \\
&= C_3 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (M_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1} \Big(1+(M_\varepsilon)^{-\frac{4}{N}}\Big) \|g_\varepsilon - g\|^2_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}.
\end{aligned}$$

It follows from (4.5) that

$$\|\mathscr{B}_1\|^2_{\mathbb{H}^s(\Omega)} \le C_3 |C(N,p)|^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1} \varepsilon^2 \Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big). \tag{4.10}$$

Let us continue to consider the term $\|\mathscr{B}_2\|^2_{\mathbb{H}^s(\Omega)}$. By applying Parseval's equality, we have that

$$
\begin{aligned}
&\|\mathscr{B}_2\|^2_{\mathbb{H}^s(\Omega)}\\
&= \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s}\left[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\left|\int_\Omega \Big(F(x,\theta)-F_\varepsilon(x,\theta)\Big)\psi_n(x)dx\right|d\theta\right]^2\\
&\le \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s}\int_t^T (\theta-t)^2e^{2(\theta-t)\lambda_n}\left|\int_\Omega \Big(F(x,\theta)-F_\varepsilon(x,\theta)\Big)\psi_n(x)dx\right|^2d\theta \qquad (4.11)\\
&\le T^2\sum_{n=1}^{n\le\mathcal{N}_\varepsilon} e^{2T\lambda_n}\lambda_n^{2s-\frac{Np-2N}{2p}}\lambda_n^{\frac{Np-2N}{2p}}\int_t^T \left|\int_\Omega \Big(F(x,\theta)-F_\varepsilon(x,\theta)\Big)\psi_n(x)dx\right|^2d\theta.
\end{aligned}
$$

From the fact that $\lambda_n \le \overline{C}n^{2/N}$ and noting that $s \ge \frac{N}{4}-\frac{N}{2p}$, we can verify that for $n \le M_\varepsilon$

$$
\lambda_n^{2s-\frac{Np-2N}{2p}}e^{2T\lambda_n} \le 2C_1e^{2T\overline{C}n^{2/N}}n^{\frac{4s}{N}+\frac{2}{p}-1} \le 2C_1e^{2T\overline{C}M_\varepsilon^{2/N}}(M_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}. \qquad (4.12)
$$

It follows from (4.11) that

$$
\begin{aligned}
\|\mathscr{B}_2\|^2_{\mathbb{H}^s(\Omega)} &\le 2C_1e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}T^2\\
&\quad\times\sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\lambda_n^{\frac{Np-2N}{2p}}\int_t^T\left|\int_\Omega \Big(F(x,\theta)-F_\varepsilon(x,\theta)\Big)\psi_n(x)dx\right|^2d\theta\\
&\le 2C_1e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}T^2\int_0^T\|F_\varepsilon(\cdot,\theta)-F(\cdot,\theta)\|^2_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}d\theta\\
&\le 2C_1T^3e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\|F_\varepsilon-F\|^2_{L^\infty(0,T;\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega))}.
\end{aligned}
$$

It follows from (4.5) that

$$
\|\mathscr{B}_2\|^2_{\mathbb{H}^s(\Omega)} \le 2C_1T^3e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2. \qquad (4.13)
$$

Combining (4.7), (4.10) and (4.13), we arrive at the following estimate

$$
\begin{aligned}
\|\mathcal{U}_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} &\le 2\|\mathscr{B}_1\|^2_{\mathbb{H}^s(\Omega)}+2\|\mathscr{B}_2\|^2_{\mathbb{H}^s(\Omega)}\\
&\le 2C_3|C(N,p)|^2e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1}\varepsilon^2\Big(1+(M_\varepsilon)^{-\frac{4}{N}}\Big)\\
&\quad+4C_1T^3e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2.
\end{aligned} \qquad (4.14)
$$

Step 2. Estimate of the term $\|\mathcal{U}_\varepsilon(\cdot,t)-u(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$.
By a combination of (4.7) and (4.3) and using Parseval's equality, we derive that

$$\begin{aligned}\|\mathcal{U}_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n>\mathcal{N}_\varepsilon}\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big)^2\\ &= \sum_{n>\mathcal{N}_\varepsilon}\lambda_n^{-2\beta}\lambda_n^{2\beta+2s}\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big)^2,\end{aligned}$$

for any $\beta>0$. If $n>\mathcal{N}_\varepsilon$ then we can check that there exists a positive constant $\widetilde{C}>0$ such that $\lambda_n^{-2\beta}\le Cn^{-\frac{4\beta}{N}}\le \widetilde{C}(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}$. Hence, we find that

$$\begin{aligned}\|\mathcal{U}_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} &\le \widetilde{C}(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\sum_{n>\mathcal{N}_\varepsilon}\lambda_n^{2\beta+2s}\Big(\int_\Omega u(x,t)\psi_n(x)dx\Big)^2\\ &\le \widetilde{C}(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\|u(\cdot,t)\|^2_{\mathbb{H}^{\beta+s}(\Omega)}\\ &\le \widetilde{C}(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\|u\|^2_{L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))}.\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15) and using the inequality $(c+d)^2\le 2c^2+2d^2$ for any c and d, we deduce that the following estimate

$$\begin{aligned}&\|\mathcal{U}_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\ &\le 2\|\mathcal{U}_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}+2\|\mathcal{U}_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\ &\le 4C_3|C(N,p)|^2e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1}\varepsilon^2\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\\ &\quad+8C_1T^3e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2+2\widetilde{C}(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\|u\|^2_{L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))}.\end{aligned} \tag{4.16}$$

By recall that $\mathcal{N}_\varepsilon=\Big(\frac{1-\theta}{T\overline{C}}\log(\frac{1}{\varepsilon})\Big)^{N/2}$ we can verify that two following equality

$$e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2=\varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\Big(\frac{1}{\varepsilon}\Big)\right)^{\frac{4s-N+\frac{2N}{p}}{2}}$$

and

$$\begin{aligned}&e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\varepsilon^2\\ &=\varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\Big(\frac{1}{\varepsilon}\Big)\right)^{\frac{4s-N+\frac{2N}{p}}{2}}\Big(1+\Big(\frac{1-\theta}{T\overline{C}}\log\Big(\frac{1}{\varepsilon}\Big)\Big)^{-2}\Big).\end{aligned}$$

Two above identities together with (4.16) complete the proof. □

In the following theorem, we provide a regularization result in the case that F has a split form.

Theorem 4.2. *Let us assume that $F(x,t) = \Phi(t)f(x)$. Let us assume that ϕ, f, g are measured by the observation data $\phi_\varepsilon, f_\varepsilon, g_\varepsilon$ respectively, such that*

$$\|\Phi - \Phi_\varepsilon\|_{L^p(0,T)} + \|g_\varepsilon - g\|_{L^p(\Omega)} + \|f_\varepsilon - f\|_{L^p(\Omega)} \le \varepsilon.$$

Let us assume that $u \in L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))$ for any $\beta > 0$. If we choose $\mathcal{N}_\varepsilon = \left(\frac{1-\theta}{T\overline{C}}\log(\frac{1}{\varepsilon})\right)^{N/2}$ for any $0 < \theta < 1$, then $\|u(\cdot,t) - v_\varepsilon(\cdot,t)\|^2_{L^{\frac{2N}{N-4s}}(\Omega)}$ is of order

$$\max\left[\varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{4s+4-N+\frac{2N}{p}}{2}}\left(1+\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{-2}\right), \left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{-2\beta}\right].$$

Proof. From (4.3), we remind the formula of u in the case $F(x,t) = \Phi(t)f(x)$ as follows

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g(x)\psi_n(x)dx\right]\psi_n(x) \\ &\quad - \sum_{n=1}^{\infty}\left[\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Phi(\theta)d\theta\right]\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x). \end{aligned}$$

Let us construct a regularized solution as follows

$$\begin{aligned} v_\varepsilon(x,t) &= \sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g_\varepsilon(x)\psi_n(x)dx\right]\psi_n(x) \\ &\quad - \sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\Big[\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Phi_\varepsilon(\theta)d\theta\Big]\Big(\int_\Omega f_\varepsilon(x)\psi_n(x)dx\Big)\psi_n(x). \end{aligned} \tag{4.17}$$

Set new function $\mathcal{V}_\varepsilon$ as follows

$$\begin{aligned} \mathcal{V}_\varepsilon(x,t) &= \sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\left[e^{(T-t)\lambda_n}\left(1-(T-t)\lambda_n\right)\int_\Omega g(x)\psi_n(x)dx\right]\psi_n(x) \\ &\quad - \sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\Big[\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Phi(\theta)d\theta\Big]\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x). \end{aligned} \tag{4.18}$$

Let us look the term $\|\mathcal{V}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$. It is treated in exactly the same way as in the proof of step 2 of the Theorem (4.1). Indeed, we have that

$$\|\mathcal{V}_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} \lesssim (\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\|u\|^2_{L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))}, \quad \beta > 0. \tag{4.19}$$

Let us now to treat the term $\|\mathcal{V}_\varepsilon(\cdot,t) - v_\varepsilon(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$. Indeed, by from the definition of (4.17) and (4.18), we verify that

$$v_\varepsilon(x,t) - \mathcal{V}_\varepsilon(x,t) = \mathscr{B}_1(x,t) + \mathscr{B}_2(x,t) \tag{4.20}$$

where $\mathscr{B}_1$ is defined in (4.7) and $\mathscr{B}_2$ is defined by

$$\begin{aligned}\mathscr{B}_2(x,t) = &\sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \Big[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Phi(\theta)d\theta\Big]\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x)\\ &- \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \Big[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Phi_\varepsilon(\theta)d\theta\Big]\Big(\int_\Omega f_\varepsilon(x)\psi_n(x)dx\Big)\psi_n(x).\end{aligned}$$

The analysis of estimating for $\mathscr{B}_1$ is given in (4.10). Let us now to estimate the term $\mathscr{B}_2$. In order to further consideration, we should divide the term $\mathscr{B}_2$ into the sum of two terms $\mathscr{B}_3$ and $\mathscr{B}_4$ as follows

$$\mathscr{B}_3(x,t) = \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \Big[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Phi_\varepsilon(\theta)d\theta\Big]\Big[\int_\Omega \Big(f(x)-f_\varepsilon(x)\Big)\psi_n(x)dx\Big]\psi_n(x)$$

and

$$\mathscr{B}_4(x,t) = \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \Big[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Big(\Phi(\theta)-\Phi_\varepsilon(\theta)\Big)d\theta\Big]\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x).$$

Let us consider the term $\|\mathscr{B}_3(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$. Using Parseval's equality, we find that

$$\begin{aligned}&\|\mathscr{B}_3(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\ &= \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s}\Big[\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Phi_\varepsilon(\theta)d\theta\Big]^2\Big[\int_\Omega \Big(f(x)-f_\varepsilon(x)\Big)\psi_n(x)dx\Big]^2\end{aligned}$$

Using Hölder inequality, we obtain that

$$\begin{aligned}\Big|\int_t^T (\theta-t)e^{(\theta-t)\lambda_n}\Phi_\varepsilon(\theta)d\theta\Big| &\le \Big(\int_0^T |\Phi_\varepsilon(\theta)|^p\,dt\Big)^{1/p}\Big(\int_t^T (\theta-t)^{p^*}e^{(\theta-t)p^*\lambda_n}dt\Big)^{1/p^*}\\ &\le \|\Phi_\varepsilon(\theta)\|_{L^p(0,T)}\Big(\int_t^T (\theta-t)^{p^*}e^{(\theta-t)p^*\lambda_n}dt\Big)^{1/p^*}\\ &\le T^{\frac{p^*+1}{p^*}}e^{T\lambda_n}\|\Phi_\varepsilon(\theta)\|_{L^p(0,T)} \le T^{\frac{p^*+1}{p^*}}e^{T\lambda_n}\widetilde{M}.\end{aligned}$$

This implies that the following estimate for $\|\mathscr{B}_3(\cdot,t)\|_{\mathbb{H}^s(\Omega)}$

$$\begin{aligned}&\|\mathscr{B}_3(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\&\le \Big(T^{\frac{p^*+1}{p^*}}\widetilde{M}\Big)^2\sum_{n=1}^{n\le\mathcal{N}_\varepsilon} e^{2T\lambda_n}\lambda_n^{2s-\frac{Np-2N}{2p}}\lambda_n^{\frac{Np-2N}{2p}}\Big[\int_\Omega\Big(f(x)-f_\varepsilon(x)\Big)\psi_n(x)dx\Big]^2.\end{aligned}$$

Because of the fact that (4.12), we infer that the following estimate for the term $\|\mathscr{B}_3(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}$

$$\begin{aligned}\|\mathscr{B}_3(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} &\le 2C_1\Big(T^{\frac{p^*+1}{p^*}}\widetilde{M}\Big)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\\&\quad\times\sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\lambda_n^{\frac{Np-2N}{2p}}\Big[\int_\Omega\Big(f(x)-f_\varepsilon(x)\Big)\psi_n(x)dx\Big]^2\\&\le 2C_1\Big(T^{\frac{p^*+1}{p^*}}\widetilde{M}\Big)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\|f_\varepsilon-f\|^2_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}\\&\le 2C_1\Big(T^{\frac{p^*+1}{p^*}}\widetilde{M}\Big)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2.\end{aligned}\tag{4.21}$$

Let us now to consider the term $\|\mathscr{B}_4(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}$. Using Hölder inequality, we obtain that

$$\begin{aligned}&\Big|\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Big(\Phi(\theta)-\Phi_\varepsilon(\theta)\Big)d\theta\Big|\\&\le\Big(\int_0^T|\Phi(\theta)-\Phi_\varepsilon(\theta)|^p\,dt\Big)^{1/p}\Big(\int_t^T(\theta-t)^{p^*}e^{(\theta-t)p^*\lambda_n}dt\Big)^{1/p^*}\\&\le\|\Phi-\Phi_\varepsilon\|_{L^p(0,T)}\Big(\int_t^T(\theta-t)^{p^*}e^{(\theta-t)p^*\lambda_n}dt\Big)^{1/p^*}\le T^{\frac{p^*+1}{p^*}}e^{T\lambda_n}\varepsilon.\end{aligned}\tag{4.22}$$

Using Parseval's equality, we get the following identity

$$\begin{aligned}&\|\mathscr{B}_4(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\&=\sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\lambda_n^{2s}\Big[\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Big(\Phi(\theta)-\Phi_\varepsilon(\theta)\Big)d\theta\Big]^2\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2\\&=\sum_{n=1}^{n\le\mathcal{N}_\varepsilon}\lambda_n^{2s-\frac{Np-2N}{2p}}\lambda_n^{\frac{Np-2N}{2p}}\Big[\int_t^T(\theta-t)e^{(\theta-t)\lambda_n}\Big(\Phi(\theta)-\Phi_\varepsilon(\theta)\Big)d\theta\Big]^2\Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2.\end{aligned}$$

This equality together with (4.22) implies that

$$\|\mathscr{B}_4(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} \le \varepsilon^2 \left(T^{\frac{p^*+1}{p^*}}\right)^2 \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{2s-\frac{Np-2N}{2p}} e^{2T\lambda_n} \lambda_n^{\frac{Np-2N}{2p}} \Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2.$$

By in view of (4.12), we arrive at the following estimate

$$\begin{aligned}
&\|\mathscr{B}_4(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\
&\le 2C_1 e^{2T\overline{C}M_\varepsilon^{2/N}} (M_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \left(T^{\frac{p^*+1}{p^*}}\right)^2 \sum_{n=1}^{n\le\mathcal{N}_\varepsilon} \lambda_n^{\frac{Np-2N}{2p}} \Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2\\
&\le 2C_1 e^{2T\overline{C}M_\varepsilon^{2/N}} (M_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \left(T^{\frac{p^*+1}{p^*}}\right)^2 \|f\|^2_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}\\
&\le 2C_1 \left(T^{\frac{p^*+1}{p^*}}\right)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (M_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \|f\|^2_{L^p(\Omega)}
\end{aligned} \tag{4.23}$$

where we have used that Sobolev embedding $L^p(\Omega) \hookrightarrow \mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)$. Combining (4.21) and (4.23) and noting that the inequality $(c+d)^2 \le 2c^2+2d^2$ for any c and d, we derive that

$$\begin{aligned}
&\|\mathscr{B}_2(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} \le 2\|\mathscr{B}_3(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} + 2\|\mathscr{B}_4(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\
&\le 4C_1 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1} \left(T^{\frac{p^*+1}{p^*}}\widetilde{M}\right)^2 \varepsilon^2\\
&\quad + 4C_1 \left(T^{\frac{p^*+1}{p^*}}\right)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \|f\|^2_{L^p(\Omega)}.
\end{aligned}$$

This inequality follows from (4.20) and (4.10), we find that

$$\begin{aligned}
\|v_\varepsilon(\cdot,t) - \mathcal{V}_\varepsilon(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} &\le 2\|\mathscr{B}_1(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} + 2\|\mathscr{B}_2(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\
&\le 2C_3|C(N,p)|^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1}\varepsilon^2\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\\
&\quad + 8C_1 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1} \left(T^{\frac{p^*+1}{p^*}}\widetilde{M}\right)^2 \varepsilon^2\\
&\quad + 8C_1 \left(T^{\frac{p^*+1}{p^*}}\right)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \|f\|^2_{L^p(\Omega)}.
\end{aligned}$$

This together with (4.19) allows us to get that

$$\begin{aligned}
&\|v_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\
&\le 2\|\mathcal{V}_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)} + 2\|v_\varepsilon(\cdot,t) - \mathcal{V}_\varepsilon(\cdot,t)\|^2_{\mathbb{H}^s(\Omega)}\\
&\lesssim 2(\mathcal{N}_\varepsilon)^{-\frac{4\beta}{N}}\|u\|^2_{L^\infty(0,T;\mathbb{H}^{\beta+s}(\Omega))} + 16C_1 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1} \left(T^{\frac{p^*+1}{p^*}}\widetilde{M}\right)^2 \varepsilon^2\\
&\quad + 4C_3|C(N,p)|^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1}\varepsilon^2\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\\
&\quad + 16C_1 \left(T^{\frac{p^*+1}{p^*}}\right)^2 e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}} (\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 \|f\|^2_{L^p(\Omega)}.
\end{aligned} \tag{4.24}$$

By recall that $\mathcal{N}_\varepsilon = \Big(\frac{1-\theta}{T\overline{C}}\log(\frac{1}{\varepsilon})\Big)^{N/2}$ we can check that some following equality

$$e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\varepsilon^2 = \varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{4s-N+\frac{2N}{p}}{2}}$$

and

$$\begin{aligned}
&e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s}{N}+\frac{2}{p}-1}\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\varepsilon^2 \\
&= \varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{4s-N+\frac{2N}{p}}{2}}\left(1+\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{-2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&e^{2T\overline{C}\mathcal{N}_\varepsilon^{2/N}}(\mathcal{N}_\varepsilon)^{\frac{4s+4}{N}+\frac{2}{p}-1}\Big(1+(\mathcal{N}_\varepsilon)^{-\frac{4}{N}}\Big)\varepsilon^2 \\
&= \varepsilon^{2\theta}\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{4s+4-N+\frac{2N}{p}}{2}}\left(1+\left(\frac{1-\theta}{T\overline{C}}\log\left(\frac{1}{\varepsilon}\right)\right)^{-2}\right)
\end{aligned}$$

From three above observations and in view of (4.24), we obtain the desired result. □

5. REGULARIZATION OF NONLINEAR BIPARABOLIC EQUATION ON L^p SPACES

In this section, we will study the initial inverse problem in the nonlinear case of source term. We consider the following problem

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = F(u(x,t)), & \text{in } \Omega\times(0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, & \text{in } \Omega, \end{cases} \tag{5.1}$$

with the following final observation

$$u_t(x,T) = 0, \quad u(x,T) = g(x) \quad \text{ in } \quad \Omega. \tag{5.2}$$

Let $\mathcal{S}(t)$ is a semigroup generated by the Laplacian $-\Delta$. The representation of $\mathcal{S}(t)f$ as a Fourier series is given by the following obvious formula

$$\mathcal{S}(t)f = \sum_{n=1}^{\infty} e^{-t\lambda_n}\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x), \quad f\in L^2(\Omega).$$

Let us provide the operator $\mathcal{H}(t)$ as follows

$$\mathcal{H}(t)f = \sum_{n=1}^{\infty}\Big(1-(T-t)\lambda_n\Big)\Big(\int_\Omega f(x)\psi_n(x)dx\Big)\psi_n(x).$$

By the same steps as in the first part of Section 4, we immediately have the formula for the mild solution of the problem (5.1)–(5.2) as follows

$$u(t) = \mathcal{S}^{-1}(T-t)\mathcal{H}(t)g - \int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)F(u(\nu))d\nu. \tag{5.3}$$

We will regularized the mild solution (5.3) by Fourier method. Our method is described as follows. For any $\mathcal{M} > 0$, let $\mathbb{Q}_{\mathcal{M}}$ be the orthogonal projection onto the eigenspace span $\{\psi_n, \lambda_n \leq \mathcal{M}\}$. Let any function $f \in L^2(\Omega)$ then we have the definition of $\mathbb{Q}_{\mathcal{M}} f$ as follows

$$\mathbb{Q}_{\mathcal{M}} f = \sum_{n=1}^{\lambda_n \leq \mathcal{M}} \Big(\int_\Omega f(x)\psi_n(x)dx \Big) \psi_n(x).$$

Let us assume that $g_\varepsilon \in L^p(\Omega)$ such that $\|g_\varepsilon - g\|_{L^p(\Omega)} \leq \varepsilon$. Let us construct a regularized solution $\mathcal{W}_\varepsilon$ as follows

$$\mathcal{W}_\varepsilon(t) = \mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon - \int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)\mathbb{Q}_{\mathcal{M}_\varepsilon} F(\mathcal{W}_\varepsilon(\nu)(\nu))d\nu. \tag{5.4}$$

Let us first provide the following theorem which shows the existence of the mild solution to regularized problem (5.4) and its well-posedness.

Theorem 5.1. *Let the terminal data $g_\varepsilon \in L^p(\Omega)$. Then the nonlinear integral equation (5.4) has a unique solution $\mathcal{W}_\varepsilon \in L^\infty(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))$. Moroever, the following statement holds*

$$\|\mathcal{W}_\varepsilon(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \leq 2\Big(1 + T\mathcal{M}_\varepsilon\Big)(\mathcal{M}_\varepsilon)^{\theta - \frac{Np-2N}{4p}} e^{(T-t)(\mathcal{M}_\varepsilon + \mu)} \|g_\varepsilon\|_{L^p(\Omega)}, \tag{5.5}$$

where $\mu \geq 2K_f(\mathcal{M}_\varepsilon)^\theta T + \mathcal{M}_\epsilon$.

Proof. Let any $f \in \mathbb{H}^s(\Omega)$. Then, for any $s' \geq s$, we have the following estimate after a simple computation

$$\begin{aligned} &\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} f\|_{\mathbb{H}^{s'}(\Omega)} \\ &= \left(\sum_{\lambda_n \leq \mathcal{M}_\epsilon} \lambda_n^{2s'-2s} \Big(1 - (T-t)\lambda_n\Big)^2 e^{2(T-t)\lambda_n} \lambda_n^{2s} \Big(\int_\Omega f(x)\psi_n(x)dx \Big)^2 \right)^{\frac{1}{2}} \\ &\leq \Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{s'-s} e^{(T-t)\mathcal{M}_\epsilon} \|f\|_{\mathbb{H}^s(\Omega)}, \end{aligned} \tag{5.6}$$

where we have used the fact that $\Big|1 - (T-t)\lambda_n\Big| \leq 1 + T\lambda_n \leq 1 + T\mathcal{M}_\varepsilon$ if $\lambda_n \leq \mathcal{M}_\varepsilon$.

By a similar argument as above, we also find that for $s' \geq s$

$$\begin{aligned}\|\mathcal{S}^{-1}(\nu-t)\mathbb{Q}_{\mathcal{M}_\varepsilon} f\|_{\mathbb{H}^{s'}(\Omega)} &= \left(\sum_{\lambda_n \leq \mathcal{M}_\epsilon} \lambda_n^{2s'-2s} e^{2(\nu-t)\lambda_n} \lambda_n^{2s} \Big(\int_\Omega f(x)\psi_n(x)dx\Big)^2\right)^{\frac{1}{2}} \\ &\leq (\mathcal{M}_\epsilon)^{s'-s} e^{(\nu-t)\mathcal{M}_\epsilon} \|f\|_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{5.7}$$

For any $\mu > 0$, denote by $L^\infty_\mu(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))$ the function space $L^\infty(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))$ associated with the following norm

$$\|w\|_\mu := \max_{0\leq t\leq T} \|\exp(-\mu(T-t))w(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)}, \quad \forall\, w \in L^{\frac{2N}{N-4\theta}}(\Omega).$$

Let us define a nonlinear map $\mathscr{P} : L^\infty_\mu(0,T; L^{\frac{2N}{N-4\theta}}(\Omega)) \to L^\infty_\mu(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))$ by

$$\mathscr{P}\psi(t) = \mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon - \int_t^T (\nu-t)\mathcal{S}^{-1}(\nu-t)\mathbb{Q}_{\mathcal{M}_\varepsilon} F(\psi(\nu)(\nu))d\nu. \tag{5.8}$$

If $\psi = 0$ then

$$\mathscr{P}(\psi = 0) = \mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon.$$

Since $0 \leq \theta \leq \frac{N}{4}$, we find that the Sobolev embedding

$$\mathbb{H}^\theta(\Omega) \hookrightarrow\ L^{\frac{2N}{N-4\theta}}(\Omega) \tag{5.9}$$

which allows us to derive that

$$\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \lesssim \|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon\|_{\mathbb{H}^\theta(\Omega)}. \tag{5.10}$$

Using (5.6) with $s' = \theta$ and $s = \frac{Np-2N}{4p}$, we get the following estimate

$$\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon\|_{\mathbb{H}^\theta(\Omega)} \leq \Big(1+T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \|g_\varepsilon\|_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)}. \tag{5.11}$$

For $1 < p < 2$, we follows from Lemma (2.2) in order to get that the Sobolev embedding

$$L^p(\Omega) \hookrightarrow\ \mathbb{H}^{\frac{Np-2N}{4p}}(\Omega).$$

Therefore, we combine two latter results to deduce that the following statement

$$\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon\|_{H^\theta(\Omega)} \lesssim \Big(1+T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \|g_\varepsilon\|_{L^p(\Omega)}.$$

This together with (5.10) yields to

$$\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \lesssim \Big(1+T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \|g_\varepsilon\|_{L^p(\Omega)}. \tag{5.12}$$

Thus, we deduce that

$$\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon} g_\varepsilon \in L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega)). \tag{5.13}$$

Let us take two functions $\psi, \widetilde{\psi} \in L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega))$. From the definition of (5.8), we find that

$$\mathscr{P}\psi(t) - \mathscr{P}\widetilde{\psi}(t) = -\int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\Big(F(\psi(\nu)) - F(\widetilde{\psi}(\nu))\Big)d\nu.$$

Therefore, take any $\mu > 0$, we arrive at

$$\begin{aligned}&\|\exp(-\mu(T-t))\Big(\mathscr{P}\psi(t) - \mathscr{P}\widetilde{\psi}(t)\Big)\|_{H^\theta(\Omega)}\\ &\le \int_t^T \exp(-\mu(T-t))(\nu - t)\|\mathcal{S}^{-1}(\nu - t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\Big(F(\psi(\nu)) - F(\widetilde{\psi}(\nu))\Big)\|_{H^\theta(\Omega)}d\nu.\end{aligned} \tag{5.14}$$

In view of (5.7) with $s' = \theta$ and $s = 0$, and noting that $\mathbb{H}^0(\Omega) = L^2(\Omega)$, one has the following bound

$$\begin{aligned}&\|\mathcal{S}^{-1}(\nu - t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\Big(F(\psi(\nu)) - F(\widetilde{\psi}(\nu))\Big)\|_{\mathbb{H}^\theta(\Omega)}\\ &\le (\mathcal{M}_\epsilon)^\theta e^{(\nu-t)\mathcal{M}_\epsilon}\|F(\psi(\nu)) - F(\widetilde{\psi}(\nu))\|_{L^2(\Omega)}\\ &\le K_f(\mathcal{M}_\epsilon)^\theta e^{(\nu-t)\mathcal{M}_\epsilon}\|\psi(\nu) - \widetilde{\psi}(\nu)\|_{L^2(\Omega)}\\ &\le K_f(\mathcal{M}_\epsilon)^\theta e^{(\nu-t)\mathcal{M}_\epsilon}\|\psi(\nu) - \widetilde{\psi}(\nu)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)},\end{aligned} \tag{5.15}$$

where in the last estimate, we have used the Sobolev embedding $L^{\frac{2N}{N-4\theta}}(\Omega) \hookrightarrow L^2(\Omega)$. Combining (5.14) and (5.15), we find that

$$\begin{aligned}&\text{the right hand side of (5.14)}\\ &\le K_f(\mathcal{M}_\epsilon)^\theta \int_t^T \exp(-\mu(\nu - t))(\nu - t)e^{(\nu-t)\mathcal{M}_\epsilon}\\ &\qquad\qquad \times \exp(-\mu(T-\nu))\|\psi(\nu) - \widetilde{\psi}(\nu)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)}d\nu.\end{aligned} \tag{5.16}$$

Since the fact that

$$\|\psi - \widetilde{\psi}\|_\mu := \operatorname*{ess\,sup}_{0\le\nu\le T} \exp(-\mu(T-\nu))\|\psi(\nu) - \widetilde{\psi}(\nu)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)}$$

we follows from (5.14) and (5.16) that

$$\begin{aligned}&\|\exp(-\mu(T-t))\Big(\mathscr{P}\psi(t) - \mathscr{P}\widetilde{\psi}(t)\Big)\|_{H^\theta(\Omega)}\\ &\le K_f(\mathcal{M}_\epsilon)^\theta\Big(\int_t^T \exp(-\mu(\nu - t))(\nu - t)e^{(\nu-t)\mathcal{M}_\epsilon}d\nu\Big)\|\psi - \widetilde{\psi}\|_\mu.\end{aligned} \tag{5.17}$$

We continue to treat the integral term on the right hand side of (5.17). It is easy to verify that for any $\mu > 2\mathcal{M}_\epsilon$

$$\begin{aligned}&\int_t^T \exp(-\mu(\nu-t))(\nu-t)e^{(\nu-t)\mathcal{M}_\epsilon}d\nu \\ &\le T\int_t^T \exp\Big(-(\mu-\mathcal{M}_\epsilon)(\nu-t)\Big)d\nu \le \frac{T}{\mu-\mathcal{M}_\epsilon}.\end{aligned} \tag{5.18}$$

A combination of two evaluations (5.17) and (5.18) and Sobolev embedding as in (5.9) allows us to give that

$$\begin{aligned}&\|\exp(-\mu(T-t))\Big(\mathscr{P}\psi(t)-\mathscr{P}\widetilde{\psi}(t)\Big)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \\ &\le \|\exp(-\mu(T-t))\Big(\mathscr{P}\psi(t)-\mathscr{P}\widetilde{\psi}(t)\Big)\|_{\mathbb{H}^\theta(\Omega)} \\ &\le \frac{K_f(\mathcal{M}_\epsilon)^\theta T}{\mu-\mathcal{M}_\epsilon}\|\psi-\widetilde{\psi}\|_\mu.\end{aligned}$$

The right hand side of the above expression is independent of t, we deduce that

$$\operatorname*{ess\,sup}_{0\le t\le T}\|\exp(-\mu(T-t))\Big(\mathscr{P}\psi(t)-\mathscr{P}\widetilde{\psi}(t)\Big)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \le \frac{K_f(\mathcal{M}_\epsilon)^\theta T}{\mu-\mathcal{M}_\epsilon}\|\psi-\widetilde{\psi}\|_\mu$$

which allows us to provide that for any $\psi,\widetilde{\psi}\in L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega))$

$$\|\mathscr{P}\psi-\mathscr{P}\widetilde{\psi}\|_\mu \le \frac{K_f(\mathcal{M}_\epsilon)^\theta T}{\mu-\mathcal{M}_\epsilon}\|\psi-\widetilde{\psi}\|_\mu. \tag{5.19}$$

Let us choose $\mu \ge 2K_f(\mathcal{M}_\epsilon)^\theta T+\mathcal{M}_\epsilon$. Then we follows from (5.19) and (5.13) that $\mathscr{P}$ is a contraction mapping from $L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega)) \to L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega))$. By applying Banach fixed point theorem, we obtain that $\mathscr{P}$ has a fixed point $\mathcal{W}_\varepsilon \in L^\infty_\mu(0,T;L^{\frac{2N}{N-4\theta}}(\Omega))$. Next, we show the regularity result for $\mathcal{W}_\varepsilon$. By looking at the estimate (5.19), we take $\psi=\mathcal{W}_\varepsilon$ and $\widetilde{\psi}=0$, and noting that $\mathcal{W}_\varepsilon=\mathscr{P}\mathcal{W}_\varepsilon$, $\mathscr{P}\widetilde{\psi}=\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}g_\varepsilon$ in order to get

$$\begin{aligned}\|\mathcal{W}_\varepsilon\|_\mu &= \|\mathscr{P}\mathcal{W}_\varepsilon\|_\mu \\ &\le \|\mathscr{P}\mathcal{W}_\varepsilon-\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}g_\varepsilon\|_\mu+\|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}g_\varepsilon\|_\mu \\ &\lesssim \frac{1}{2}\|\mathcal{W}_\varepsilon\|_\mu+\Big(1+T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta-\frac{Np-2N}{4p}}e^{(T-t)\mathcal{M}_\epsilon}\|g_\varepsilon\|_{L^p(\Omega)},\end{aligned}$$

where in the last above estimate, we have used (5.12). Thus, we obtain that

$$\|\mathcal{W}_\varepsilon\|_\mu \lesssim 2\Big(1+T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta-\frac{Np-2N}{4p}}e^{(T-t)\mathcal{M}_\epsilon}\|g_\varepsilon\|_{L^p(\Omega)},$$

which allows us to give that the desired result (5.5). □

The following theorem provides the error between the regularized solution and the exact solution in space the space of L^p type when the noisy data in L^p.

Theorem 5.2. *Let us assume that Problem* (1.1)–(1.2) *has a unique solution* $u \in L^\infty(0,T;\mathbb{H}^{b+\theta}(\Omega))$ *for any* $b > 0$ *and* $0 \le \theta < \frac{N}{4}$. *In addition, we assume that there exists a postive constant* $\overline{M} > 0$ *such that*

$$\overline{M} = \operatorname*{ess\,sup}_{0\le t\le T} \left(\sum_{j=1}^{\infty} \lambda_j^{2\theta_0} \exp\left(2t\lambda_j\right) \langle u(t), e_j\rangle^2 \right)^{\frac{1}{2}},$$

where $\theta_0 > \theta$. *Let us take the noisy data* $g_\varepsilon \in L^p(\Omega)$ *such that*

$$\|g_\varepsilon - g\|_{L^p(\Omega)} \le \varepsilon, \quad 1 < p < 2.$$

Let us choose $\mathcal{M}_\varepsilon$ *such that for any* $t > 0$

$$\lim_{\epsilon\to 0} \mathcal{M}_\varepsilon = \infty, \;\; \lim_{\varepsilon\to 0} (\mathcal{M}_\varepsilon)^{1+\theta-\frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon}\varepsilon = 0. \tag{5.20}$$

Then the error $\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)}$ *is of order*

$$\max\left((\mathcal{M}_\varepsilon)^{-b}, (\mathcal{M}_\varepsilon)^{\theta-\theta_0}, (\mathcal{M}_\varepsilon)^{1+\theta-\frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon}\varepsilon \right).$$

Remark 5.3. Let us choose $\mathcal{M}_\varepsilon = \frac{1-\alpha}{T}\log(\frac{1}{\varepsilon})$ for any $0 < \alpha < 1$. Then it is easy to check that (5.20) holds. The error $\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)}$ is of order

$$\max\left(\left[\log\left(\frac{1}{\varepsilon}\right)\right]^{-b}, \left[\log\left(\frac{1}{\varepsilon}\right)\right]^{\theta-\theta_0}, \left[\log\left(\frac{1}{\varepsilon}\right)\right]^{1+\theta-\frac{Np-2N}{4p}} \varepsilon^\alpha \right).$$

Proof. Our purpose is to provide the upper bound for the term $\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)}$. We use the inequality to obtain that

$$\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \le \|\mathcal{W}_\varepsilon(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{L^2(\Omega)}.$$

From the definition of the mild solution as in (5.3), one has

$$\mathbb{Q}_{\mathcal{M}_\varepsilon} u(t) = \mathcal{S}^{-1}(T-t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\mathcal{H}(t)g - \int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)\mathbb{Q}_{\mathcal{M}_\varepsilon} F(u(\nu))d\nu.$$

From (5.4), we know that

$$\begin{aligned} \mathcal{W}_\varepsilon(t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(t) &= \mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\left(g_\varepsilon - g\right) \\ &\quad - \int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)\Big[\mathbb{Q}_{\mathcal{M}_\varepsilon} F(\mathcal{W}_\varepsilon(\nu)) - \mathbb{Q}_{\mathcal{M}_\varepsilon} F(u(\nu))\Big] d\nu. \end{aligned}$$

This implies that

$$\begin{aligned}
&\|\mathcal{W}_\varepsilon(t) - u(t)\|_{L^2(\Omega)} \\
&\leq \|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\big(g_\varepsilon - g\big)\|_{L^2(\Omega)} + \|u(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{L^2(\Omega)} \\
&\quad + \left\| \int_t^T (\nu - t)\mathcal{S}^{-1}(\nu - t)\Big[\mathbb{Q}_{\mathcal{M}_\varepsilon} F(\mathcal{W}_\varepsilon(\nu)) - \mathbb{Q}_{\mathcal{M}_\varepsilon} F(u(\nu))\Big] d\nu \right\|_{L^2(\Omega)} \\
&= \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3.
\end{aligned} \tag{5.21}$$

Let us first consider the first quantity of the above expression. For $1 < p < 2$, we follows from Lemma (2.2) in order to get that the Sobolev embedding

$$L^p(\Omega) \hookrightarrow \mathbb{H}^{\frac{Np-2N}{4p}}(\Omega). \tag{5.22}$$

Using a similar technique as in (5.11) and in view of Sobolev embedding (5.22), we arrive at

$$\begin{aligned}
\mathscr{D}_1 &= \|\mathcal{S}^{-1}(T-t)\mathcal{H}(t)\mathbb{Q}_{\mathcal{M}_\varepsilon}\big(g_\varepsilon - g\big)\|_{L^2(\Omega)} \\
&\leq \Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \|g_\varepsilon - g\|_{\mathbb{H}^{\frac{Np-2N}{4p}}(\Omega)} \\
&\leq C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \|g_\varepsilon - g\|_{L^p(\Omega)} \\
&\leq C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\epsilon} \varepsilon.
\end{aligned} \tag{5.23}$$

For the second term on the right hand side of (5.21), we derive that

$$\begin{aligned}
\mathscr{D}_2 &= \|u(t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(t)\|_{L^2(\Omega)} \\
&= \left(\sum_{\lambda_j > \mathcal{M}_\varepsilon} \exp\left(-2t\lambda_j\right) \lambda_j^{-2\theta_0} \exp\left(2t\lambda_j\right) \lambda_j^{2\theta_0} \left\langle u(t), e_j \right\rangle^2 \right)^{\frac{1}{2}} \\
&\leq (\mathcal{M}_\epsilon)^{-\theta_0} \exp\left(-t\mathcal{M}_\varepsilon\right) \overline{M},
\end{aligned}$$

where we have used that $\lambda_j^{-2\theta_0} \leq (\mathcal{M}_\epsilon)^{-2\theta_0}$ for $\lambda_j > \mathcal{M}_\varepsilon$ and we also denote by

$$\overline{M} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \left(\sum_{j=1}^{\infty} \lambda_j^{2\theta_0} \exp\left(2t\lambda_j\right) \left\langle u(t), e_j \right\rangle^2 \right)^{\frac{1}{2}}.$$

By using the similar argument as in (5.15), we also find that

$$\begin{aligned}
\mathscr{D}_3 &= \|\mathcal{S}^{-1}(\nu - t)\Big[\mathbb{Q}_{\mathcal{M}_\varepsilon} F(\mathcal{W}_\varepsilon(\nu)) - \mathbb{Q}_{\mathcal{M}_\varepsilon} F(u(\nu))\Big]\|_{L^2(\Omega)} \\
&\leq e^{(\nu - t)\mathcal{M}_\epsilon} \|F(\mathcal{W}_\varepsilon(\nu)) - F(u(\nu))\|_{L^2(\Omega)} \\
&\leq K_f e^{(\nu - t)\mathcal{M}_\epsilon} \|W_\varepsilon(\nu) - u(\nu)\|_{L^2(\Omega)}.
\end{aligned}$$

This implies that the second term on the right hand side of (5.21) is bounded by

$$\mathscr{D}_3 \leq K_f \int_t^T (\nu - t) e^{(\nu - t)\mathcal{M}_\epsilon} \|W_\varepsilon(\nu) - u(\nu)\|_{L^2(\Omega)} d\nu. \tag{5.24}$$

Combining (5), (5.21), (5.23), (5.24) and (5), we deduce that

$$\begin{aligned} \|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} &\leq \|\mathcal{W}_\varepsilon(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{L^2(\Omega)} \\ &\leq C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{(T-t)\mathcal{M}_\varepsilon}\varepsilon \\ &\quad + (\mathcal{M}_\epsilon)^{-\theta_0} e^{-t\mathcal{M}_\varepsilon}\overline{M} \\ &\quad + K_f \int_t^T (\nu - t) e^{(\nu - t)\mathcal{M}_\epsilon} \|W_\varepsilon(\nu) - u(\nu)\|_{L^2(\Omega)} d\nu. \end{aligned}$$

Multiplying both sides by $e^{t\mathcal{M}_\varepsilon}$ and then applying Gronwall's inequality, we find that

$$\begin{aligned} & e^{t\mathcal{M}_\varepsilon}\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \\ & \leq \left[C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon}\varepsilon + (\mathcal{M}_\epsilon)^{-\theta_0}\overline{M}\right] \exp\Big(K_f T(T-t)\Big). \end{aligned}$$

This leads to

$$\begin{aligned} & \|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \\ & \leq e^{-t\mathcal{M}_\varepsilon}\left[C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{-\frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon}\varepsilon + (\mathcal{M}_\epsilon)^{-\theta_0}\overline{M}\right] \exp\Big(K_f T(T-t)\Big). \end{aligned} \tag{5.25}$$

By in view of the Sobolev embedding $\mathbb{H}^\theta(\Omega) \hookrightarrow L^{\frac{2N}{N-4\theta}}(\Omega)$, we know that the following inequality

$$\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \leq C(N,\theta)\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)}.$$

In the following, we continue to estimate the term $\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{H^\theta(\Omega)}$. Indeed, it is easy to observe that

$$\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)} \leq \underbrace{\|\mathcal{W}_\varepsilon(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)}}_{\widetilde{D}_1} + \underbrace{\|u(\cdot,t) - \mathbb{Q}_{\mathcal{M}_\varepsilon} u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)}}_{\widetilde{D}_2}. \tag{5.26}$$

Now, we treat the first term on the right hand side of (5.26). Indeed, we find that

$$\begin{aligned}\widetilde{D}_1 &\le \sqrt{\sum_{n=1}^{\lambda_n \mathcal{M}_\varepsilon} \lambda_j^{2\theta} \Big(\int_\Omega (\mathcal{W}_\varepsilon(x,t) - u(x,t))\psi_n(x)dx \Big)^2} \\ &\le (\mathcal{M}_\varepsilon)^\theta \|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \\ &\le e^{-t\mathcal{M}_\varepsilon} \Bigg[C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta - \frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon} \varepsilon \\ &\qquad + (\mathcal{M}_\epsilon)^{\theta-\theta_0}\overline{M}\Bigg] \exp\Big(K_f T(T-t)\Big)\end{aligned}$$

where in the last estimate, we have used (5.25). And we also note that $0 < \theta < \theta_0$. The second term on the right hand side of (5.26) is bounded by

$$\widetilde{D}_2 = \Bigg(\sum_{\lambda_j > \mathcal{M}_\varepsilon} \lambda_j^{-2b} \lambda_j^{2b} \lambda_j^{2\theta} \langle u(t), e_j \rangle^2 \Bigg)^{\frac{1}{2}} \le (\mathcal{M}_\varepsilon)^{-b} \|u\|_{L^\infty(0,T;\mathbb{H}^{b+\theta}(\Omega))}.$$

From some above observations, we deduce that

$$\begin{aligned}&\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \\ &\le C(N,\theta)\|\mathcal{W}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{\mathbb{H}^\theta(\Omega)} \\ &\le C(N,\theta)(\mathcal{M}_\varepsilon)^{-b}\|u\|_{L^\infty(0,T;\mathbb{H}^{b+\theta}(\Omega))} \\ &+C(N,\theta)e^{-t\mathcal{M}_\varepsilon} \Bigg[C(N,p)\Big(1 + T\mathcal{M}_\epsilon\Big)(\mathcal{M}_\epsilon)^{\theta - \frac{Np-2N}{4p}} e^{T\mathcal{M}_\varepsilon} \varepsilon \\ &\qquad + (\mathcal{M}_\epsilon)^{\theta-\theta_0}\overline{M}\Bigg] \exp\Big(K_f T(T-t)\Big).\end{aligned}$$

6. CONCLUSION

In this work, we focus on the regularized problem for bi-parabolic equation when the observed data are obtained in L^p. By applying the Fourier series truncation method, we introduce the error between the Fourier regularized solution and the exact solution in L^p spaces. Moreover, under some reasonable smoothness assumptions of the exact solution, the error between the the regularized solution and the exact solution are derived in the space L^p. □

Acknowledgements

This research is funded by Vietnam National University Ho Chi Minh City (VNU-HCM) under grant number B2020-18-03.

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Received: October 7, 2021.
Revised: January 5, 2022.
Accepted: January 8, 2022.