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ON THE NUMERICAL SOLUTION OF ONE INVERSE PROBLEM FOR A LINEARIZED TWO-DIMENSIONAL SYSTEM OF NAVIER–STOKES EQUATIONS

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Abstract. The paper studies the numerical solution of the inverse problem for a linearized two-dimensional system of Navier–Stokes equations in a circular cylinder with a final overdetermination condition. For a biharmonic operator in a circle, a generalized spectral problem has been posed. For the latter, a system of eigenfunctions and eigenvalues is constructed, which is used in the work for the numerical solution of the inverse problem in a circular cylinder with specific numerical data. Graphs illustrating the results of calculations are presented.

Keywords: Navier–Stokes equations, inverse problem, numerical solution.

Mathematics Subject Classification: 35Q30, 35R30, 65N21.

1. INTRODUCTION

The theory of the Navier–Stokes equations attracts unrelenting interest and attention not only from theoreticians but also from applied scientists. The literature on inverse problems for the Navier–Stokes system of equations is quite extensive and numerous. As is known, these equations describe the motion of a fluid, taking into account its pressure, as well as many accompanying processes, such as thermal conductivity, electromagnetic phenomena, etc. To date, although many qualitative issues of solutions to initial boundary value problems have been studied, at the same time, a number of open problems remain unresolved [13, 16, 24]. In the last three or four decades, inverse problems for the system of Navier–Stokes equations have been actively studied. First of all, we want to note a series of remarkable and, to a certain extent, pioneering works devoted to the theory of inverse problems for several equations of mathematical physics [20, 21, 28], including linearized and nonlinear Navier–Stokes equations, and which are directly related to the subject of our work. One should also point to the works [2, 3, 8–10, 12, 18, 19, 22], devoted to inverse problems for parabolic equations. Note that various issues related to the solvability and approximate solution of inverse problems for partial differential equations, including for the two-dimensional Navier–Stokes system, are considered in [4, 5, 14, 26, 27]. In [23], the spectral problem for the Stokes operator is solved in the case of periodic boundary conditions (on a torus). We also note the monograph [25] devoted to the general theory of ill-posed problems and various questions about its connection with inverse problems.

In this work, in a circular cylinder, we consider an inverse problem for a linearized two-dimensional system of Navier–Stokes equations with a final overdetermination condition. The main purpose of our work is: to algorithmically implement the numerical solution of the inverse problem, because it is known that such algorithms play an important role in applied problems. We are solving a generalized spectral problem for a biharmonic operator with Dirichlet conditions in a circular domain: we are finding the eigenfunctions that make up the orthonormal basis in the space of solutions of the Navier–Stokes equations, and their corresponding eigenvalues. For the numerical solution of the inverse problem, we use an optimization method associated with minimizing the discrepancy of the solution of the direct problem with the final condition and based on the Pontryagin maximum principle [6, 17].

This paper is organized as follows. In Section 2, the formulation of an inverse problem is given. In Section 3, using a stream function, we give a statement of the generalized spectral problem for a biharmonic operator with Dirichlet conditions corresponding to a two-dimensional linearized system of Navier–Stokes equations. Section 4 is devoted to the solution of the spectral problem under consideration, in which a system of eigenfunctions is constructed and the corresponding eigenvalues are found. Section 5 contains the results of applying an optimization method based on the optimality conditions of the Pontryagin maximum principle. An example of a numerical solution to the inverse problem with specific numerical data is given in Section 6. The results of the numerical solution in the form of graphs are also given here.

In this paper, we restrict ourselves to considering only the linearized Navier–Stokes model. As for the more complex inverse problem for the complete two-dimensional nonlinear system of Navier–Stokes equations, we propose to devote a separate paper to it. In this case, of course, we will actively use the results of the presented work, especially those related to the generalized spectral problem for the biharmonic operator.

In the notation of spaces, we adhere to works [1, 16, 24].

2. STATEMENT OF THE INVERSE PROBLEM

Let $\Omega = \{|y| < 1\} \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial\Omega$. We introduce the notation of spaces $\mathbf{V}$, $\mathbf{H}$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^2(\Omega)$:

$$\begin{aligned}
&\mathbf{V} = \{v : \, v \in \mathbf{H}_0^1(\Omega) = \left(H_0^1(\Omega)\right)^2, \operatorname{div} v = 0\},\\
&\mathbf{H} = \left\{v : \, v \in \mathbf{L}^2(\Omega), \operatorname{div} v = 0\right\},\\
&\quad \mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^2, \quad \mathbf{H}^2(\Omega) = \left(H^2(\Omega)\right)^2.
\end{aligned}$$

The following dense embeddings take place

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}', \;\; \mathbf{H}_0^1(\Omega) \subset \mathbf{L}^2(\Omega) \equiv \big(\mathbf{L}^2(\Omega)\big)' \subset \mathbf{H}^{-1}(\Omega),$$

and $(\cdot,\cdot)$, $((\cdot,\cdot))$ are scalar products in spaces $\mathbf{H}$, $\mathbf{L}^2(\Omega)$ and $\mathbf{V}$, $\mathbf{H}_0^1(\Omega)$, respectively. The Helmholtz decomposition of space $\mathbf{L}^2(\Omega)$: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^{\perp}$,

$$\mathbf{H}^{\perp} \text{ is an orthogonal complement to } \mathbf{H} \text{ in the space } \mathbf{L}^2(\Omega),$$

$$\mathbf{H}^{\perp} = \{v : \, v \in \mathbf{L}^2(\Omega), \, v = \nabla u, \, u \in H^1(\Omega)\},$$

$$\big(\mathbf{H} \oplus \mathbf{H}^{\perp}\big)' \equiv \big(\mathbf{L}^2(\Omega)\big)' \equiv \mathbf{L}^2(\Omega) \equiv \mathbf{H} \oplus \mathbf{H}^{\perp}.$$

Further, let $Q_{yt} = \Omega \times (0,T)$, $\Sigma_{yt} = \partial\Omega \times (0,T)$. The following inverse problem of determining functions $\{w(y,t), P(y,t), f(y)\}$ is considered:

$$\partial_t w - \nu \Delta w = g(t) f(y) - \nabla P, \quad (y,t) \in Q_{yt}, \tag{2.1}$$

$$\operatorname{div} w = 0, \quad (y,t) \in Q_{yt}, \tag{2.2}$$

$$w(y,t) = 0, \quad (y,t) \in \Sigma_{yt}, \tag{2.3}$$

$$w(y,0) = 0, \quad y \in \Omega, \tag{2.4}$$

with overdetermination condition:

$$w(y,T) = w_T(y), \tag{2.5}$$

where $g(t) = \{g_1(t), g_2(t)\}$ and $w_T(y)$ are given functions.

So, in the weak version of the problem (2.1)–(2.5), the vector function $w = (w_1, w_2) \in L^2(0,T; \mathbf{V} \cap \mathbf{H}^2(\Omega))$, the scalar functions $P \in L^2(0,T; \mathbf{H}^1(\Omega))$, $g \in L^2(0,T)$ and the function $f(y) = \{f_1(y), f_2(y)\} \in \mathbf{H}$ are unknown.

Let us formulate the inverse problem: Disperse the liquid filling the domain Ω from the initial state of rest, and described by (2.1)–(2.4), and bring it to the desired state $w_T(y) \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$ (2.5) at time $t = T$.

3. GENERALIZED SPECTRAL PROBLEM FOR A BIHARMONIC OPERATOR IN A CIRCLE

Further, without loss of generality, for simplicity, we take $\nu = 1$. We transform boundary value problem (2.1)–(2.4). For this purpose, in the domain Q_{yt}, we introduce the scalar stream function $U(y,t)$, defined up to an additive constant, by the following equations [16, 24]:

$$\partial_{y_1} U = -w_2, \quad \partial_{y_2} U = w_1. \tag{3.1}$$

Differentiating with respect to y_2 the first equation and with respect to y_1 the second equation of system (2.1), and summing them taking into account (3.1), we obtain the following equation for $U(y,t)$:

$$-\partial_t \Delta U + \Delta^2 U = G(y,t) \equiv -\partial_{y_2} F_1 + \partial_{y_1} F_2, \quad (y,t) \in Q_{yt}, \tag{3.2}$$

with boundary and initial conditions

$$U = 0, \quad \partial_{\vec{n}} U = 0, \quad (y,t) \in \Sigma_{yt}, \tag{3.3}$$

$$U = 0, \quad y \in \Omega, \quad t = 0, \tag{3.4}$$

where $F_j = g_j(t) f_j(y)$, $j = 1, 2$, and $\vec{n}$ is the unit vector of the outward normal to the circle $|y| = 1$.

To solve problem (2.1)–(2.5) numerically, we must be able to approximately solve boundary value problem (3.2)–(3.4). For this we will use the Faedo–Galerkin method, which is provided by solving the following spectral problem:

$$\Delta^2 \mathcal{V}(y) = \mu^2 (-\Delta \mathcal{V}(y)), \quad y \in \Omega = \{|y| < 1\}, \tag{3.5}$$

$$\partial_{\vec{n}} \mathcal{V}(y) = 0, \text{ at } |y| = 1, \tag{3.6}$$

$$\mathcal{V}(y) = 0, \text{ at } |y| = 1. \tag{3.7}$$

4. SOLVING GENERALIZED SPECTRAL PROBLEM (3.5)–(3.7)

We rewrite equation (3.5) as a system for unknown functions $\{\mathcal{V}(y), \mathcal{W}(y)\}$:

$$-\Delta \mathcal{V}(y) = \mathcal{W}(y), \quad -\Delta \mathcal{W}(y) = \mu^2 \mathcal{W}(y), \quad y \in \Omega. \tag{4.1}$$

So, we get spectral problem (4.1), (3.6) and (3.7). We write this problem in the polar coordinate system $\{y_1 = r\cos\theta, \ y_2 = r\sin\theta\}$ in the domain $\Omega_1 = \{0 < r < 1, \ 0 \le \theta < 2\pi\}$:

$$-\frac{1}{r}\partial_r (r\partial_r Z(r,\theta)) - \frac{1}{r^2}\partial_\theta^2 Z(r,\theta) = Y(r,\theta), \quad (r, \ \theta) \in \Omega_1, \tag{4.2}$$

$$-\frac{1}{r}\partial_r (r\partial_r Y(r,\theta)) - \frac{1}{r^2}\partial_\theta^2 Y(r,\theta) = \mu^2 Y(r,\theta), \quad (r, \ \theta) \in \Omega_1, \tag{4.3}$$

$$Z(r,\theta) \text{ is bounded in the neighborhood of the point } r = 0, \tag{4.4}$$

$$\partial_r Z(r,\theta) = 0, \quad \text{at} \quad r = 1, \tag{4.5}$$

$$Z(r,\theta) = 0, \quad \text{at} \quad r = 1, \tag{4.6}$$

where $Z(r,\theta) = \mathcal{V}(r\cos\theta, r\sin\theta)$, $Y(r,\theta) = \mathcal{W}(r\cos\theta, r\sin\theta)$.

Problem (4.2)–(4.6) will be solved by the method of separation of variables:

$$Z(r,\theta) = \sum_j R_{Zj}(r)\,\Theta_{Zj}(\theta), \quad Y(r,\theta) = \sum_j R_{Yj}(r)\,\Theta_{Yj}(\theta). \tag{4.7}$$

Substituting (4.7) into (4.2)–(4.6), we get

$$-\Theta_{Zj}^{''}(\theta) = \mu_{Zj}^2 \Theta_{Zj}(\theta), \quad \theta \in (0, 2\pi), \quad \Theta_{Zj}(0) = \Theta_{Zj}(2\pi), \tag{4.8}$$

$$-\Theta''_{Yj}(\theta) = \mu_{Yj}^2 \Theta_{Yj}(\theta), \quad \theta \in (0, 2\pi), \quad \Theta_{Yj}(0) = \Theta_{Yj}(2\pi), \tag{4.9}$$

$$r^2 R''_{Zj}(r) + r R'_{Zj}(r) - \mu_{Zj}^2 R_{Zj}(r) = -r^2 R_{Yj}(r), \tag{4.10}$$

$$r^2 R''_{Yj}(r) + r R'_{Yj}(r) + \left(\mu^2 r^2 - \mu_{Yj}^2\right) R_{Yj}(r) = 0, \tag{4.11}$$

$$R_{Zj}(r) \text{ is bounded in a neighborhood of zero,} \tag{4.12}$$

$$R_{Zj}(1) = 0, \qquad R'_{Zj}(1) = 0. \tag{4.13}$$

The solutions of problems (4.8) and (4.9) coincide and are equal to:

$$\begin{aligned} &\Theta_{Z0}^{(-)}(\theta) = \Theta_{Y0}^{(-)}(\theta) = 1, \quad \theta \in [0, 2\pi), \\ &\Theta_{Zj}^{(-)}(\theta) = \Theta_{Yj}^{(-)}(\theta) = \cos j\theta, \ \Theta_{Zj}^{(+)}(\theta) = \Theta_{Yj}^{(+)}(\theta) = \sin j\theta, \quad \theta \in [0, 2\pi), \\ &\mu_{Zj}^2 = \mu_{Yj}^2 = j^2, \quad j \in \{1, 2, \ldots\}. \end{aligned} \tag{4.14}$$

It is known that the equation (4.11) is a special case of the Bessel equation and its general solution has the form $R_{Yj} = C_1 J_j(\mu r) + C_2 \tilde{J}_j(\mu r)$, where $\tilde{J}_j(\mu r) = J_{-j}(-\mu r)$ and J_j, $j = 0, 1, 2, \ldots$, is the Bessel function of the first order. Due to the fact that $j = 1, 2, \ldots$, according to formulas (12)–(14) from [15, Chapter 7, §3] we have $J_{-j}(-\mu r) \equiv J_j(\mu r)$ for all $j = 1, 2, \ldots$ Thus, it suffices for us to consider the system of functions $\{J_0(\mu r),\ J_j(\mu r),\ j = 1, 2, \ldots\}$.

Further, if we make the substitution $\rho = \mu r$, then by the definition of a cylindrical function for equation (4.11) the following statement is true.

Lemma 4.1. *Equation* (4.11) *has a general solution in the form of a cylindrical function* $R_{Yj}(r) = J_j(\mu r), j = 0, 1, 2, \ldots$

Substituting this solution from Lemma 4.1 into equation (4.10), we will have a boundary value problem for an nonhomogeneous ordinary differential equation of the second order:

$$r^2 R''_{Zj}(r) + r R'_{Zj}(r) - j^2 R_{Zj}(r) = -r^2 J_j(\mu r), \quad r \in (0, 1), \tag{4.15}$$

$$R_{Zj}(r) \text{ are bounded in a neighborhood of zero,} \tag{4.16}$$

$$R_{Zj}(1) = 0, \quad R'_{Zj}(1) = 0, \tag{4.17}$$

where $j = 0, 1, 2, \ldots$

For boundary value problem (4.15)–(4.17), we establish the following lemma.

Lemma 4.2. *For all* $j \in \{0, 1, 2, \ldots\}$ *boundary value problem* (4.15)–(4.17) *has a countable family of solutions*

$$\left\{ R_{Zjk}(r) = \int_0^1 G_j(r, \rho) J_j(\mu_{j+1,k}\rho) d\rho,\ \mu_{j+1,k}^2 \right\}, \quad k = 1, 2, \ldots,$$

where $\mu_{j+1,k}$ *are roots of equations* $J_{j+1}(\mu) = 0$ *and* G_j, $j = 0, 1, \ldots$ *is the suitable Green function.*

Proof. We seek fundamental solutions for (4.15) in the form of $R_{j\text{f.s.}}(r) = r^m$, where m is an unknown number for now. Substituting r^m into equation (4.15), we find $m = \pm j$, i.e. fundamental solutions are equal

$$z_{1j}(r) = r^{-j},\ z_{2j}(r) = r^j \text{ for any } j \in \{1, 2, \ldots\},\ z_{10}(r) = \ln r,\ z_{20}(r) = 1. \quad (4.18)$$

Thus, the general solution of homogeneous equation (4.15), according to (4.18), is written in the form

$$R_{j\text{g.s.}}(r) = C_{1j} r^{-j} + C_{2j} r^j,\ j \in \{1, 2, \ldots\},\ R_{0\text{g.s.}}(r) = C_{10} \ln r + C_{20}, \quad (4.19)$$

from the boundedness conditions in the neighborhood of the point $r = 0$ from (4.16) it follows that

$$C_{1j} = 0, \quad j = 0, 1, 2, \ldots \quad (4.20)$$

Thus, the general solutions for equation (4.15), obtained on the basis of fundamental solutions (4.18)–(4.20) ([7, Chapter 1, §5]) and according to the first condition from (4.17), have the form:

$$R_{j\text{g.s.}}(r) = \begin{cases} C_{2j} r^j + R_{j\text{p.s.}}(r), & j \neq 0 \\ C_{20} + R_{0\text{p.s.}}(r), & j = 0 \end{cases} = \int_0^1 G_j(r, \rho) J_j(\mu\rho) d\rho, \quad j = 0, 1, \ldots, \quad (4.21)$$

where

$$G_j(r, \rho) = \begin{cases} -\frac{1}{2j} r^j \left[\rho^{j+1} - \rho^{-j+1}\right], & 0 < r < \rho < 1, \\ -\frac{1}{2j} \rho^{j+1} \left[r^j - r^{-j}\right], & 0 < \rho < r < 1, \end{cases} \quad j = 1, 2, \ldots, \quad (4.22)$$

$$G_0(r, \rho) = \begin{cases} -\rho \ln\rho, & 0 < r < \rho < 1, \\ -\rho \ln r, & 0 < \rho < r < 1, \end{cases} \quad j = 0, \quad (4.23)$$

$$C_{2j} = \begin{cases} -\frac{1}{2j} \int_0^1 \left[\rho^{j+1} - \rho^{-j+1}\right] J_j(\mu\rho) d\rho, & j = 1, 2, \ldots, \\ -\int_0^1 \rho \ln\rho\, J_0(\mu\rho) d\rho, & j = 0. \end{cases} \quad (4.24)$$

Finally, for eigenfunctions $R_{Zj}(r)$ from (4.21)–(4.24) we obtain:

$$R_{Zj}(r) = \int_0^1 G_j(r, \rho) J_j(\mu\rho) d\rho, \quad j = 1, 2, \ldots,$$

$$R_{Z0}(r) = \int_0^1 G_0(r, \rho) J_0(\mu\rho) d\rho.$$

Further, from the second condition from (4.17) according to formula (20) from [15, Chapter 7, §3] we obtain

$$R'_{Zj}(1) = 0 \;\Leftrightarrow\; \begin{cases} \int\limits_0^1 \rho J_0(\mu\rho)d\rho = 0 & \Leftrightarrow J_1(\mu) = 0, \\ \int\limits_0^1 \rho^{j+1} J_j(\mu\rho)d\rho = 0 & \Leftrightarrow J_{j+1}(\mu) = 0,\ j = 1, 2, \dots \end{cases}$$

We denote by μ_{jk}, $k \in \{1, 2, \dots\}$, the roots of equations $J_j(\mu) = 0$, $j \in \{1, 2, \dots\}$. Thus we get that the system of functions $\{J_j(\mu_{jk}r),\ k = 1, 2, \dots\}$ satisfies the orthogonality condition with weight r, i.e.

$$\int\limits_0^1 J_j(\mu_{jk}r)J_j(\mu_{jl}r)\, r\, dr = \begin{cases} 0, & \text{at } k \neq l, \\ d_{jk}^2, & \text{at } k = l, \end{cases} \qquad \text{where } d_{jk}^2 = \int\limits_0^1 J_j^2(\mu_{jk}r)\, r\, dr.$$

Thus the solutions to initial boundary value problem (4.15)–(4.17) have the following form

$$\left\{ R_{Zjk}(r) = \int\limits_0^1 G_j(r, \rho)J_j(\mu_{j+1,k}\rho)d\rho,\ \mu_{j+1,k}^2 \right\},\ j = 0, 1, 2, \dots,\ k = 1, 2, \dots, \quad (4.25)$$

where $\mu_{j+1,k}$ are roots of equations $J_{j+1}(\mu) = 0$.

□

Remark 4.3. We will assume that $\theta \in [0,\ 2\pi)$. We represent the interval $[0,\ 2\pi)$ as a union of sets

$$[0,\ \pi/2) \cup (3\pi/2,\ 2\pi) \cup (\pi/2,\ 3\pi/2) \cup \{\pi/2\} \cup \{3\pi/2\}\,. \quad (4.26)$$

Then on the set (4.26) to define θ we get the following formula:

$$\theta \in \begin{cases} [0,\ \pi/2)\,, & y_1 > 0,\ y_2 \geq 0, \\ (3\pi/2,\ 2\pi)\,, & y_1 > 0,\ y_2 < 0, \\ (\pi/2,\ 3\pi/2)\,, & y_1 < 0, \\ \pi/2, & y_1 = 0,\ y_2 > 0, \\ 3\pi/2, & y_1 = 0,\ y_2 < 0. \end{cases} \quad (4.27)$$

Note that (4.27) takes into account the fact that an angle θ is determined using function $\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)$, taking into account the signs of coordinates y_1, y_2 and values of $\arctan\left(\frac{y_2}{y_1}\right)$. This function will be denoted by $\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)$, i.e. $\theta = \widetilde{\arctan}\left(\frac{y_2}{y_1}\right)$.

Theorem 4.4. *From solutions* (4.14) *and* (4.25) *of boundary value problems* (4.8)–(4.9) *and* (4.10)–(4.13)*, respectively, we obtain the following system of eigenfunctions and the corresponding eigenvalues:*

$$\left\{R_{Z0k}(r),\, R_{Zjk}(r)\cos j\theta,\, R_{Zjk}(r)\sin j\theta,\, \mu_{1k}^2,\, \mu_{j+1,k}^2\right\}, \quad j,k\in\{1,2,\dots\}. \tag{4.28}$$

In the Cartesian coordinate system, respectively, we obtain

$$\begin{aligned}\Bigg\{ & U_{jk}^{(-)}(y)\equiv R_{Zjk}(|y|)\cos\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right),\\ & U_{jk}^{(+)}(y)\equiv R_{Zjk}(|y|)\sin\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right),\ \mu_{j+1,k}^2\Bigg\},\\ & j\in\{1,2,\dots\},k\in\{1,2,3,\dots\},|y|<1,\end{aligned} \tag{4.29}$$

$$\left\{U_{0k}^{(-)}(y)\equiv R_{Z0k}(|y|),\mu_{1k}^2\right\}, \quad j=0,k\in\{1,2,3,\dots\},|y|<1. \tag{4.30}$$

Now, according to formulas (3.1) and (4.29)–(4.30) we define a system of eigenfunctions $w(y)=\{w_1(y),w_2(y)\}$ for the weak form of the spectral problem (3.5)–(3.7), which (it is not difficult to verify) is equivalent to the following:

$$a(w,v)=\mu^2(w,v), \quad v\in\mathbf{V}, \tag{4.31}$$

where

$$a(w,v)=((w,v))\equiv\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial w_j}{\partial y_i}\frac{\partial v_j}{\partial y_i}dy, \quad w,v\in\mathbf{V}.$$

Remark 4.5. Spectral problem (4.31) is posed for the self-adjoint Stokes operator. Therefore, its solution, i.e. system of eigenfunctions, constitutes an orthogonal basis in space $\mathbf{V}$ with real eigenvalues (see, for example, [24, Chapter 1, 2.6]).

Further, from Theorem 4.4 we finally get the following result.

Theorem 4.6. *For all* $j\in\{0,1,2,\dots\}$, $k\in\{1,2,3,\dots\}$, $|y|<1$, *the eigenfunctions*

$$\left\{w_{1jk}^{(-)}(y)\equiv\partial_{y_2}U_{jk}^{(-)}(y),\ w_{2jk}^{(-)}(y)\equiv-\partial_{y_1}U_{jk}^{(-)}(y)\right\}, \tag{4.32}$$

$$\left\{w_{1jk}^{(+)}(y)\equiv\partial_{y_2}U_{jk}^{(+)}(y),\ w_{2jk}^{(+)}(y)\equiv-\partial_{y_1}U_{jk}^{(+)}(y)\right\}, \tag{4.33}$$

$$w_{10k}^{(-)}(y)\equiv\partial_{y_2}U_{0k}^{(-)}(y)=R'_{Z0k}(|y|)\frac{y_2}{|y|}, \tag{4.34}$$

$$w_{20k}^{(-)}(y)\equiv-\partial_{y_1}U_{0k}^{(-)}(y)=-R'_{Z0k}(|y|)\frac{y_1}{|y|}, \tag{4.35}$$

in pairs constitute an orthogonal basis in space $\mathbf{V}$*, where*

$$\begin{aligned}\partial_{y_2}U_{jk}^{(-)}(y)=R'_{Zjk}(|y|)\frac{y_2}{|y|}\cos\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right)\\ -R_{Zjk}(|y|)\sin\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right)j\frac{y_1}{|y|^2},\end{aligned} \tag{4.36}$$

$$\partial_{y_2}U_{jk}^{(+)}(y) = R'_{Zjk}(|y|)\frac{y_2}{|y|}\sin\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right) + R_{Zjk}(|y|)\cos\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right)j\frac{y_1}{|y|^2}\ , \tag{4.37}$$

$$-\partial_{y_1}U_{jk}^{(-)}(y) = -R'_{Zjk}(|y|)\frac{y_1}{|y|}\cos\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right) + R_{Zjk}(|y|)\sin\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right)j\frac{y_2}{|y|^2}\ , \tag{4.38}$$

$$-\partial_{y_1}U_{jk}^{(+)}(y) = -R'_{Zjk}(|y|)\frac{y_1}{|y|}\sin\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right) - R_{Zjk}(|y|)\cos\left(j\,\widetilde{\arctan}\left(\frac{y_2}{y_1}\right)\right)j\frac{y_2}{|y|^2}\ , \tag{4.39}$$

$$R'_{Zjk}(|y|) = \int\limits_{|y|}^{1}\partial_{|y|}G_j(|y|,\rho)\,J_j(\mu_{j+1,k}\rho)\,d\rho + \int\limits_{0}^{|y|}\partial_{|y|}G_j(|y|,\rho)\,J_j(\mu_{j+1,k}\rho)\,d\rho, \tag{4.40}$$

$$R_{Zjk}(|y|) = \int\limits_{|y|}^{1}G_j(|y|,\rho)\,J_j(\mu_{j+1,k}\rho)\,d\rho + \int\limits_{0}^{|y|}G_j(|y|,\rho)\,J_j(\mu_{j+1,k}\rho)\,d\rho, \tag{4.41}$$

$$R'_{Z0k}(|y|) = \int\limits_{|y|}^{1}\partial_{|y|}G_0(|y|,\rho)\,J_0(\mu_{1k}\rho)\,d\rho + \int\limits_{0}^{|y|}\partial_{|y|}G_0(|y|,\rho)\,J_0(\mu_{1k}\rho)\,d\rho, \tag{4.42}$$

$$R_{Z0k}(|y|) = \int\limits_{|y|}^{1}G_0(|y|,\rho)\,J_0(\mu_{1k}\rho)\,d\rho + \int\limits_{0}^{|y|}G_0(|y|,\rho)\,J_0(\mu_{1k}\rho)\,d\rho. \tag{4.43}$$

According to Remark 4.5 the system of eigenfunctions (4.32)–(4.43) must be orthogonal and, accordingly, it constitutes an orthogonal basis. However, the orthogonality of the system of eigenfunctions (4.32)–(4.43) also follows from its construction, and this property is verified by direct calculations.

Remark 4.7. In Section 6, in the numerical example, for convenience, we introduce the following notation for the eigenfunctions that correspond to the eigenvalues $\mu_{j+1,k}^2$,

$$w_{0k}(y) = w_{0k}^{(-)}(y), \quad w_{-jk}(y) = w_{jk}^{(-)}(y), \quad w_{jk}(y) = w_{jk}^{(+)}(y), \quad k = 1, 2, 3, \dots, \tag{4.44}$$

where $w_{jk}^{(-)}(y)$, $w_{jk}^{(+)}(y)$ and $w_{0k}(y)$ are defined according to (4.32)–(4.35).

5. AN OPTIMIZATION METHOD FOR SOLVING THE INVERSE PROBLEM. OPTIMALITY CONDITIONS

We proceed to solving inverse problem (2.1)–(2.5) by the optimization method. We replace the fulfillment of the overdetermination conditions (2.5) by the minimization of the following regularized functional

$$J[w,f]=\int_{|y|<1}|w(y,T)-w_T(y)|^2\,dy+\gamma\int_{|y|<1}|f(y)|^2dy, \tag{5.1}$$

on a pair of functions $\{w(y,t),f(y)\}$ that satisfies boundary value problem (2.1)–(2.4), where $\gamma=\alpha\int_0^T|g(t)|^2dt$, α is the initial regularization parameter. We introduce the Hamilton–Pontryagin function:

$$\begin{aligned}H[w,\psi,f,y]=&-w(y,T)\psi(y,T)-|w(y,T)-w_T(y)|^2\\&+f(y)B_T[g(t)\psi(y,t)]-\gamma|f(y)|^2,\end{aligned} \tag{5.2}$$

According to (5.1)–(5.2) and in accordance with the results of the Pontryagin maximum principle ([6, 17]), we need to solve the following optimality conditions (5.3)–(5.11): the direct boundary value problem

$$\partial_t w-\Delta w=\frac{g(t)}{2\gamma}B_T[g(t)\psi(y,t)]-\nabla P,\quad (y,t)\in Q_{yt}, \tag{5.3}$$

$$\operatorname{div} w=0,\quad (y,t)\in Q_{yt}, \tag{5.4}$$

$$w=0,\quad (y,t)\in\Sigma_{yt}, \tag{5.5}$$

$$w\,(y,\;0)=0,\quad |y|<1, \tag{5.6}$$

and the adjoint boundary value problem

$$-\partial_t\psi-\Delta\psi=-\nabla S,\quad (y,t)\in Q_{yt}, \tag{5.7}$$

$$\operatorname{div}\psi{=}0,\quad (y,t)\in Q_{yt}, \tag{5.8}$$

$$\psi=0,\quad (y,t)\in\Sigma_{yt}, \tag{5.9}$$

$$\psi(y,T)=-2[w(y,T)-w_T(y)],\quad |y|<1, \tag{5.10}$$

where the unknown function $f(y)$ is determined by formula

$$f(y)=\frac{1}{2\gamma}B_T[g(t)\psi(y,t)]. \tag{5.11}$$

Using the Riccati transform [17] (this is an artificial trick for splitting (5.3)–(5.6) and (5.7)–(5.10)) we get

$$\begin{cases}\psi_i(y,t;w_i)=E_i(y,t)w_i+r_i(y,t),\quad i=1,2,\\ S(y,t;w)=(\tilde S(y,t),w)+S_3(y,t),\end{cases} \tag{5.12}$$

where

$$\tilde{S} = \{S_1, S_2\}, \quad (\tilde{S}, w) = S_1 w_1 + S_2 w_2,$$

$$E = \{E_1,\ E_2\}, \quad r = \{r_1,\ r_2\}, \quad S_1, S_2, S_3,$$

are unknown functions to be determined. We will split optimality conditions (5.3)–(5.10):

$$-\partial_t E(y,t) - \Delta E(y,t) = -\nabla(S_1 + S_2), \quad (y,t) \in Q_{yt}, \tag{5.13}$$

$$\operatorname{div} E = 0, \quad (y,t) \in Q_{yt}, \tag{5.14}$$

$$E(y,t) = 0, \quad (y,t) \in \Sigma_{yt}, \tag{5.15}$$

$$E(y,T) = -2, \quad y \in \Omega, \tag{5.16}$$

$$\begin{aligned} &- \partial_t r(y,t) - \Delta r(y,t) + \frac{g(t)}{2\gamma} E(y,t) B_T[g(t)E(y,t)\widetilde{w}(y,t) \\ &+ r(y,t)] = -\nabla S_3, \quad (y,t) \in Q_{yt}, \end{aligned} \tag{5.17}$$

$$\operatorname{div} r = 0, \quad (y,t) \in Q_{yt}, \tag{5.18}$$

$$r(y,t) = 0, \quad (y,t) \in \Sigma_{yt}, \tag{5.19}$$

$$r(y,T) = 2w_T(y), \quad y \in \Omega, \tag{5.20}$$

where in equation (5.17) the function $\widetilde{w}(y,t)$ is given at the first iteration only, instead of which in subsequent iterations we will take the solution of the following boundary value problem (5.21)–(5.24).

After solving boundary value problems (5.13)–(5.20), by using (5.12), from (5.3)–(5.6) we obtain

$$\partial_t w - \Delta w = \frac{g(t)}{2\gamma} B_T[g(t)(E(y,t)w + r(y,t))] - \nabla P, \quad (y,t) \in Q_{yt}, \tag{5.21}$$

$$\operatorname{div} w = 0, \quad (y,t) \in Q_{yt}, \tag{5.22}$$

$$w = 0, \quad (y,t) \in \Sigma_{yt}, \tag{5.23}$$

$$w(y,0) = 0, \quad |y| < 1. \tag{5.24}$$

Note that in boundary value problem (5.21)–(5.24) only the functions $w(y,t)$, $P(y,t)$ are unknown.

6. EXAMPLE: CALCULATION RESULTS AND GRAPHS

For a numerical example, assume that $g(t) = \{g_1(t), g_2(t)\} = \{1, 1\}$. Above, based on relations (4.32)–(4.44), we have constructed an orthogonal basis

$$\{w_{jk}(y), \quad j = 0, \pm 1, \pm 2, \ldots,\ k = 1, 2, 3, \ldots\}$$

in space $\mathbf{V}$. By using this basis, we introduce an approximate solution

$$w_N(y,t) = \sum_{j=-N,k=1}^{N} g_{jkN}(t) w_{jk}(y), \quad E_N(y,t) = \sum_{j=-N,k=1}^{N} e_{jkN}(t) w_{jk}(y),$$
$$r_N(y,t) = \sum_{j=-N,k=1}^{N} h_{jkN}(t) w_{jk}(y),$$

for direct boundary value problem (5.21)–(5.24) and adjoint boundary value problem (5.13)–(5.20), formulated in a weak form:

$$(\partial_t w_N, w_{lm}) + ((w_N, w_{lm})) = \frac{1}{2\gamma} \left(B_T[E_N w_N + r_N], w_{lm}\right), \qquad -N \le l \le N,\ m = 1, \ldots, N, \tag{6.1}$$

$$w_N(y, 0) = 0, \tag{6.2}$$

$$-(\partial_t E_N, w_{lm}) + ((E_N, w_{lm})) = 0, \quad -N \le l \le N,\ m = 1, \ldots, N, \tag{6.3}$$

$$E_N(y, T) = -2, \tag{6.4}$$

$$-(\partial_t r_N, w_{lm}) + ((r_N, w_{lm})) = \frac{1}{2\gamma} E_N \left(B_T[E_N w_N + r_N]_N, w_{lm}\right), \qquad -N \le l \le N,\ m = 1, \ldots, N, \tag{6.5}$$

$$r_N(y, T) = 2 w_{TN}. \tag{6.6}$$

Functions $g_{jkN}(t)$, $e_{jkN}(t)$, $h_{jkN}(t)$, $-N \le j \le N$, $1 \le k \le N$ are scalar functions defined on $[0, T]$, (6.1), (6.3) and (6.5) are a system of ordinary differential equations with respect to these functions. For each $-N \le l \le N$, $m = 1, \ldots, N$ from (6.1)–(6.2), (6.3)–(6.4) and (6.5)–(6.6) we have

$$\begin{aligned} &|w_{lm}(y)|^2 g'_{lmN}(t) + \sum_{j=-N,k=1}^{N} ((w_{jk}(y), w_{lm}(y))) g_{jkN}(t) \\ &= \frac{1}{2\gamma} \sum_{n=-N,s=1}^{N} \sum_{j=-N,k=1}^{N} (B_T[e_{jkN}(t) g_{nsN}(t) w_{jk}(y) w_{ns}(y) \\ &\qquad + h_{nsN}(t) w_{ns}(y)], w_{lm}(y)), \end{aligned} \tag{6.7}$$

$$g_{lmN}(0) = 0, \tag{6.8}$$

$$-\left|w_{lm}\left(y\right)\right|^{2} e'_{lmN}(t) + \sum_{j=-N,k=1}^{N} \left(\left(w_{jk}\left(y\right), w_{lm}\left(y\right)\right)\right) e_{jkN}\left(t\right) = 0, \tag{6.9}$$

$$e_{lmN}(T) = -2 \int\limits_{|y|<1} w_{lm}(y) dy. \tag{6.10}$$

$$\begin{aligned} &-|w_{lm}(y)|^2 h'_{lmN}(t) + \sum_{j=-N,k=1}^{N} (w_{jk}(y), w_{lm}(y) h_{jkN}(t)) \\ &= \frac{1}{2\gamma} \sum_{j=-N,k=1}^{N} e_{jkN}(t) w_{jk}(y) \Bigg(B_T \Bigg[\sum_{j=-N,k=1}^{N} e_{jkN}(t) w_{jk}(yt) \\ &\times \sum_{j=-N,k=1}^{N} g_{jkN}(t) w_{jk}(yt) + \sum_{j=-N,k=1}^{N} h_{jkN}(t) w_{jk}(y) \Bigg], w_{lm}(y) \Bigg), \end{aligned} \tag{6.11}$$

$$h_{lmN}(T) = 2\left(w_T, w_{lmN}\right). \tag{6.12}$$

Based on the relations given by formulas (6.7)–(6.12), numerical experiments were carried out for $N = 0$ and $N = 1$.

Figures 1–4 show the graphs of the specified and calculated final functions (for the first component of the fluid velocity $w_1(y,t)$). To obtain graphs of the second component $w_2(y,t)$ in Figures 1–4, it is only necessary to swap the axes y_1 and y_2. Moreover, the y_2 axis is located in figures, from left to right, and the y_1 axis is vertical to the drawing plane. Note that when plotting graphs, the program automatically uses the formula $\theta = \widetilde{\arctan}\left(\frac{y_2}{y_1}\right)$ specified in (4.26)–(4.27).

The calculations were carried out for $T = 2$, when Ω is a circle of unit radius centered at the origin, and for different values of the regularizing parameter γ in the minimized functional:

$$\|w(y,T) - w_T\|^2 + \gamma \|f(y)\|^2, \quad \gamma = 4\alpha. \tag{6.13}$$

Figure 5 shows a graph of the change in the values of minimized functional (6.13) (excluding the regularizing term) depending on the values of the γ parameter.

Thus, we get a decrease in the difference between the final values of the calculated and given functions $\|w(y,T) - w_T(y)\|$ in L^2-norm with a decrease of γ parameter.

Remark 6.1. The convergence and stability of the proposed algorithm for solving the optimization problem is ensured by the corresponding result from the book [11] (see Chapter 2, Theorems 2.5.1–2.5.3 therein) for the Tikhonov regularization.

Note that the elements $\{q, f\}$ in Theorem 2.5.1 from [11] in our case, namely, in the functional (6.13), respectively, mean $\{f, w_T\}$, i.e. $Af = w(y,T)$, $f = w_T$.

Therefore, from [11, Theorem 2.5.1] there exists a unique $f_\gamma \in \mathbf{H}$, since the operator $A : \mathbf{H} \to \mathbf{V} \cap \mathbf{H}^2(\Omega)$, defined by relation $Af = \left[A_1^{-1}(gf)\right]_{|t=T}$, is a linear completely continuous operator from $\mathbf{H}$ to $\mathbf{H}$, where $A_1 w = gf$ is the notation of the initial boundary value problem (2.1)–(2.4) in operator form.

When numerically solving the example, we took into account the statements of Theorems 2.5.2–2.5.3 from [11]. Namely, we ensured the fact that the accuracy of the approximation of the element w_T by $w_{T\delta}$, in a sense, should be higher than the decreasing order of the regularizing parameter $\gamma(\delta)$ at $\delta \to 0$ (for example, $\lim\limits_{\delta\to 0} \delta^2/\gamma(\delta) \to 0$, where $\|w_T - w_{T\delta}\|_{\mathbf{V}\cap\mathbf{H}^2(\Omega)} \le \delta$).

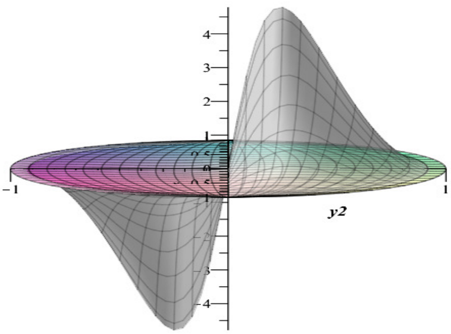

Fig. 1. Graphs of the given and calculated final functions, $\gamma = 1$

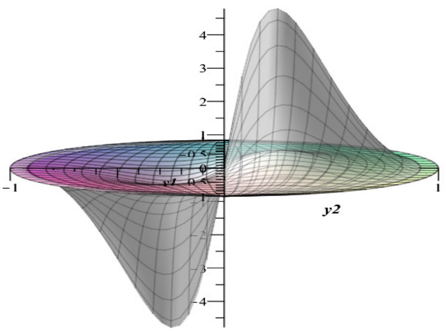

Fig. 2. Graphs of the given and calculated final functions, $\gamma = 0.13$

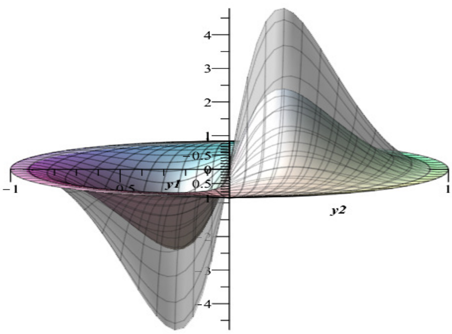

Fig. 3. Graphs of the given and calculated final functions, $\gamma = 0.026$

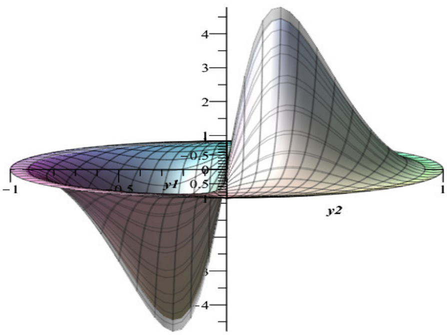

Fig. 4. Graphs of the given and calculated final functions, $\gamma = 0.014$

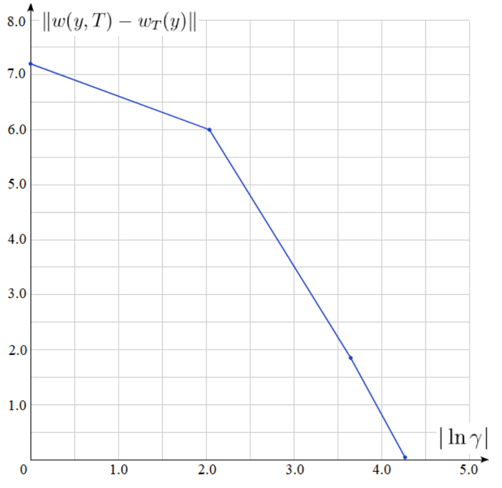

Fig. 5. Graph of the values of the minimized functional (excluding the regularizing term)

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