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POSITIVE STATIONARY SOLUTIONS OF CONVECTION-DIFFUSION EQUATIONS FOR SUPERLINEAR SOURCES

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Abstract. We investigate the existence and multiplicity of positive stationary solutions for a certain class of convection-diffusion equations in exterior domains. This problem leads to the following elliptic equation

$$\Delta u(x) + f(x, u(x)) + g(x)x \cdot \nabla u(x) = 0,$$

for $x \in \Omega_R = \{x \in \mathbb{R}^n, \|x\| > R\}$, $n > 2$. The goal of this paper is to show that our problem possesses an uncountable number of nondecreasing sequences of minimal solutions with finite energy in a neighborhood of infinity. We also prove that each of these sequences generates another solution of the problem. The case when $f(x, \cdot)$ may be negative at the origin, so-called semipositone problem, is also considered. Our results are based on a certain iteration schema in which we apply the sub and supersolution method developed by Noussair and Swanson. The approach allows us to consider superlinear problems with convection terms containing functional coefficient g without radial symmetry.

Keywords: semipositone problems, positive stationary solutions, minimal solutions with finite energy, sub and supersolutions methods.

Mathematics Subject Classification: 35B09, 35B40, 35J15, 35J61.

1. INTRODUCTION

The main purpose of this paper is to formulate sufficient conditions which guarantee the existence of a large number of positive solutions of the semilinear elliptic equation

$$\Delta u(x) + f(x, u(x)) + g(x)x \cdot \nabla u(x) = 0, \quad \text{for } x \in \Omega_R, \tag{1.1}$$

where $n > 2$, $R > 1$, $\Omega_R = \{x \in \mathbb{R}^n, \|x\| > R\}$, in the case when a sign of f is not fixed. We discuss solutions decaying in a certain neighborhood of infinity, namely

$$\lim_{\|x\| \to \infty} u(x) = 0. \tag{1.2}$$

In the literature such solutions are often called evanescent solutions. The lack of a boundary condition on $\partial\Omega$ allows us to expect the existence of many solutions.

Let us note that such problems naturally arise when we search for positive stationary solutions for the below convection-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + f(x,u) + g(x)x \cdot \nabla u. \tag{1.3}$$

Equations like (1.3) appear in models associated with many physical phenomena such as problems concerning fluid flows in chemical reactors [1], water pollution [30], oil extraction from underground reservoirs [16] or convection of heat transport for the large number of Pèclet [21] (see also, e.g., [20, 24]).
Problem (1.1)–(1.2) was formulated by Constantin in the 90s (see, e.g., [4–6]). The author considered (1.1) with radial g and proved the existence of at least one positive solution for (1.1)–(1.2) in $\Omega_{\widetilde{R}}$, for some $\widetilde{R} \geq R$. Constantin's results (see, e.g., [4, 5]) were devoted to the case when there exist $Q > 0$, $\widetilde{a} \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a nonincreasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$0 \leq f(x,u) \leq Q\widetilde{a}(\|x\|)w(|u|) \quad \text{for } x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}.$$

In [4], it was assumed additionally that $w(s) > 0$ for $s > 0$, $\int_1^\infty \frac{ds}{w(s)} = +\infty$, g was bounded and the following condition took place

$$\int_0^{+\infty} r[\widetilde{a}(r) + |g(r)|]dr < +\infty. \tag{1.4}$$

In 2005, Constantin published the paper [6] where the assumptions on w were simplified to $w \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $w(0) = 0$. One year later, Ehrnström [12] obtained similar result neglecting (1.4) for nonnegative g. In [9] and [15], Djebali *et al.* and Ehrnström and Mustafa proved the existence results for evanescent solutions of (1.1)–(1.2) without the conditions associated with the integrability of $\widetilde{a}$. The case of radial nonlinearity was discussed, among others, in [13] and [15]. The authors showed that if f depended radially on its first argument and f was nonincreasing with respect to the second variable, boundary conditions made the unique solution radial. Further results devoted to such problems can be found also in [33], where the nonlinearity f was positive and estimated by a radial function $\widetilde{f}(r,u)$ nondecreasing in u.

In more recent papers [10] and [11], the authors relaxed the assumptions concerning the growth of the nonlinearity f with respect to the second variable and described more precisely the speed of vanishing of solutions of (1.1)–(1.2). In the above papers, g is a radial function. Moreover, many of these results are based on the assumptions concerning the sublinear growth of $f(x,\cdot)$ at zero and the fact that $f(x,0)$ is nonnegative for $x \in \mathbb{R}$ with the norm $\|x\|$ sufficiently large. The latter condition plays the crucial role, especially in papers based on the sub- and supersolution methods, because it gives the trivial subsolution. The results concerning the case when the nonlinearity $f(x,\cdot)$ may change its sign at zero and (1.4) is omitted can be found in Deng's papers devoted to the same problem (see [7] and [8]). To obtain the existence of positive solutions the author had to assume some sophisticated integral inequalities which allowed him to construct sub- and supersolution of the problem with g being a radial function.

The sign conditions on f and g were relaxed also in [14], where the author assumed either that $f(x,u) \geq 0$ for $u \geq 0$ or that $f(x,0) = 0$. However, both conditions allow us to consider zero as a subsolution. Moreover, in [14], g has to be radial.

Similar problems with $g \equiv 0$ appear, among others, in combustion theory (see, e.g., [18]) and are widely discussed in the literature (see, e.g., [2, 23, 31] and references therein). In those papers the authors deal with nonlinearities f which are also negative at the origin. Such problems are known in the literature as semipositone problems. The investigation of the existence of positive solutions to semipositone problems is considerably more challenging than in the case $f(\cdot,0) > 0$, namely positone problems (see, e.g., [2, 22] and references therein).

The positone case of (1.1)–(1.2) was discussed in the recent papers [28] and [29]. It is worth emphasizing that assumption $f(\cdot,0) \geq 0$ makes the problem much easier, because it enables us to use the trivial function as a subsolution of our problem. In the case discussed in this paper we have to choose a subsolution from positive functions.

The main contribution of this paper is the method which allows us to neglect assumptions concerning the sign of nonlinearities. The novelty is the possibility of employing the same approach to positone ($f(x,0) > 0$) as well as semipositone ($f(x,0) < 0$) problems. Moreover, in our investigation $f(\cdot,0)$ may also change its sign. Therefore, we obtain the existence of uncountable set of minimal solutions with finite energy for a wide class of problems with f being superlinear at the origin. We want to emphasize that our results cover the case when nonlinearity $f(x,\cdot)$ does not need to satisfy any growth conditions at infinity. We have to control only the behavior of $f(x,\cdot)$ in a certain neighborhood of zero. In particular, we can investigate the generalized superlinear Lane–Emden–Fowler equation in a certain exterior domain.

As in many of papers mentioned above, we employ the sub- and supersolution method presented by Noussair and Swanson in [25]. Therefore, for reader's convenience, we recall the most important elements of Noussair and Swanson's results. We start with definitions of a solution, a subsolution and a supersolution for the following PDE

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u = F(x,u,\nabla u), \quad x \in \Omega_A. \tag{1.5}$$

We say that u is a solution of (1.5) in Ω_A if $u \in C^{2+\alpha}(\overline{M})$ for every bounded domain $M \subset \Omega_A$, and u satisfies (1.5) at every point $x \in \Omega_A$. As a subsolution of (1.5) we understand a function w satisfying $Lw \geq F(x,w,\nabla w)$, and as a supersolution – a function v satisfying $Lv \leq F(x,v,\nabla v)$.

Noussair and Swanson proved the following theorem (see [25]):

Theorem 1.1. *Assume that* $a_{ij} \in C^{2+\alpha}(\overline{M})$, *the matrix* $[a_{ij}(x)]_{i,j}$ *is uniformly positive definite in every bounded domain* $M \subset \Omega_A$, *where* $A > 0$, $\alpha \in (0,1)$, $F \in C^{\alpha}(\overline{M} \times \overline{J} \times \overline{N})$ *for all bounded domains* $M \subset \Omega_A$, $N \subset \mathbb{R}^n$ *and bounded interval* $J \subset \mathbb{R}$ *and moreover, for every bounded subdomain* M *of* Ω_A, *there exists a nonnegative continuous function* g_M *such that* $|F(x,u,p)| \leq g_M(|u|)\left(1+|p|^2\right)$ *for all* $x \in \overline{M}$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ *(Nagumo condition). If there exist a nonnegative subsolution* w *of* (1.5) *and a positive supersolution* v *of* (1.5) *such that* $w(x) \leq v(x)$ *for* $x \in \overline{\Omega}_A$ *then there exists a solution* u *of* (1.5) *in* Ω_A *such that* $w(x) \leq u(x) \leq v(x)$ *for all* $x \in \overline{\Omega}_A$. *Moreover,* $u(x) = v(x)$ *for all* $x \in \mathbb{R}^n$ *such that* $\|x\| = A$.

In our case, w is a subsolution of (1.1)–(1.2) when w satisfies (1.2) and

$$\Delta w(x) + f(x, w(x)) + g(x)x \cdot \nabla w(x) \geq 0$$

and v is a supersolution of (1.1)–(1.2) when v satisfies (1.2) and

$$\Delta v(x) + f(x, v(x)) + g(x)x \cdot \nabla v(x) \leq 0.$$

Let us recall also the definitions of minimal solutions and solutions with finite energy in a neighborhood of infinity (see, e.g., [26, 27]).

As a minimal solution we understand a positive solution u to our problem for which the following function $x \mapsto \|x\|^{n-2}u(x)$ is bounded above and below by a positive constant in a certain exterior domain.

We say that a positive solution u is a finite energy solution (or u is a solution with finite energy) in a neighborhood of infinity when there exists a nonnegative radial function $\psi \in C^1(\Omega_R)$ with $\psi(x) \equiv 1$ for $\|x\|$ sufficiently large and such that $\psi u \in D_0^{1,2}(\Omega_R)$, where $D_0^{1,2}(\Omega_R)$ denotes the completion of $C_0^{\infty}(\Omega_R)$ in the norm $\|\varphi\| := \|\nabla\varphi\|_{L^2(\Omega_R)}$.

Since we cannot consider the trivial function as a subsolution which is associated with the lack of the assumption concerning the nonnegativity of $f(x,0)$, we have to find a positive subsolution $\underline{u}$. Next the existence of a positive supersolution $\overline{u}$ such that $\underline{u} \leq \overline{u}$ in a certain exterior domain is proved. Then, the Noussair–Swanson theorem (Theorem 1.1) leads to the existence of a positive solution $\widehat{u}$ of (1.1)–(1.2) which is squeezed between $\underline{u}$ and $\overline{u}$. Moreover, estimates satisfied by $\underline{u}$ and $\overline{u}$ allow us to infer that $\widehat{u}$ is minimal and has finite energy. In the next part of the paper we show that a little stronger estimate for f allows us to get the existence of k positive solutions for our problem in both positone and semipositone cases, where $k \in \mathbb{N} := \{1, 2, \ldots\}$. Precisely, we start our consideration with the construction of k supersolutions $\overline{u}_1, \ldots, \overline{u}_k$ as radial positive solutions of k auxiliary linear boundary value problems in $\Omega_{\overline{R}_i}$, $i = 1, \ldots, k$. Moreover, we show that $\overline{u}_1(x) < \ldots < \overline{u}_k(x)$ in $\Omega_{\overline{R}}$, with $\overline{R} := \max_{i \in \{1,\ldots,k\}} \overline{R}_i$, namely in the intersection of $\Omega_{\overline{R}_1}, \ldots, \Omega_{\overline{R}_k}$. Then, the Noussair–Swanson theorem (Theorem 1.1) is applied k times to obtain the existence of k positive solutions to (1.1)–(1.2). Here, we want to stress that k can be an arbitrary positive integer. Thus, the natural questions arise: whether it is possible to construct the infinite number of solutions and whether this number is countable only or can be also uncountable. Let us note that if we apply the above approach in the infinite case we have the intersection of an infinite number of exterior domains $\Omega_{\overline{R}_i}$. In general, we cannot exclude the situation when this intersection is empty. Thus, the main problem is associated with the fact that we have to guarantee the existence of an infinite number of supersolutions in the same exterior domain. It appears that for nonlinearities satisfying some additional conditions concerning the behavior with respect to the independent variable, we can obtain the existence of an uncountable set of nondecreasing sequences of positive solutions in the exterior domain with sufficiently large radius. The fact that Theorem 1.1 guarantees the equality between solution and supersolution on the sphere $\|x\| = R$ plays the crucial role in the reasoning concerning the multiplicity. Since the supersolutions will be constructed explicitly, we get the additional information concerning the value of our solution on this sphere. Another aim of our discussion is to investigate more precisely the asymptotics of solutions and their gradients. We apply these results for a certain class of supelinear nonlinearities which may be negative at the origin.

Since we are focused on the stationary solutions of convection-diffusion equations associated with ecology and chemistry mainly, it is natural to treat an unknown function u as a concentration of a certain substance. Thus we look for positive solutions. Moreover, taking into account the fact that our solutions vanish at infinity, we can consider the existence of a threshold value $d > 0$ which estimates the solutions from above. Therefore we will investigate solutions in $[0, d]$ range and consequently, consider the source term f in a proper interval. We obtain these results without the assumptions concerning the radial symmetry of g. Precisely, our approach is based on the following assumptions:

(Ag) $g : \Omega_1 \to \mathbb{R}$ is C^1-*function, where* $\Omega_1 = \{x \in \mathbb{R}^n, \|x\| > 1\}$, which is nonpositive in $\Omega_{l_0} = \{x \in \mathbb{R}^n, \|x\| > l_0\}$ and

$$\int_1^\infty r^{n-1} \sup_{\|x\|=r} |g(x)| dr := \frac{a}{4} < 1.$$

(Af) $f : \Omega_1 \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous and there exists $d > 0$ such that

$$\sup_{u\in[0,d]} \sup_{\|x\|=r} f(x, u) \leq (4 - a)d(n - 2)r^{-n-1} \quad \text{in } [1, +\infty). \tag{1.6}$$

$(Af1)$ *There exists* $c, b, k, l > 0$ *such that* $l \geq n + 1$, $k > n + l - 2$ *and for all* $x \in \Omega_1$ and $u \in [0, 1]$,

$$f(x, u) \geq \frac{cu}{\|x\|^l} - \frac{b}{\|x\|^k}.$$

Remark 1.2. Condition (Af) implies immediately the following inequality

$$\int_1^\infty r^{n-1} \sup_{u\in[0,d]} \sup_{\|x\|=r} f(x, u) dr \leq (4 - a)(n - 2)d. \tag{1.7}$$

Moreover, our approach can be employed in the case when the inequality in (Af) is replaced by the below condition in the asymptotic form:

$$\overline{\lim_{d\to\infty}} \sup_{u\in[0,d]} \sup_{\|x\|=r} \frac{f(x,u)}{d} < (4-a)(n-2)r^{-n-1} \quad \text{in } [1,+\infty).$$

Since f is continuous, the requirement that f is bounded uniformly in $[0,d]$ by a function which depends on the norm of x (see (1.6)), is not very restrictive. Thus, it turns out that our conditions are satisfied by many f being polynomials, rational or exponential functions with respect to the second variable. Let us emphasize that $(Af1)$ includes also the cases when $f(x,0)<0$ or $f(\cdot,0)$ may change its sign. An explicit example with details is described at the end of this paper.

2. EXISTENCE OF A POSITIVE SOLUTION WITH THE FINITE ENERGY

In this section we show the existence of at least one positive solution of (1.1)–(1.2) and investigate the asymptotic behavior of this solution and its gradient. Our main tool is the sub- and supersolution theorem (Theorem 1.1). We start with the following auxiliary linear problem

$$\begin{cases} -\Delta v(x) = M(\|x\|) & \text{for } x\in\Omega_1,\\ v(x)=0 & \text{for } \|x\|=1,\\ \lim_{\|x\|\to\infty} v(x)=0, \end{cases} \tag{2.1}$$

where $M:[1,+\infty)\to(0,+\infty)$ is a certain continuous function. It is a well-known fact that the investigation of the existence of radial solutions of (2.1) leads, via a suitable transformation, to the solvability of the following Dirichlet problem with singularity at the end-point 1

$$\begin{cases} -z''(t)=h(t) & \text{in } (0,1),\\ z(0)=z(1)=0, \end{cases} \tag{2.2}$$

where

$$h(t)=(n-2)^{-2}(1-t)^{\frac{2n-2}{2-n}}M((1-t)^{\frac{1}{2-n}}).$$

Precisely, having a solution z of (2.2) we can derive that $v(x)=z(1-\|x\|^{2-n})$ is a solution of (2.1). Conversely, if $v(x)=\overline{z}(\|x\|)$, with $\overline{z}:[1,+\infty)\to\mathbb{R}$, is a radial solution of (2.1), then $z(t)=\overline{z}((1-t)^{\frac{1}{2-n}})$ satisfies (2.2). Now we focus on the solution of (2.2) and its properties.

Lemma 2.1. *If $h:(0,1)\to(0,+\infty)$ is a continuous function and $\int_0^1 h(t)dt\le 4d$ then there exists a unique positive classical solution z of (2.2), such that $z(t)\le d$ in $[0,1]$ and $z'(t)\le 0$ for all $t\in(\overline{t},1)$, where $\overline{t}\in(0,1)$ maximizes z. Moreover,*

$$z(t)=O(1-t) \quad \textit{for} \quad t\to 1^-, \tag{2.3}$$

and

$$z(t)=o(\phi(t)) \quad \textit{for} \quad t\to 1^- \tag{2.4}$$

for any function $\phi\in C^1(0,1)$ such that $\lim_{t\to 1^-}\phi(t)=0$ and $\lim_{t\to1^-}\phi'(t)=+\infty$.

Proof. We obtain immediately the solution of (2.2) which is given by the following formula

$$z(t) = \int_0^1 \mathbf{G}(s,t)h(s)ds \tag{2.5}$$

with Green's function

$$\mathbf{G}(s,t) := \begin{cases} (1-t)s & \text{for } 0 \le s \le t, \\ t(1-s) & \text{for } t < s \le 1. \end{cases} \tag{2.6}$$

We also derive that $z \in C([0,1]) \cap C^2(0,1)$, $z(0) = z(1) = 0$ and

$$0 \le z(t) \le \frac{1}{4} \int_0^1 h(s)ds \le d.$$

The fact that h is positive in $(0,1)$ leads to the conclusion that z is strictly concave. Thus z is positive, attains its global maximum at certain $\bar{t} \in (0,1)$ and for all $t \in (\bar{t}, 1)$, $z'(t) < 0$.

Taking into account the definition of z, we can apply L'Hospital's rule, which allows us to obtain the following chain of assertions

$$(0,+\infty) \ni \int_0^1 sh(s)ds = -\lim_{t\to 1^-} z'(t) = \lim_{t\to 1^-} \frac{z(t)}{(1-t)},$$

consequently,

$$z(t) = O(1-t) \quad \text{for } t \to 1^-. \tag{2.7}$$

Now we take an arbitrary function $\phi \in C^1(0,1)$ satisfying both conditions: $\lim_{t\to 1^-} \phi(t) = 0$ and $\lim_{t\to 1^-} \phi'(t) = +\infty$. Then, by the boundedness of z', we infer that $\lim_{t\to 1^-} \frac{z'(t)}{\phi'(t)} = 0$. In consequence, L'Hospital's rule gives $\lim_{t\to 1^-} \frac{z(t)}{\phi(t)} = 0$ and further, (2.4) is proved. □

We can formulate the existence theorem covering superlinear problem in positone as well as semipositone cases.

Theorem 2.2. *Let (Ag), (Af) and $(Af1)$ hold. Then problem (1.1)–(1.2) possesses at least one minimal solution $\widehat{u}$ in $\overline{\Omega}_{\widehat{R}}$, for a certain $\widehat{R} > 1$, such that $\widehat{u} \le d$ in $\overline{\Omega}_{\widehat{R}}$ and*

$$\widehat{u}(x) = o\left(\widetilde{\phi}(\|x\|)\right) \quad \textit{as } \|x\| \to +\infty \tag{2.8}$$

for any $\widetilde{\phi} \in C^1(1,+\infty)$ such that $\lim_{r\to+\infty} \widetilde{\phi}(r) = 0$ and $\lim_{r\to+\infty} \widetilde{\phi}'(r)r^{n-1} = +\infty$. Moreover, $\widehat{u}$ has finite energy in a certain neighborhood of infinity.

Proof. Step 1 (*subsolution*). Let us consider the harmonic function of the form $\underline{u}(x) = \alpha\|x\|^{2-n}$ with $0 < \alpha \leq \frac{(4-a)d}{4(n-2)}$. To show that $\underline{u}$ is a subsolution of our problem in a certain exterior domain we have to note that

$$\frac{\partial}{\partial x_i}\underline{u}(x) = \alpha\,(2-n)\,\|x\|^{1-n}\frac{x_i}{\|x\|} = \alpha\,(2-n)\,\frac{x_i}{\|x\|^n}$$

and further

$$x \cdot \nabla\underline{u}(x) = -\alpha(n-2)\sum_{i=1}^{n}\frac{x_i}{\|x\|^n}x_i = -\alpha(n-2)\|x\|^{2-n} < 0.$$

Taking into account the fact that $g(x) \leq 0$ for all for $\|x\| > l_0$, we get

$$g(x)x \cdot \nabla\underline{u}(x) \geq 0 \text{ in } \Omega_{l_0}.$$

Then for all $x \in \mathbb{R}^n$ with

$$\|x\| > R_0 := \max\left\{l_0, \left(\frac{b}{c\alpha}\right)^{\frac{1}{k-(n+l-2)}}\right\},$$

the following chain of inequalities holds

$$\Delta\underline{u}(x) + g(x)x \cdot \nabla\underline{u}(x) + f(x,\underline{u}(x)) \geq f(x,\underline{u}(x)) \geq c\alpha\|x\|^{-(n+l-2)} - b\|x\|^{-k} \geq 0,$$

where the last assertion follows from the choice of R_0. This means that $\underline{u}$ is a subsolution of (1.1)–(1.2) with $R = R_0$.

Step 2 (*supersolution*). Let us consider (2.1) with

$$M(r) := d(n-2)[(4-a)r^{-n-1} + 4\widetilde{g}(r)],$$

where $\widetilde{g}(r) := \sup_{\|x\|=r}|g(x)|$, and note that

$$h(t) = (n-2)^{-2}(1-t)^{\frac{2n-2}{2-n}}M((1-t)^{\frac{1}{2-n}})$$

satisfies the assumptions of Lemma 2.1. To this effect it suffices to integrate by substitution ($r = (1-t)^{\frac{1}{2-n}}$) which gives what follows

$$\int_0^1 h(t)dt = (4-a)d\int_1^\infty r^{-2}dr + 4d\int_1^\infty r^{n-1}\widetilde{g}(r)dr \leq 4d.$$

Therefore, owing to Lemma 2.1 we get the existence of the unique solution z of (2.2) such that $z(t) \leq d$ in $[0,1]$, for all $t \in (\bar{t},1)$, $z'(t) \leq 0$, where $\bar{t} \in (0,1)$ maximizes z, and assertions (2.3) and (2.4) hold. Now we derive that $\overline{u}$, given by the formula $\overline{u}(x) = z(1-\|x\|^{2-n})$, is a positive solution of (2.1). Moreover, the substitution $t = 1-\|x\|^{2-n}$ and the estimate (2.3) implies

$$\overline{u}(x) = O\left(\|x\|^{2-n}\right) \quad \text{as } \|x\| \to +\infty.$$

The above assertion gives the existence of $A_1 > 0$, $L_1 > 1$ such that for all $x \in \mathbb{R}^n$, $\|x\| \geq L_1$,

$$\overline{u}(x) \leq A_1 \|x\|^{2-n}. \tag{2.9}$$

As a simple consequence of (2.4) we obtain

$$\overline{u}(x) = o\left(\widetilde{\phi}(\|x\|)\right) \quad \text{as } \|x\| \to +\infty \tag{2.10}$$

for any $\widetilde{\phi} \in C^1(1, +\infty)$ such that $\lim_{r \to +\infty} \widetilde{\phi}(r) = 0$ and $\lim_{r \to +\infty} \widetilde{\phi}'(r) r^{n-1} = +\infty$.

Our task is now to show that $\overline{u}(x) = z(1 - \|x\|^{2-n})$ is a supersolution of (1.1)–(1.2) in a certain exterior domain. Since z is nonincreasing in $(\overline{t}, 1)$ as the solution of (2.2) (see Lemma 2.1), we can take

$$\overline{R} := \left(1 - \overline{t}\right)^{\frac{1}{2-n}} > 1$$

and obtain for all $x \in \mathbb{R}^n$ such that $\|x\| \geq \overline{R}$, the following estimate

$$x \cdot \nabla \overline{u}(x) = z'(1 - \|x\|^{2-n})(n-2)\|x\|^{2-n} \leq 0$$

and, on the other hand,

$$x \cdot \nabla \overline{u}(x) = z'(1 - \|x\|^{2-n})(n-2)\|x\|^{2-n} \geq -4d(n-2).$$

Thus, by (Ag), we obtain for $x \in \overline{\Omega}_{\max\{\overline{R}, l_0\}}$,

$$g(x) x \cdot \nabla \overline{u}(x) \leq 4d(n-2)\widetilde{g}(\|x\|).$$

Finally, we get

$$\begin{aligned} \Delta \overline{u}(x) + f(x, \overline{u}(x)) + g(x) x \cdot \nabla \overline{u}(x) &\leq \Delta \overline{u}(x) + f(x, \overline{u}(x)) + 4d(n-2)\widetilde{g}(\|x\|) \\ &\leq \Delta \overline{u}(x) + d(n-2)[(4-a)\|x\|^{-n-1} + 4\widetilde{g}(\|x\|)] \\ &= \Delta \overline{u}(x) + M(\|x\|) = 0. \end{aligned}$$

Hence $\overline{u}$ is a positive supersolution of (1.1)–(1.2)) in $\overline{\Omega}_{\max\{\overline{R}, l_0\}}$.

Step 3 (*inequality between subsolution and supersolution*). Our task is now to show that for all $x \in \overline{\Omega}_{\widehat{R}}$, where

$$\widehat{R} := \max\left\{\overline{R}, R_0, 2^{\frac{1}{n-2}}\right\},$$

we have $\underline{u}(x) \leq \overline{u}(x)$. To this effect we consider another auxiliary linear problem

$$\begin{cases} -\Delta v(x) = m\|x\|^{-2n+2} & \text{for } x \in \Omega_1, \\ v(x) = 0 & \text{for } \|x\| = 1, \\ \lim_{\|x\| \to \infty} v(x) = 0, \end{cases} \tag{2.11}$$

where $m = (4-a)d(n-2)$.

It is easy to check that its solution has the following form

$$\overline{v}(x) = \frac{1}{2}(4-a)d(n-2)^{-1}\|x\|^{2-n}(1-\|x\|^{2-n}).$$

Moreover, we have for all $x \in \Omega_1$,

$$\begin{aligned}-\Delta(\overline{u}(x) - \overline{v}(x)) &= M(\|x\|) - (4-a)d\,(n-2)\,\|x\|^{-2n+2} \\ &\geq 4d(n-2)\widetilde{g}(\|x\|) > 0,\end{aligned}$$

$$\overline{u}(x) - \overline{v}(x) = 0 \quad \text{for all } \|x\| = 1$$

and

$$\lim_{\|x\|\to\infty} (\overline{u}(x) - \overline{v}(x)) = 0.$$

Thus, the maximum principle leads to the inequality $\overline{u}(x) \geq v(x)$ for all $x \in \overline{\Omega}_1$. Finally, we get for all $x \in \mathbb{R}^n$, $\|x\| \geq \widehat{R}$,

$$\overline{u}(x) \geq \overline{v}(x) \geq \frac{1}{4}(4-a)d\,(n-2)^{-1}\,\|x\|^{2-n} \geq \underline{u}(x).$$

Finally, we have the required inequality.

Step 4 (*minimal solution*). In the previous steps we have proved the existence of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (1.1)–(1.2) such that $\underline{u}(x) \leq \overline{u}(x)$ in $\overline{\Omega}_{\widehat{R}}$. Thus, the Noussair and Swanson theorem leads to the existence of at least one solution $\widehat{u}$ of (1.1)–(1.2) such that

$$\underline{u}(x) \leq \widehat{u}(x) \leq \overline{u}(x) \quad \text{in } \overline{\Omega}_{\widehat{R}} \tag{2.12}$$

and

$$\widehat{u}(x) = \overline{u}(x) \quad \text{in } \partial\Omega_{\widehat{R}}. \tag{2.13}$$

Let us note that (2.12), together with the definition of $\underline{u}$ and estimate (2.9), implies that

$$\alpha\|x\|^{2-n} \leq \widehat{u}(x) \leq A_1\|x\|^{2-n}$$

for all $x \in \mathbb{R}^n$ such that $\|x\| \geq \max\{L_1, \widehat{R}\}$, namely, $\widehat{u}$ is the minimal solution. (2.8) is a simple consequences of (2.10) and (2.12).

Step 5 (*finite energy*). Let us consider the positive solution $\widehat{u}$ described in the previous step. Here we apply the standard approach (see, e.g., [26]) based on the classical estimates for solutions of elliptic problems ([17, Theorem 6.2]). Let us take $x \in \Omega_{\widehat{R}}$ such that $\|x\| \geq 2\widehat{R}$, and consider the ball $B(x; r/2)$ of center x and radius $r/2$, where $r = \|x\|$. Taking into account the estimates mentioned above, we derive the existence of $C > 0$ such that

$$\begin{aligned}\frac{r}{2}\|\nabla\widehat{u}(x)\| &\leq C\left(\|\widehat{u}\|_{C(B(x;r/2))} + \frac{3}{4}r^2\|f\|_{C(B(x;r/2))}\right) \\ &\leq C\left(A_1 + \frac{3}{4}(4-a)d(n-2)\right)r^{2-n},\end{aligned}$$

where a, d are given in (Af), and further we obtain

$$\|\nabla\widehat{u}(x)\| \leq 2C\left(A_1 + \frac{3}{4}(4-a)d\,(n-2)\right)\|x\|^{1-n}.$$

The above assertion implies that $\widehat{u}$ has finite energy according to the definition given in Section 1. □

3. EXISTENCE OF AN ARBITRARY NUMBER OF SOLUTIONS

Here we apply the idea concerning the construction of an arbitrary number of solutions presented, for example, in [28] or [29]. Thus, we have to describe a finite number of supersolutions in the same exterior domain. The natural question is whether we have to assume additional conditions. It turns out that it suffices to replace (1.6) by the stronger inequality. Precisely, (Af) has to be replaced by (Af') in the following form:

(Af') $f : \Omega_1 \times \mathbb{R} \to \mathbb{R}$ *is locally Hölder continuous and there exist* $d > 0$ *and* $\varepsilon \in (0, 4-a)$ *such that*

$$\sup_{u\in[0,d]}\sup_{\|x\|=r} f(x,u) < (4-a-\varepsilon)(n-2)dr^{-n-1} \quad \text{in } [1,+\infty). \tag{3.1}$$

Now we formulate the proposition in which we construct k functions M_i such that each

$$h_i(t) = (n-2)^{-2}(1-t)^{\frac{2n-2}{2-n}} M_i((1-t)^{\frac{1}{2-n}})$$

satisfies the assumptions of Lemma 2.1.

Proposition 3.1. *Under condition* (Af'), *there exist continuous functions* $M_i : [1,+\infty) \to (0,+\infty)$, $i = 1,\ldots,k$, *such that for all* $i = 1,\ldots,k-1$, *the following inequalities hold*

$$\sup_{u\in[0,d]}\sup_{\|x\|=r} f(x,u) + 4d(n-2)\widetilde{g}(r) \leq M_i(r) < M_{i+1}(r) \tag{3.2}$$

for all $r \geq 1$ *and*

$$\int_1^\infty r^{n-1}M_i(r)dr \leq 4(n-2)d. \tag{3.3}$$

Proof. Let us consider $c_1, c_2, ,\ldots, c_{k-1}, c_k \in (0,\varepsilon]$ such that $c_1 < c_2 < \ldots < c_{k-1} < c_k$. We start the proof with the definition of each function M_i:

$$M_i(r) := d(n-2)[(4-a-\varepsilon+c_i)r^{-n-1} + 4\widetilde{g}(r)]$$

for all $r \in [1,+\infty)$, $i = 1,2,3,\ldots,k$. Thus (3.2) holds. Moreover, the following estimate holds:

$$\int_1^\infty r^{n-1}M_i(r)dr \leq d(n-2)\,[4-\varepsilon+c_i] \leq 4(n-2)d. \qquad \square$$

Now we consider k auxiliary linear elliptic problems

$$\begin{cases} -\Delta v(x) = M_i(\|x\|) & \text{for } x \in \Omega_1, \\ v(x) = 0 & \text{for } \|x\| = 1, \\ \lim_{\|x\|\to\infty} v(x) = 0, \end{cases} \tag{3.4}$$

$i \in \{1, \dots, k\}$, which leads to k independent Dirichlet problems

$$\begin{cases} -z''(t) = h_i(t) & \text{in } (0,1), \\ z(0) = z(1) = 0, \end{cases} \tag{3.5}$$

$i = 1, 2, \dots, k$, where

$$h_i(t) = (n-2)^{-2}(1-t)^{\frac{2n-2}{2-n}} M_i((1-t)^{\frac{1}{2-n}}).$$

For each $i = 1, 2, \dots, k$, we can apply the approach presented in the previous section. Thus, we obtain immediately k ordered solutions of (3.5) with $i \in \{1, \dots, k\}$, described in the below lemma.

Lemma 3.2. *Assume that (Af') holds. For each $i = 1, \dots, k$, there exists a positive classical solution z_i of (3.5), such that $0 \le z_i(t) \le d$ in $[0,1]$ and $z_i'(t) \le 0$ for all $t \in (t_i, 1)$, where $t_i \in (0,1)$ maximizes z_i. For all $i = 1, \dots, k$,*

$$z_i(t) = O(1-t) \quad for \quad t \to 1^-, \tag{3.6}$$

and for any function $\phi \in C^1(0,1)$ such that $\lim_{t\to1^-} \phi(t) = 0$ and $\lim\limits_{t\to1^-} \phi'(t) = +\infty$, we have

$$z_i(t) = o(\phi(t)) \quad for \quad t \to 1^-. \tag{3.7}$$

Moreover, the solutions are ordered, i.e.

$$z_1(t) < \ldots < z_i(t) < z_{i+1}(t) < \ldots < z_k(t). \tag{3.8}$$

Proof. Applying Proposition 3.1 and Lemma 2.1 for each problem (3.5) we obtain the existence of solutions $z_1, \dots, z_k$ satisfying the required estimates. Thus, it suffices to prove the last part of the theorem, namely chain of inequalities (3.8). To this end, we use the inequality $h_{i+1}(t) > h_i(t)$ between right-hand sides of these problems which holds for each $i = 1, 2, \dots, k-1$. Therefore,

$$-\left(z_{i+1}(t) - z_i(t)\right)'' = h_{i+1}(t) - h_i(t) > 0$$

for all $t \in (0,1)$. Taking into account the boundary condition

$$z_{i+1}(0) = z_i(0) = z_{i+1}(1) = z_i(1) = 0,$$

we derive that $z_{i+1}(t) - z_i(t) > 0$ for all $t \in (0,1)$, what we have claimed. □

The above lemma allows us to construct k ordered supersolution in the same exterior domain. Now we can prove the following result concerning the multiplicity of minimal solutions with finite energy.

Theorem 3.3. *Let* (Ag), (Af') *and* $(Af1)$ *hold. For each* $k \in \mathbb{N}$, *problem* (1.1)–(1.2) *possesses at least* k *minimal solutions* $\widetilde{u}_i$, $i = 1, \ldots, k$, *in* $\overline{\Omega}_{\widetilde{R}}$, *for a certain* $\widetilde{R} > 1$, *having finite energy in a certain neighborhood of infinity and satisfying the following estimates*

$$\widetilde{u}_i \leq d \quad in \quad \overline{\Omega}_{\widetilde{R}}, \tag{3.9}$$

and for any $\widetilde{\phi} \in C^1(1, +\infty)$ *such that* $\lim_{r\to+\infty} \widetilde{\phi}(r) = 0$ *and* $\lim_{r\to+\infty} \widetilde{\phi}'(r) r^{n-1} = +\infty$,

$$\widetilde{u}_i(x) = o\left(\widetilde{\phi}(\|x\|)\right) \quad as \ \|x\| \to +\infty. \tag{3.10}$$

Moreover,

$$\widetilde{u}_1(x) \leq \ldots \leq \widetilde{u}_i(x) \leq \widetilde{u}_{i+1}(x) \leq \ldots \leq \widetilde{u}_k(x) \quad for\ all \quad x \in \overline{\Omega}_{\widetilde{R}} \tag{3.11}$$

and

$$\widetilde{u}_1(x) < \ldots < \widetilde{u}_i(x) < \widetilde{u}_{i+1}(x) < \ldots < \widetilde{u}_k(x) \quad for\ all \quad x \in \partial\overline{\Omega}_{\widetilde{R}}. \tag{3.12}$$

Proof. Step 1 (*subsolution*). As in the proof of Theorem 2.2 we show that the function $\underline{u}(x) = \alpha\|x\|^{2-n}$, with $0 < \alpha \leq \frac{1}{4}(4-a-\varepsilon)d(n-2)^{-1}$, can be considered as a subsolution of (1.1).

Step 2 (*k supersolutions*). Let k be an arbitrary positive integer. Then for each $i = 1, \ldots, k$, M_i satisfies (3.3). It suffices to apply (3.3) and note that the following estimate holds:

$$\int_0^1 h_i(t)dt = (n-2)^{-1} \int_1^\infty r^{n-1} M_i(r)dr \leq 4d.$$

Now Lemma 3.2 leads to the existence of solution z_i of (3.5) such that $z_i'(t) \leq 0$ for all $t \in (\overline{t}_i, 1)$ and satisfying (3.6), (3.7) and (3.8). Thus, the function $\overline{u}_i(x) = z_i(1 - \|x\|^{2-n})$ is a solution of (3.4). Taking into account the definition of M_i and applying the same reasoning as in the second step of the proof of Theorem 2.2, we can show that $\overline{u}_i$ is a supersolution of (1.1)–(1.2) in $\overline{\Omega}_{\overline{R}_i}$, where

$$\overline{R}_i := \max\left\{\left(1 - \overline{t}_i\right)^{\frac{1}{2-n}}, l_0\right\} > 1.$$

Thus, we have k supersolutions of (1.1) $\overline{u}_1, \ldots, \overline{u}_k$ in the same exterior domain $\overline{\Omega}_{\overline{R}}$, where $\overline{R} := \max\{\overline{R}_1, \ldots, \overline{R}_k\}$. Moreover, we obtain also the following estimates:

$$\overline{u}_i \leq d \quad \text{in } \overline{\Omega}_{\overline{R}}, \tag{3.13}$$

$$\overline{u}_i(x) = O\left(\|x\|^{2-n}\right) \ \text{ as } \ \|x\| \to +\infty \tag{3.14}$$

and

$$\overline{u}_i(x) = o\left(\widetilde{\phi}(\|x\|)\right) \text{ as } \|x\| \to +\infty \tag{3.15}$$

for any $\widetilde{\phi} \in C^1(1,+\infty)$ satisfying conditions $\lim_{r\to+\infty}\widetilde{\phi}(r) = 0$ and $\lim_{r\to+\infty}\widetilde{\phi}'(r)r^{n-1} = +\infty$. Moreover, by (3.8) we get

$$\overline{u}_1(x) \le \ldots \le \overline{u}_i(x) \le \overline{u}_{i+1}(x) \le \ldots \le \overline{u}_k(x) \quad \text{for all } x \in \overline{\Omega}_{\max\{\overline{R}_1,\ldots,\overline{R}_k\}}, \tag{3.16}$$

Step 3 (*inequality between subsolution and supersolution*). Let us consider (2.11) with $m = (4-a-\varepsilon)d(n-2)$. Then, the function

$$\overline{\overline{v}}(x) = \frac{1}{2}(4-a-\varepsilon)d(n-2)^{-1}\|x\|^{2-n}(1-\|x\|^{2-n})$$

is the solution of this problem and further $\overline{\overline{v}}(x) \ge \underline{u}(x)$ in $\Omega_{\sqrt[n-2]{2}}$. Moreover, we can obtain the following estimates for all $x \in \Omega_1$,

$$\begin{aligned}-\Delta(\overline{u}_1(x) - \overline{\overline{v}}(x)) &= M_1(\|x\|) - (4-a-\varepsilon)d(n-2)\|x\|^{-2n+2}\\ &\ge d(n-2)c_1\|x\|^{-2n+2} + 4d(n-2)\widetilde{g}(\|x\|) > 0.\end{aligned}$$

Since $\overline{u}_1(x) = \overline{\overline{v}}(x) = 0$ for $\|x\| = 1$ and

$$\lim_{\|x\|\to\infty} \overline{u}_1(x) = 0 = \lim_{\|x\|\to\infty} \overline{\overline{v}}(x),$$

the maximum principle leads to the inequality $\overline{u}_1(x) \ge \overline{\overline{v}}(x)$ in Ω_1. Finally, for $x \in \Omega_{\sqrt[n-2]{2}}$ we get

$$\underline{u} \le \overline{\overline{v}} \le \overline{u}.$$

Step 4 (*k solutions*). Applying the Noussair and Swanson theorem (Theorem 1.1), we obtain the existence of a certain positive solution $\widetilde{u}_1$ of (1.1)–(1.2) in $\overline{\Omega}_{\widetilde{R}}$, where $\widetilde{R} := \max\{\overline{R}_1,\ldots,\overline{R}_k,\widehat{R}\}$, such that $\underline{u}(x) \le \widetilde{u}_1(x) \le \overline{u}_1(x)$ for all $x \in \overline{\Omega}_{\widetilde{R}}$ and $\widetilde{u}_1(x) = \overline{u}_1(x)$ for $x \in \partial\overline{\Omega}_{\widetilde{R}}$. We can treat $\widetilde{u}_1$as a subsolution to (1.1)–(1.2) and consider supersolution $\overline{u}_2(x)$. It is clear that $\widetilde{u}_1(x) \le \overline{u}_1(x) < \overline{u}_2(x)$ on $\Omega_{\widetilde{R}}$, where the latter inequality follows from (3.16). The Noussair and Swanson theorem leads to the existence of the second solution $\widetilde{u}_2$ of (1.1)–(1.2) in $\overline{\Omega}_{\widetilde{R}}$, such that $\widetilde{u}_1(x) \le \widetilde{u}_2(x) \le \overline{u}_2(x)$ in $\overline{\Omega}_{\widetilde{R}}$ and $\widetilde{u}_2(x) = \overline{u}_2(x)$ in $\partial\overline{\Omega}_{\widetilde{R}}$. Consequently, we infer that for all $x \in \partial\overline{\Omega}_{\widetilde{R}}$, $\widetilde{u}_1(x) = \overline{u}_1(x) < \overline{u}_2(x) = \widetilde{u}_2(x)$. Thus, $\widetilde{u}_1$ and $\widetilde{u}_2$ are different functions. We can proceed with this process. Having constructed $i-th$ solution $\widetilde{u}_i$, for $i = 1,2,\ldots,k-1$, we know that $\widetilde{u}_i \le \overline{u}_i < \overline{u}_{i+1}$ in $\Omega_{\widetilde{R}}$. Treating $\widetilde{u}_i$ as a subsolution of (1.1)–(1.2) and applying again the Noussair and Swanson theorem, we derive the existence of $\widetilde{u}_{i+1}$ satisfying our problem. Moreover, we know that $\widetilde{u}_i(x) \le \widetilde{u}_{i+1}(x) \le \overline{u}_{i+1}(x)$ in $\overline{\Omega}_{\widetilde{R}}$ and $\widetilde{u}_i(x) = \overline{u}_i(x) < \overline{u}_{i+1}(x) = \widetilde{u}_{i+1}(x)$ in $\partial\overline{\Omega}_{\widetilde{R}}$. Finally, we obtain k different minimal solutions of our problem and state that (3.11) and (3.12) hold. Assertion (3.10) is a consequence of (3.15).

Step 4 and 5 (*estimates*). Finally, following the fourth and fifth steps of the proof of Theorem 2.2, we can prove that each $\widetilde{u}_i$ is a minimal solution with finite energy in a certain neighborhood of infinity. □

4. UNCOUNTABLE SET OF NONDECREASING SEQUENCES OF POSITIVE SOLUTIONS

In this section we answer the question how to obtain the infinite number of positive minimal solutions with finite energy and state that they are ordered. Following the reasoning from the previous section we show that it suffices to guarantee the existence of infinite number of positive supersolutions in the common exterior domain. Then the procedure described in the proof of Theorem 3.3 can be employed to obtain the existence of uncountable number of such solutions. To this end we assume stronger estimates concerning behavior of f and g with respect to x and add the condition which guarantees that the radius R is sufficiently large. Precisely, instead of (Af) (or (Af')) and (Ag) we assume (Af'') and (Ag') in the following form:

(Ag') $g : \Omega_1 \to \mathbb{R}$ *belongs to* $C^1(\Omega_1)$, there exist $l_0 > 1$, $a \in (0, 4)$ such that

$$g(x) \leq 0 \quad \textit{and} \quad |g(x)| \leq \frac{a}{4}\|x\|^{2-2n} \quad \textit{for all} \|x\| > l_0.$$

(Af'') $f : \Omega_1 \times \mathbb{R} \to \mathbb{R}$, *is locally Hölder continuous and there exist* $d > 0$ and $\varepsilon \in (0, 4-a)$ such that

$$\sup_{u\in[0,d]} \sup_{\|x\|=r} f(x,u) \leq (4-a-\varepsilon)d(n-2)r^{2-2n} \quad \text{in } [1,+\infty) \tag{4.1}$$

and there exists $K > 1$ such that

$$\max\left\{ l_0, 2^{\frac{1}{n-2}}, \left[\frac{1}{2}\left(1-K^{2-n}\right)^2 \frac{1}{K^2}\right]^{-\frac{1}{n-2}} \right\} \leq R. \tag{4.2}$$

For each $p \in \mathbb{R}$, $k \in \mathbb{N}$, we consider the sequence of uncountable sets of auxiliary problems

$$\begin{cases} -\Delta v(x) = M_{p,k}(\|x\|) & \text{for } x \in \Omega_1, \\ v(x) = 0 & \text{for } \|x\| = 1, \\ \lim_{\|x\|\to\infty} v(x) = 0, \end{cases} \tag{4.3}$$

where

$$M_{p,k}(r) := \frac{(n-2)d}{r^{2n}} \left[(4-\varepsilon)\, r^2 + \varepsilon \left(1 - \frac{e^p}{e^p+1}\frac{1}{k} \right) \right].$$

Now we describe some useful properties of functions $M_{p,k}(r)$, where $p \in \mathbb{R}$ and $k \in \mathbb{N}$.

Remark 4.1. For all $p \in \mathbb{R}$, $k \in \mathbb{N}$, we have:

(1) $M_{p,k}(r) > 0$ in Ω_1,
(2) $M_{p,k+1}(r) > M_{p,k}(r)$ in Ω_1,
(3) $\int_1^\infty r^{n-1} M_{p,k}(r)dr \leq 4(n-2)d$.

Proof. The definition of $M_{p,k}$ gives immediately the first and second properties. To obtain the third one it suffices to estimate the following integral

$$\int_1^\infty r^{n-1} M_{p,k}(r)dr = \int_1^\infty r^{n-1} \frac{(n-2)d}{r^{2n}} \left[(4-\varepsilon) r^2 + \varepsilon \left(1 - \frac{e^p}{e^p+1} \frac{1}{k} \right) \right] dr$$
$$\leq (4-\varepsilon)(n-2)d \frac{1}{n-2} + \varepsilon (n-2) d \frac{1}{n} \leq 4(n-2)d,$$

where the last inequality follows from the fact that $n \geq 3$. □

Now we formulate lemma in which the existence of continua of positive supersolutions of (1.1)–(1.2) in Ω_R will be proved, where R satisfies inequality (4.2). It is worth emphasizing that we obtain all these supersolutions in the same exterior domain. In this sections, the auxiliary lemma describes supersolutions of our problem, because their construction in the same exterior domain is much more complicated than in the finite case.

Lemma 4.2. *Suppose that* (Ag'), $(Af1)$ *and* (Af'') *hold. Then for each* $k \in \mathbb{N}$, $p \in \mathbb{R}$, *problem* (1.1)–(1.2) *possesses at least one positive supersolution* $\overline{u}_{p,k}$ *in* Ω_R, *such that for all* $p \in \mathbb{R}$, *the sequence* $\{\overline{u}_{p,k}\}_{k\in\mathbb{N}}$ *is increasing. Moreover, for each* $p \in \mathbb{R}$ *and* $k \in \mathbb{N}$ *the following estimates hold*

$$\overline{u}_{p,k}(x) = O\left(\|x\|^{2-n}\right) \quad as \|x\| \to +\infty \tag{4.4}$$

and for all $\phi \in C^1(1,+\infty)$ *such that* $\lim_{r\to+\infty} \phi(r) = 0$ *and* $\lim\limits_{r\to+\infty} \phi'(r) r^{n-1} = +\infty$,

$$\overline{u}_{p,k}(x) = o\left(\phi(\|x\|)\right) \quad as\ \|x\| \to +\infty. \tag{4.5}$$

Proof. Basing ourselves on the reasoning presented in the previous sections, we construct for each $k \in \mathbb{N}$ and $p \in \mathbb{R}$, fixed the positive solution of problem (4.3) in the following form

$$\overline{u}_{p,k}(x) = z_{p,k}(1 - \|x\|^{2-n}),$$

where

$$z_{p,k}(t) = \int_0^1 \mathbf{G}(s,t)(n-2)^{-2} (1-s)^{\frac{2n-2}{2-n}} M_{p,k}((1-s)^{\frac{1}{2-n}}) ds,$$

and $\mathbf{G}$ is given by (2.6). Moreover, we can notice that for all $k \in \mathbb{N}$ and $p \in \mathbb{R}$,

$$z_{p,k+1}(t) > z_{p,k}(t) \quad \text{for all } t \in (0,1). \tag{4.6}$$

Now we show the crucial fact that each $z_{p,k}$ is nonincreasing in $(\bar{t},1)$, where $\bar{t} = 1 - R^{2-n}$. Indeed, we can calculate

$$\begin{aligned} z'_{p,k}(\bar{t}) &= \frac{1}{(n-2)^2}\left[\int_{\bar{t}}^{1}(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})ds \right.\\ &\qquad \left. - \int_0^1 s\,(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})ds\right] \\ &\le \frac{1}{(n-2)^2}\left[\sup_{s\in(0,1)}[(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})](1-\bar{t}) \right.\\ &\qquad \left. - \inf_{s\in(0,1-K^{2-n})}[(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})]\frac{1}{2}\left(1-K^{2-n}\right)^2\right]. \end{aligned}$$

Since

$$\sup_{s\in(0,1)}[(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})] = \sup_{l\in[1,+\infty)}\left[l^{2n-2}M_{p,k}(l)\right]$$

and

$$\inf_{s\in(0,1-K^{2-n})}[(1-s)^{\frac{2n-2}{2-n}}\,M_{p,k}((1-s)^{\frac{1}{2-n}})] = \inf_{l\in[1,K]}\left[l^{2n-2}M_{p,k}(l)\right],$$

we obtain

$$\begin{aligned} z'_{p,k}(\bar{t}) &\le \frac{1}{(n-2)^2}\left[\sup_{l\in[1,+\infty)}\left[l^{2n-2}M_{p,k}(l)\right]R^{2-n} \right.\\ &\qquad \left. - \inf_{l\in[1,K]}\left[l^{2n-2}M_{p,k}(l)\right]\frac{1}{2}\left(1-K^{2-n}\right)^2\right] \\ &\le \frac{1}{2(n-2)^2}\left(1-K^{2-n}\right)^2\left[\sup_{l\in[1,+\infty)}\left[l^{2n-2}M_{p,k}(l)\right]\frac{1}{K^2} - \inf_{l\in[1,K]}\left[l^{2n-2}M_{p,k}(l)\right]\right] \\ &= \frac{1}{2(n-2)^2}\left(1-K^{2-n}\right)^2(n-2)\,d\,(4-\varepsilon)\left[\frac{1}{K^2}-1\right] \le 0. \end{aligned}$$

Taking into account the inequality $z''_{p,k} < 0$ in $(0,1)$ we get that for all $t \in \left(\bar{t},1\right)$, $z'_{p,k}(t) < 0$. As in the finite case, we obtain the estimates for all $x \in \mathbb{R}^n$ such that $\|x\| \ge R$,

$$-4d(n-2) \le x\cdot\nabla\overline{u}_{p,k}(x) = z'_{p,k}(1-\|x\|^{2-n})(n-2)\|x\|^{2-n} \le 0$$

and further $\left(M_{p,k}(\|x\|) = (n-2)d\,(4-\varepsilon)\,\|x\|^{2-2n} + \varepsilon(n-2)d\left(1 - \frac{e^p}{e^p+1}\frac{1}{k}\right)\|x\|^{-2n}\right)$

$$\begin{aligned}
&\Delta\overline{u}_{p,k}(x) + f(x,\overline{u}_{p,k}(x)) + g(x)x\cdot\nabla\overline{u}_{p,k}(x)\\
&\le \Delta\overline{u}_{p,k}(x) + f(x,\overline{u}_{p,k}(x)) + 4d(n-2)|g(x)|\\
&\le \Delta\overline{u}_{p,k}(x) + (4-a-\varepsilon)d\,(n-2)\,\|x\|^{2-2n} + d(n-2)a\|x\|^{2-2n}\\
&= \Delta\overline{u}_{p,k}(x) + (4-\varepsilon)d(n-2)\|x\|^{2-2n}\\
&\le \Delta\overline{u}_{p,k}(x) + M_{p,k}(\|x\|) = 0.
\end{aligned}$$

Finally, we state that each $\overline{u}_{p,k}$ is a positive supersolution of our problem in the same exterior domain Ω_R. □

Now we can formulate our main result.

Theorem 4.3. *Assume that* (Ag'), (Af'') *and* $(Af1)$ *hold. Then for each* $p \in \mathbb{R}$, *problem* (1.1)–(1.2) *possesses a sequence* $\{u_{p,k}\}_{k\in\mathbb{N}}$ *of minimal solutions with finite energy which is increasing on* $\partial\Omega_R$ *and nondecreasing in* Ω_R. *Each such a sequence generates another solution* u_p *of our problem in* Ω_R *which is also minimal and has finite energy. Moreover, for each* $p \in \mathbb{R}$ *and* $k \in \mathbb{N}$, *the following estimate holds*

$$u_{p,k}(x) = o\left(\phi(\|x\|)\right) \quad \text{as } \|x\| \to +\infty, \quad i = 1,2 \tag{4.7}$$

for all $\phi \in C^1(1,+\infty)$ *such that* $\lim_{r\to+\infty}\phi(r) = 0$ *and* $\lim_{r\to+\infty}\phi'(r)r^{n-1} = +\infty$.

Proof. Step 1 (*nondecreasing sequences of solutions*). By Lemma 4.2 we get the uncountable set of supersolutions $\{\overline{u}_{p,k} : p \in \mathbb{R},\ k \in \mathbb{N}\}$, where each $\overline{u}_{p,k}$ is the unique radial solution of (4.3). Fix $p \in \mathbb{R}$. Let us consider the increasing sequence $\{\overline{u}_{p,k}\}_{k\in\mathbb{N}}$ of supersolutions of our problem in Ω_R. The first task is to investigate the relations between the supersolution $\overline{u}_{p,1}$ and the subsolution $\underline{u}(x) = \alpha\|x\|^{2-n}$ with $0 < \alpha \le \frac{1}{4}(4-a-\varepsilon)d(n-2)^{-1}$ of (1.1)–(1.2). Let us recall that $\overline{u}_{p,1}$ satisfies the problem

$$\begin{cases} -\Delta v(x) = M_{p,1}(\|x\|), & \text{for } x \in \Omega_1,\\ v(x) = 0, & \text{for } \|x\| = 1,\\ \lim_{\|x\|\to\infty} v(x) = 0, \end{cases}$$

with

$$M_{p,1}(r) = \frac{4(n-2)d}{r^{2n}}\left[r^2 + \frac{n-3}{n-2}\left(1 - \frac{e^p}{e^p+1}\right)\right].$$

Now we compare $\overline{u}_{p,1}$ and $\overline{\overline{v}}$ which is the solution of (2.11) with the right-hand side containing $m = (4-a-\varepsilon)d(n-2)$ (as in the proof of the previous theorem). Since

$$M_{p,1}(r) = \frac{(n-2)d}{r^{2n}}\left[(4-\varepsilon)\,r^2 + \varepsilon\left(1 - \frac{e^p}{e^p+1}\right)\right] > \frac{(4-\varepsilon-a)d(n-2)}{r^{2n-2}},$$

$\overline{\overline{v}}(x) = 0 = \overline{u}_{p,1}(x)$ for $\|x\| = 1$ and $\lim_{\|x\|\to\infty}\overline{\overline{v}}(x) = 0 = \lim_{\|x\|\to\infty}\overline{u}_{p,1}$, the maximum principle gives $\overline{u}_{p,1}(x) \ge \overline{\overline{v}}(x)$. Thus, we get for all $x \in \mathbb{R}^n$, $\|x\| \ge R \ge 2^{\frac{1}{n-2}}$,

$$\underline{u}(x) \le \frac{1}{4}(4-a-\varepsilon)d(n-2)^{-1}\|x\|^{2-n} \le \overline{\overline{v}}(x) \le \overline{u}_{p,1}(x).$$

Now we can apply the same iterative schema as in the proof of Theorem 3.3. Bearing in mind the above inequality, the Noussair–Swanson theorem leads to the existence of a solution $u_{p,1}$ for (1.1)–(1.2) in Ω_R such that

$$\underline{u}(x) \leq u_{p,1}(x) \leq \overline{u}_{p,1}(x) \text{ in } \Omega_R \text{ and } u_{p,1}(x) = \overline{u}_{p,1}(x) \text{ in } \partial\Omega_R.$$

We treat $u_{p,1}$ as a subsolution to (1.1)–(1.2) and consider supersolution $\overline{u}_{p,2}$. Taking into account Lemma 4.2, we know that $u_{p,1}(x) \leq \overline{u}_{p,1}(x) < \overline{u}_{p,2}(x)$ in Ω_R. The Noussair and Swanson theorem leads to the existence of a second solution $u_{p,2}$ of (1.1)–(1.2) in $\overline{\Omega}_R$, such that $u_{p,1}(x) \leq u_{p,2}(x) \leq \overline{u}_{p,2}(x)$ in $\overline{\Omega}_R$ and $u_{p,2}(x) = \overline{u}_{p,2}(x)$ on $\partial\overline{\Omega}_R$. Therefore for all $x \in \partial\overline{\Omega}_R$, $u_{p,1}(x) = \overline{u}_{p,1}(x) < \overline{u}_{p,2}(x) = u_{p,2}(x)$. Thus, $u_{p,1}$ and $u_{p,2}$ are not the same. Iterating this procedure we construct the sequence $\{u_{p,k}\}_{k\in\mathbb{N}}$ which is increasing on $\partial\Omega_R$ and nondecreasing in Ω_R.

To sum up, we have proved the existence of uncountable set of nondecreasing sequences of positive evanescent solutions of (1.1)–(1.2). It is clear that each of them is minimal and has finite energy.

Step 2 (*another uncountable set of solutions*). Coming to the last part of the proof we fix $p \in \mathbb{R}$ and follow the reasoning presented, e.g., in [25] or [29]. For each $j \in \mathbb{N}$, we consider the annulus

$$\Omega_{j,R} := \left\{ x \in \mathbb{R}^n : R + \frac{1}{2j} < \|x\| < R + j \right\}.$$

Taking into account the classical estimate for solutions of elliptic PDE (see, e.g., [25, Lemma 3.2]) in bounded domains, we obtain the existence of $D > 0$ independent of k such that the following inequality holds

$$\|u_{p,k}\|_{C^{2,\alpha}(\overline{\Omega}_{j,R})} \leq D \quad \text{for all } j \geq 1.$$

Bearing in mind the compactness of the injection $C^{2,\alpha}(\overline{\Omega}_{1,R}) \to C^2(\overline{\Omega}_{1,R})$, we derive that there exists a subsequence $\{u_{p,k_l}\}_{l\in\mathbb{N}}$ of $\{u_{p,k}\}_{k\in\mathbb{N}}$ converging to a certain element $\widetilde{u}_p^1$ in the $C^2(\overline{\Omega}_{1,R})$ norm. This implies that $\widetilde{u}_p^1$ is also a solution of (1.1)–(1.2) in $\overline{\Omega}_{1,R}$. Since for the subsequence $\{u_{p,k_l}\}_{l\in\mathbb{N}}$ the above estimate also holds, there exists a subsequence of $\{u_{p,k_l}\}_{l\in\mathbb{N}}$ which tends in the $C^2(\overline{\Omega}_{2,R})$-norm to $\widetilde{u}_p^2$ being a solution of (1.1)–(1.2) in $\overline{\Omega}_{2,R}$ and such that for all $x \in \overline{\Omega}_{1,R}$ we have $\widetilde{u}_p^2 = \widetilde{u}_p^1$. This schema can be iterated, which allows us to construct inductively a sequence $\{\widetilde{u}_p^j\}_{j\in\mathbb{N}}$ of solutions of (1.1)–(1.2) in $\overline{\Omega}_{j,R}$ with the property

$$\widetilde{u}_p^{j+1} = \widetilde{u}_p^j \quad \text{in } \overline{\Omega}_{j,R}. \tag{4.8}$$

Now we define the function u^p given in the following way

$$u^p := \widetilde{u}_p^j \text{ in } \Omega_{j,R} \quad \text{for all } j \geq 1.$$

Equality (4.8) guarantees that u^p is well-defined. Bearing in mind the construction of each $\widetilde{u}_p^j$ we can state that u^p satisfies (1.1)–(1.2). Our task is now to show that

u^p is sufficiently smooth. Let us consider an arbitrary bounded set $\overline{M} \subset \Omega_R$ and $j \in \mathbb{N}$ such that $\overline{M} \subset \Omega_{j,R}$. The above reasoning applied for $\Omega_{j,R}$ allows us to derive that $u^p \in C^2(\overline{M})$ and further, the regularity arguments associated with the Schauder estimates imply that $u^p \in C^{2,\alpha}(\overline{M})$. Finally, $u^p \in C^{2,\alpha}_{loc}(\Omega_R)$ and satisfies (1.1)–(1.2). Thus, the set $\{u^p, p \in \mathbb{R}\}$ is uncountable and contains solutions of (1.1)–(1.2). The properties of $u_{p,k}$ lead to the conclusion that u^p is also minimal and has finite energy. □

5. EXAMPLES AND ADDITIONAL REMARKS

Let us consider the following problem

$$\begin{cases} \Delta u(x) + \left[cu + A(x)ue^{B(x)u}\right] \|x\|^{-l} - b\|x\|^{-k} \\ +\dfrac{1}{4}\dfrac{2-\|x\|}{\|x\|^6 + 1 + (\cos x)^2} x \cdot \nabla u(x) = 0, & \text{for } x \in \Omega_R, \\ \lim_{\|x\|\to\infty} u(x) = 0, \end{cases} \tag{5.1}$$

where $n = 3$, $l \geq 4$, $k > l+1$, $R \geq 32$, $b, c > 0$, $\Omega_R = \left\{x \in \mathbb{R}^3, \|x\| > R\right\}$, $A, B : \Omega_1 \to \mathbb{R}$ are positive, locally Hölder continuous functions with the following properties

$$\overline{A} := \sup_{x\in\Omega_R} A(x), \quad \overline{B} := \sup_{x\in\Omega_R} B(x)$$

and

$$\left(c + \overline{A}e^{\overline{B}}\right) < 2.$$

Here we have

$$g(x) := \frac{1}{4}\frac{2-\|x\|}{\|x\|^6 + 1 + (\cos x)^2}$$

and

$$f(x,u) := \left[cu + A(x)ue^{B(x)u}\right] \|x\|^{-l} - b\|x\|^{-k}.$$

We see that for all $\|x\| > 2$, $g(x) < 0$. It is easy to check that for $x \in \Omega_R$,

$$|g(x)| = \frac{1}{4}\frac{\|x\|-2}{\|x\|^6 + 1 + (\cos x_1)^2} \leq \frac{1}{4}\|x\|^{-5}$$

which guarantees that (Ag') holds with $l_0 = 2$ and $a = 1$. It is obvious that f satisfies $(Af1)$. Now we show that f satisfies also (Af'') for $d = 1$ and $\varepsilon = 1$. To this end we note that for all $x \in \mathbb{R}^n$, such that $\|x\| > 1$ and $u \in [0,1]$, we have

$$\sup_{u\in[0,d]} \sup_{\|x\|=r} f(x,u) \leq \left[c + \overline{A}e^{\overline{B}}\right] r^{-l} \leq \left(c + \overline{A}e^{\overline{B}}\right) r^{-4} \leq (4-1-1)r^{-3-1}.$$

Finally, (Af'') holds. Moreover, we derive that $\underline{u}(x) = \alpha\|x\|^{-1}$ with $0 < \alpha \leq \frac{1}{2}$ is a positive subsolution of (5.1). To sum up, all the assumptions are satisfied. Thus, Theorem 4.3 leads to the conclusion that there exists an uncountable set of nondecreasing sequences of positive minimal solutions of (5.1) with finite energy in a neighborhood of infinity. Moreover, the sequences are increasing on $\partial\Omega_R$.

Remark 5.1. It is worth emphasizing that $f(x,0) = -\frac{b}{\|x\|^k} < 0$. Moreover, our approach allows us to consider also equations of Lane–Emden–Fowler type and many other nonlinearities which are superlinear with respect to the second variable. On the other hand, we have to admit that this method works when the function $x \mapsto f(x,u)$ decreases sufficiently fast as $\|x\| \to \infty$ for each u fixed. Such conditions appear in many papers, see, e.g., [31] and references therein.

Remark 5.2. Let us note that in the case of large n assumption (4.2) is not very strong. Taking into account the properties of the function

$$K \to j_n(K) := \left[\frac{1}{2}\left(1 - K^{2-n}\right)^2 \frac{1}{K^2}\right]^{-\frac{1}{n-2}}$$

we get

$$\lim_{n\to\infty}\left[\frac{1}{2}\left(1 - K^{2-n}\right)^2 \frac{1}{K^2}\right]^{-\frac{1}{n-2}} = 1.$$

Thus, in case when $l_0 = 1$, we can obtain the existence of uncountable number of nondecreasing sequences in the exterior domain which is "close" to the complement of the unit ball in $\mathbb{R}^n$.

Proof. The main task is to find a number K for which (4.2) holds. We are going to choose it as a minimizer of $j_n(\cdot)$ in $(1,+\infty)$. To this effect, for given $n \in \mathbb{N}$, $n \geq 3$, we calculate the derivative

$$\frac{d}{dK} j_n(K) = \frac{K^{2n} + K^4 n - K^4 - K^2 K^n n}{\left(\frac{1}{2}\right)^{\frac{n-1}{n-2}} K^{2n+3}\left(\frac{1}{K^2}\left(K^{2-n} - 1\right)^2\right)^{\frac{n-1}{n-2}} (n-2)}.$$

We state that $K_n = (n-1)^{\frac{1}{n-2}} \in (1,+\infty)$ is a stationary point of the function $j_n(\cdot)$ and minimizes $j_n(\cdot)$ in $(1,+\infty)$. We have to admit that for n small, we can only consider the exterior domain $\Omega_R := \{x \in \mathbb{R}^n, \|x\| > R\}$ with large R (e.g. for $n = 3$ we have $R \geq 32$). However, the final conclusion is that the bigger n the smaller radius R can be considered, namely when n increases then R gets closer and closer to 1 (e.g. for $n = 6$, $R \geq 1.626$; for $n = 10$, $R \geq 1.203$; for $n = 20$, $R \geq 1.065$). □

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REFERENCES

[1] K. Alhumaizi, *Flux-limiting solution techniques for simulation of reaction-diffusion--convection system*, Commun. Nonlinear Sci. Numer. Simul. **12** (2007), 953–965.

[2] A. Castro, L. Sankar, R. Shivaji, *Uniqueness of nonnegative solutions for semipositone problems on exterior domains*, J. Math. Anal. Appl. **394** (2012), 432–437.

[3] I.P. Clement, G. Sweers, *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, Ann. Sc. Norm. Sup. Pisa (4) **14** (1987), no. 1, 97–121.

[4] A. Constantin, *Existence of positive solutions of quasilinear elliptic equations*, Bull. Austral. Math. Soc. **54** (1996), 147–154.

[5] A. Constantin, *Positive solutions of quasilinear elliptic equations*, J. Math. Anal. Appl. **213** (1997), 334–339.

[6] A. Constantin, *On the existence of positive solutions of second order differential equations*, Ann. Mat. Pura Appl. **184** (2005), 131–138.

[7] J. Deng, *Bounded positive solutions of semilinear elliptic equations*, J. Math. Anal. Appl. **336** (2007), 1395–1405.

[8] J. Deng, *Existence of bounded positive solutions of semilinear elliptic equations*, Nonlinear Anal. **68** (2008), 3697–3706.

[9] S. Djebali, T. Moussaoui, O.G. Mustafa, *Positive evanescent solutions of nonlinear elliptic equations*, J. Math. Anal. Appl. **333** (2007), 863–870.

[10] S. Djebali, A. Orpel, *A note on positive evanescent solutions for a certain class of elliptic problems*, J. Math. Anal. Appl. **353** (2009), 215–223.

[11] S. Djebali, A. Orpel, *Continuous dependence on parameters of solutions for a class of elliptic problems on exterior domains*, Nonlinear Anal. **73** (2010), 660–672.

[12] M. Ehrnström, *Positive solutions for second-order nonlinear differential equation*, Nonlinear Anal. **64** (2006), 1608–1620.

[13] M. Ehrnström, *On radial solutions of certain semi-linear elliptic equations*, Nonlinear Anal. **64** (2006), 1578–1586.

[14] M. Ehrnström, *Existence of global solutions of semilinear elliptic equations*, Nonlinear Anal. **71** (2009), 1606–1610.

[15] M. Ehrnström, O.G. Mustafa, *On positive solutions of a class of nonlinear elliptic equations*, Nonlinear Anal. **67** (2007), 1147–1154.

[16] R.E. Ewing (ed.), *The Mathematics of Reservoir Simulation*, vol. 1 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1983.

[17] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin–Heidelberg, New York–Tokyo, 1983.

[18] P.V. Gordon, E. Ko, R. Shivaji, *Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion*, Nonlinear Anal. Real World Appl. **15** (2014), 51–57.

[19] P. Habets, F. Zanolin, *Upper an lower solutions for a generalized Emden–Fowler equation*, J. Math. Anal. Appl. **181** (1994), 684–700.

[20] C. Hirsch, *Numerical Computation of Internal and External Flows*, vol. 1, Wiley, Chichester, 1988.

[21] M. Jakob, *Heat Transfer*, Wiley, New York, 1959.

[22] E. Ko, E. Lee, R. Shivaji, *Multiplicity results for classes of singular problems on an exterior domain*, Discrete Contin. Dyn. Syst. **33** (2013), 5153–5166.

[23] E. Ko, M. Ramaswamy, R. Shivaji, *Uniqueness of positive radial solutions for a class of semipositone problems on the exterior of a ball*, J. Math. Anal. Appl. **423** (2015), 399–409.

[24] H.O. Kreiss, J. Lorenz, *Initial-boundary Value Problems and the Navier–Stokes Equations*, vol. 47 of Classics in Appl. Math. SIAM, Philadelphia, PA, 2004, Reprint of the 1989 edition.

[25] E.S. Noussair, C.A. Swanson, *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal. Appl. **75** (1980), 121–133.

[26] E.S. Noussair, C.A. Swanson, *Asymptotics for semilinear elliptic systems*, Canad. Math. Bull. **34** (1991), no. 4, 514–519.

[27] E.S. Noussair, C.A. Swanson, *Multiple finite energy solutions of critical semilinear field equations*, J. Math. Anal. Appl. **195** (1995), 278–293.

[28] A. Orpel, *Increasing sequences of positive evanescent solutions of nonlinear elliptic equations*, J. Differential Equations **259** (2015), no. 5, 1743–1756.

[29] A. Orpel, *Uncountable sets of finite energy solutions for semilinear elliptic problems in exterior domains*, J. Math. Anal. Appl. **472** (2019), 1175–1188.

[30] A. Rap, L. Elliott, D.B. Ingham, D. Lesnic, X. Wen, *The inverse source problem for the variable coefficients convection-diffusion equation*, Inverse Probl. Sci. Eng. **15** (2007), 413–440.

[31] L. Sankar, S. Sasi, R. Shivaji, *Semipositone problems with falling zeros on exterior domains*, J. Math. Anal. Appl. **401** (2013), 146–153.

[32] J. Serrin, *Nonlinear equations of second order*, [in:] A.M.S. Sympos. Partial Differential Equations, Berkeley, August 1971.

[33] Z. Yin, *Monotone positive solutions of second-order nonlinear differential equations*, Nonlinear Anal. **54** (2003), 391–403.

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