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NOTES ON APPLICATIONS OF THE DUAL FOUNTAIN THEOREM TO LOCAL AND NONLOCAL ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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Abstract. Using the Dual Fountain Theorem we obtain some existence of infinitely many solutions for local and nonlocal elliptic equations with variable exponent. Our results correct some of the errors that have appeared recently in the literature.

Keywords: dual fountain theorem, $p(\cdot)$-Laplacian, fractional $p(\cdot)$-Laplacian, infinitely many solutions.

Mathematics Subject Classification: 35J60, 35D30, 35J20.

1. INTRODUCTION

The nonlinear problems driven by variable exponent operators appear in numerous physical models such as, for example, the model of image restoration [3] or the model of motion of electrorheological fluids [16]. This is a very active field recently (see for example [1, 4, 8, 10, 13, 15, 17]). In the monograph [14], Rădulescu and Repovš provided a thorough introduction to the theory of nonlinear partial differential equations with a variable exponent.

In this paper, we discuss the existence of a sequence of solutions for two types of problems: a local one driven by $p(\cdot)$-Laplace type operator with a variable exponent, and a nonlocal one, driven by fractional $p(\cdot,\cdot)$-Laplace type operator with a variable exponent. These problems were considered in [9] and [12] respectively, but the authors incorrectly applied the Dual Fountain Theorem in their proofs. The aim of this note is to give correct arguments.

In paper [9], the authors discussed the existence and multiplicity of weak solutions for a general class of local quasilinear problems involving $p(\cdot)$-Laplace type operators, with Dirichlet boundary conditions involving variable exponents

$$\begin{cases} -\operatorname{div}(a\left(|\nabla u|^{p(x)}\right)|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_1$}$$

where $\lambda > 0$ is a real parameter, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, Ω is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\bar{\Omega})$, and $1 < p(x) < N$, for all $x \in \bar{\Omega}$. The function $a : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following hypotheses:

(A) (i) the function $a : \mathbb{R}_+ \to \mathbb{R}$ is continuous and the mapping $Y : \mathbb{R}^N \to \mathbb{R}$, given by $Y(\eta) := A(|\eta|^{p(x)})$ is strictly convex for all $x \in \Omega$, where A is the primitive of a, that is $A(t) = \int_0^t a(s)ds$,

(ii) there are constants $\alpha, \beta > 0$, $0 < \alpha \leq \beta$ such that $\alpha \leq a(s) \leq \beta$ for all $s \geq 0$,

(iii) there exists $\varpi > 1$ such that $(0, \infty) \ni t \longmapsto \frac{1}{p(x)} A(t) - \frac{1}{\varpi} a(t)t$ is nondecreasing for all $x \in \Omega$.

An important case is given by $a(t) \equiv 1$. In this case, we have the $p(x)$-Laplacian operator

$$\Delta_{p(\cdot)}(u) = \nabla \cdot \left(|\nabla u|^{p(\cdot)-2} \nabla u \right)$$

and $\varpi = p^+$, where $p^+ = \max_{x \in \bar{\Omega}} p(x)$.

In paper [12], the authors discussed the existence results of solutions to the nonlocal elliptic problem involving the fractional $p(\cdot,\cdot)$-Laplacian

$$\begin{cases} -\mathcal{L}_K u + |u|^{p(x)-2} u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P$_2$}$$

where $\lambda > 0$ is a real parameter, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\mathcal{L}_K$ is a nonlocal operator defined pointwise as

$$\mathcal{L}_K u(x) = 2 \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x, y) dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, \infty)$ is a kernel function with the following properties:

(K) (i) $mK \in L^1 \left(\mathbb{R}^N \times \mathbb{R}^N \right)$, where $m(x, y) = \min\{1, |x - y|^{p(x,y)}\}$;

(ii) there exist $\theta_0 > 0$ and $0 < s < 1$ such that $K(x, y) |x - y|^{N+sp(x,y)} \geq \theta_0$ for almost all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $x \neq y$;

(iii) $K(x, y) = K(y, x)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$,

where $p \in C\left(\mathbb{R}^N \times \mathbb{R}^N\right)$ satisfies $p(x, y) = p(y, x)$ for all $x, y \in \mathbb{R}^N$,

$$1 < \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y) \leq \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y) < \frac{N}{s}$$

and $p(x)$ is the abbreviation for $p(x, x)$ for all $x \in \mathbb{R}^N$. An important case is given by $K(x, y) = |x - y|^{-(N+sp(x,y))}$. In this case, we have the fractional $p(\cdot,\cdot)$-Laplacian operator

$$(-\Delta)^s_{p(\cdot,\cdot)} u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy \quad \text{for all } x \in \mathbb{R}^N.$$

For both problems we introduce the hypotheses on the nonlinearity f:

(f_1) f is a Carathéodory function and there exists $c > 0$ such that

$$|f(x,t)| \leq c\left(1 + |t|^{\nu(x)-1}\right)$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $\nu \in C(\bar{\Omega})$, $1 < p^+ < \nu^- \leq \nu(x) \leq \nu^+ < p^*(x)$ for $x \in \bar{\Omega}$, and $p^*(x)$ denotes the critical variable exponent related to $p(x)$, which is defined for all $x \in \bar{\Omega}$ by the pointwise relation $p^*(x) = \frac{Np(x)}{N-sp(x)}$, where $s = 1$ for problem (P_1), $p^- = \min_{x\in\bar{\Omega}} p(x)$, $p^+ = \max_{x\in\bar{\Omega}} p(x)$.

(f_2) $\lim_{t\to+\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$ uniformly for almost all $x \in \Omega$, where the function F is the primitive of f with respect to the second variable, that is, $F(x,t) := \int_0^t f(x,s)ds$, $x \in \Omega$.

(f_3) There exists $\eta > 0$ such that

$$\mathcal{F}(x,t) \leq \mathcal{F}(x,s) + \eta$$

for any $x \in \Omega$ and all $0 \leq t \leq s$ or $s \leq t \leq 0$, where $\mathcal{F}(x,t) = f(x,t)t - \omega F(x,t)$ and $\omega = p^+$ for problem (P_1), and $\omega = \varpi$ for problem (P_2), where ϖ occurs in (A)(iii).

(f_4) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and all $t \in \mathbb{R}$.

(f_5) $\liminf\limits_{|t|\to 0} \frac{F(x,t)}{|t|^{p^-}} = +\infty$ uniformly for almost all $x \in \Omega$.

We can now formulate our main results.

Theorem 1.1. *Assume that (A), (f_1)–(f_5) hold. Then, for each $\lambda > 0$, the problem (P_1) possesses infinitely many small negative energy solutions.*

Theorem 1.2. *Assume that (K), (f_1)–(f_5) hold. Then, for each $\lambda > 0$, the problem (P_2) possesses infinitely many small negative energy solutions.*

As is shown in [18, p. 18], the papers [9] and [12], which discuss problems (P_1) and (P_2), respectively, contain incorrect reasonings, which use the Dual Fountain Theorem (see Theorem 2.4 below). The authors assume only (f_1)–(f_4) of the nonlinearity f and try to find some unbounded sequences $\{\delta_k\}, \{\gamma_k\}$ for which (B_1)–(B_4) are satisfied. But showing $(B_1), (B_2)$ is inconsistent and showing (B_3) is non conclusive. In our approach to this problems we find null sequences $\{\delta_k\}, \{\gamma_k\}$ for which (B_1)–(B_4) are satisfied. To do this, we additionally impose a further constraint on the nonlinearity f, i.e. (f_5), that controls its behaviour for small t. Let us note that we also widen the range of λ to any positive number. Since we used the same list of assumptions for nonlinearity f in both (P_1) and (P_2) problems, we are able to prove both theorems simultaneously.

2. MATHEMATICAL BACKGROUND

In this section we define solution spaces for problems (P_1) and (P_2). Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$ and $p \in C(\bar{\Omega})$ with $1 \leq p(x) < N$. The variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(\Omega)$, is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the modular $\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$ is finite, that is,

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int\limits_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

If we endow this space with the so-called Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \int\limits_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}$$

it becomes a separable Banach space (see [5, Theorems 3.2.7 and 3.4.4]). If $1 < p(x)$ for $x \in \Omega$, then $L^{p(\cdot)}(\Omega)$ is reflexive (see [5, Theorem 3.4.7]). Note that, if p is a constant function, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ and the space $L^{p(\cdot)}(\Omega)$ coincide with the standard norm $\|\cdot\|_p$ and the standard Lebesgue space $L^p(\Omega)$, respectively. We will need the following relation between the modular $\varrho_{p(\cdot)}$ and the norm $\|\cdot\|_{p(\cdot)}$ (see [9, Proposition 2.2]).

Lemma 2.1. *Let $u \in L^{p(\cdot)}(\Omega)$, then, we have*

(a) *if $\|u\|_{p(\cdot)} < 1$, then $\|u\|_{p(\cdot)}^{p^+} \leq \varrho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$,*
(b) *if $\|u\|_{p(\cdot)} > 1$, then $\|u\|_{p(\cdot)}^{p^-} \leq \varrho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$.*

Next, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

and it is equipped with the norm

$$\|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)} + \|u\|_{p(\cdot)}$$

for all $u \in W^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Since $\|u\|_{p(\cdot)} \leq C \||\nabla u|\|_{p(\cdot)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and some constant $C > 0$, $\|u\|_{1,p(\cdot)}$ and $\||\nabla u|\|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. For a solution space for problem (P_1) we use

$$E_1 := W_0^{1,p(\cdot)}(\Omega)$$

equipped with the norm $\|u\|_{E_1} := \||\nabla u|\|_{p(\cdot)}$ for all $u \in E_1$. E_1 is a reflexive and separable Banach space (see [9, Proposition 2.4]).

Under the assumption (A), the functional $\mathcal{A}_1 : E_1 \to \mathbb{R}$ defined by

$$\mathcal{A}_1(u) := \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx \quad \text{for all } u \in E_1$$

is of class $C^1(E_1, \mathbb{R})$ with the derivative given by

$$\langle \mathcal{A}_1'(u), v\rangle = \int_\Omega a(|\nabla u|^{p(x)})\, |\nabla u|^{p(x)-2}\, \nabla u \cdot \nabla v \quad \text{for all } u, v \in E_1$$

(see [9, Lemma 2.5]). Moreover, under the assumption (f_1), the functional $\mathcal{B}_1 : E_1 \to \mathbb{R}$ defined by

$$\mathcal{B}_1(u) := \int_\Omega F(x, u)dx \quad \text{for all } u \in E_1$$

is of class $C^1(E_1, \mathbb{R})$ with the derivative given by $\langle \mathcal{B}_1'(u), v\rangle = \int_\Omega f(x, u)v$ for all $u, v \in E_1$. Hence, if we define $\mathcal{E}_1 : E_1 \to \mathbb{R}$ by

$$\mathcal{E}_{1,\lambda} := \mathcal{A}_1 - \lambda \mathcal{B}_1,$$

then $\mathcal{E}_{1,\lambda} \in C^1(E_1, \mathbb{R})$ and its critical points are the so-called weak solutions to the problem (P_1), i.e. $u \in E_1$ such that

$$\int_\Omega a(|\nabla u|^{p(x)})\, |\nabla u|^{p(x)-2}\, \nabla u \cdot \nabla v - \lambda \int_\Omega f(x, u)v = 0 \quad \text{for all } v \in E_1.$$

Now, we define a solution space for problem (P_2). Let us denote with $W_K^{s,p(\cdot,\cdot)}(\Omega)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left\|u_{|\Omega}\right\|_{p(\cdot)} + [u]_{p(\cdot,\cdot)}$$

where

$$[u]_{p(\cdot,\cdot)} = \inf \left\{ \mu > 0 : \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \frac{u(x) - u(y)}{\mu} \right|^{p(x,y)} K(x, y)dxdy \le 1 \right\}.$$

For a solution space for problem (P_2) we use

$$E_2 := \left\{ u \in W_K^{s,p(\cdot,\cdot)}(\Omega) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

with the norm $\|\cdot\|_{E_2} = \|\cdot\|$. E_2 is a reflexive and separable Banach space (see [2, Lemma 8 and Remark 6]).

Let us define the functionals $\mathcal{A}_2, \mathcal{B}_2 : E_2 \to \mathbb{R}$ by

$$\mathcal{A}_2(u) := \int\limits_{\mathbb{R}^N}\int\limits_{\times\mathbb{R}^N} \frac{1}{p(x,y)} |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + \int\limits_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx$$

and

$$\mathcal{B}_2(u) := \int\limits_{\Omega} F(x,u)dx$$

for all $u \in E_2$.

Standard arguments imply that $\mathcal{A}_2, \mathcal{B}_2 \in C^1(E_2, \mathbb{R})$ and their Fréchet derivatives are given by

$$\begin{aligned}&\langle \mathcal{A}_2'(u), v\rangle \\ &= \int\limits_{\mathbb{R}^N}\int\limits_{\times\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2} (u(x)-u(y))(v(x)-v(y)) K(x,y)dxdy \\ &\quad + \int\limits_{\Omega} |u|^{p(x)-2} uvdx\end{aligned}$$

and $\langle \mathcal{B}_2'(u), v\rangle = \int_\Omega f(x,u)v$ for all $u, v \in E_2$. If we define $\mathcal{E}_2 : E_2 \to \mathbb{R}$ by

$$\mathcal{E}_{2,\lambda} := \mathcal{A}_2 - \lambda\mathcal{B}_2,$$

then $\mathcal{E}_{2,\lambda} \in C^1(E_2, \mathbb{R})$ and its critical points are the so-called weak solutions to the problem (P_2), i.e. $u \in E_2$ such that

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\int\limits_{\times\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2} (u(x)-u(y))(v(x)-v(y)) K(x,y)dxdy \\ &+ \int\limits_{\Omega} |u|^{p(x)-2} uvdx - \lambda \int\limits_{\Omega} f(x,u)v = 0\end{aligned}$$

for all $v \in E_2$.

Now we formulate some facts we will need.

Lemma 2.2. *Let $\tau \in C(\bar{\Omega})$ with $1 \le \tau(x) < p^*(x)$. Then the embedding $E_i \hookrightarrow L^{\tau(\cdot)}(\Omega)$, $i = 1, 2$, is continuous and compact.*

Proof. By [9, Proposition 2.4] for $i = 1$, and [2, Theorem 2] for $i = 2$, the embedding $E_i \hookrightarrow L^{\max\{\tau(\cdot),p^-\}}(\Omega)$ is continuous and compact. By [11, Theorem 2.2], the embedding $L^{\max\{\tau(\cdot),p^-\}}(\Omega) \hookrightarrow L^{\tau(\cdot)}(\Omega)$ is continuous. Thus Lemma follows by superposition $E_i \hookrightarrow L^{\max\{\tau(\cdot),p^-\}}(\Omega) \hookrightarrow L^{\tau(\cdot)}(\Omega)$. $\square$

Lemma 2.3. *For $i = 1, 2$, we have:*

(a) *if* $\|u\|_{E_i} < 1$, *then* $\frac{\alpha_i}{p^+} \|u\|_{E_i}^{p^+} \le \mathcal{A}_i(u) \le \frac{\beta_i}{p^-} \|u\|_{E_i}^{p^-}$,

(b) *if* $\|u\|_{E_i} > 1$, *then* $\frac{\alpha_i}{p^-} \|u\|_{E_i}^{p^-} \le \mathcal{A}_i(u) \le \frac{\beta_i}{p^+} \|u\|_{E_i}^{p^+}$,

where $\alpha_1 = \alpha$, $\alpha_2 = 1$, $\beta_1 = \beta$ *and* $\beta_2 = 1$ *and* α, β *are constants from* (*A*)(ii).

Proof. It follows from (A)(ii) and [9, Proposition 2.2] for the functional $\mathcal{A}_1$ and from [12, Lemma 2.3] for the functional $\mathcal{A}_2$. □

Finally, we recall the Dual Fountain Theorem. Let X be a reflexive and separable Banach space. By [7, Theorem 1.22], there exist sequences $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

$$X = \overline{\text{span}}\{e_n, n \in \mathbb{N}\}, \quad X^* = \overline{\text{span}}^{w^*}\{e_n^*, n \in \mathbb{N}\}$$

and $\langle e_i^*, e_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Write

$$X_n = \text{span}\{e_n\}, \quad Y_k = \bigoplus_{n=1}^{k} X_n, \quad Z_k = \overline{\bigoplus_{n=k}^{\infty} X_n}. \tag{2.1}$$

We say $J \in C^1(X, \mathbb{R})$ satisfies $(C)_c^*$-condition at level $c \in \mathbb{R}$ if any sequence $\{u^{(n)}\}_{n \in \mathbb{N}} \subset X$ satisfying

$$u^{(n)} \in Y_n, \quad J(u^{(n)}) \to c, \quad \left(1 + \left\|u^{(n)}\right\|\right) \left\|(J|_{Y_n})'(u^{(n)})\right\|_{Y_n^*} \to 0,$$

where Y_m are subspaces defined in (2.1), contains a subsequence converging to a critical point of J.

We are ready to recall the Dual Fountain Theorem [19, Theorem 3.18] (see also [9, Theorem 3.11]).

Theorem 2.4. *Let X be a reflexive and separable Banach space, $J \in C^1(X, \mathbb{R})$ an even functional, and Y_k, Z_k the subspaces defined in* (2.1)*. Assume that there is $k_0 > 0$ such that for each $k \ge k_0$, there exist $\delta_k > \gamma_k > 0$ such that*

(B_1) $a_k = \inf\{J(u) : u \in Z_k, ||u|| = \delta_k\} \ge 0$,
(B_2) $b_k = \max\{J(u) : u \in Y_k, ||u|| = \gamma_k\} < 0$,
(B_3) $d_k = \inf\{J(u) : u \in Z_k, ||u|| \le \delta_k\} \to 0$ *as* $k \to +\infty$,
(B_4) *J satisfies $(C)_c^*$-condition for every $c \in [d_{k_0}, 0)$.*

Then J has a sequence of negative critical values converging to 0.

3. PROOFS OF THE MAIN RESULTS

Since E_i, $i = 1, 2$, are reflexive and separable Banach spaces, we can obtain $Y_{k,i}, Z_{k,i} \subset E_i$, $k \in \mathbb{N}$, which satisfy (2.1). We will need the following lemma (see [9, Lemma 3.7], [6, Lemma 4.9]).

Lemma 3.1. *For $\tau \in C(\bar{\Omega})$ with $1 \le \tau(x) < p^*(x)$ for all $x \in \mathbb{R}^N$, $i = 1, 2$ and $k \in \mathbb{N}$ define*

$$\beta_{k,\tau,i} = \sup\{\|u\|_{\tau(\cdot)} : \|u\|_{E_i} = 1, u \in Z_{k,i}\}. \tag{3.1}$$

Then $\lim_{k\to\infty} \beta_{k,\tau,i} = 0$.

Proof. Fix $i \in \{1, 2\}$ and $\tau \in C(\bar{\Omega})$ with $1 \le \tau(x) < p^*(x)$ for all $x \in \mathbb{R}^N$. Obviously, $0 < \beta_{k+1,\tau,i} \le \beta_{k,\tau,i}$, so $\{\beta_{k,\tau,i}\}_{k\in\mathbb{N}}$ converges to some $\beta \ge 0$. For every $k \in \mathbb{N}$ choose $u_k \in Z_{k,i}$ such that $\|u_k\|_{E_i} = 1$ and $\|u_k\|_{\tau(\cdot)} > \frac{\beta_{k,\tau,i}}{2}$. As E_i is reflexive, $\{u_k\}$, up to a subsequence, weakly converges to some $u \in E_i$. Since $\langle e_n^*, u_k\rangle = 0$ if $k > n$, we have $0 = \lim_{k\to\infty} \langle e_n^*, u_k\rangle = \langle e_n^*, u\rangle$ for all $n \in \mathbb{N}$. Thus $u = 0$. By Lemma 2.2, E_i compactly embeds into $L^{\tau(\cdot)}(\Omega)$, which implies $\|u_k\|_{\tau(\cdot)} \to 0$ and so $\beta_{k,\tau,i} \to 0$, which proves the lemma. □

Using the notations introduced in the previous section, we can provide proof of Theorems 1.1 and 1.2 simultaneously.

Proof of Theorems 1.1 and 1.2. Fix $\lambda > 0$. Since the functional $\mathcal{E}_{i,\lambda}$ is even and belongs to $C^1(E_i, \mathbb{R})$, by Theorem 2.4 it suffices to show that if k is large enough, then there exist $\delta_k > \gamma_k > 0$ such that (B_1)–(B_4) hold.

Verification of (B_1). By (f_1), there exists $C > 0$ such that

$$F(x,t) \le C\left(|t| + |t|^{\nu(x)}\right) \text{ for all } x \in \Omega \text{and } t \in \mathbb{R}. \tag{3.2}$$

Then, by Lemma 2.3 and (3.1), for any $u \in Z_{k,i}$ with $\|u\|_{E_1} < 1$, we have

$$\begin{aligned}
\mathcal{E}_{i,\lambda}(u) = \mathcal{A}_i(u) - \lambda\mathcal{B}_i(u) &\ge \frac{\alpha_i}{p^+}\|u\|_{E_i}^{p^+} - C\lambda\int_\Omega |u|\,dx - C\lambda\int_\Omega |u|^{\nu(x)}\,dx \\
&\ge \frac{\alpha_i}{p^+}\|u\|_{E_i}^{p^+} - C\lambda\left(\|u\|_1 + \max\{\|u\|_{\nu(\cdot)}^{\nu^+}, \|u\|_{\nu(\cdot)}^{\nu^-}\}\right) \\
&\ge \frac{\alpha_i}{p^+}\|u\|_{E_i}^{p^+} - C\lambda\left(\beta_{k,1,i}\|u\|_{E_i} + \max\{\beta_{k,\nu,i}^{\nu^+}\|u\|_{E_i}^{\nu^+}, \beta_{k,\nu,i}^{\nu^-}\|u\|_{E_i}^{\nu^-}\}\right) \\
&\ge \frac{\alpha_i}{p^+}\|u\|_{E_i}^{p^+} - C\lambda\left(\beta_{k,1,i} + \max\{\beta_{k,\nu,i}^{\nu^+}, \beta_{k,\nu,i}^{\nu^-}\}\right)\|u\|_{E_i}
\end{aligned}$$

Since $\{\beta_{k,1,i}\}, \{\beta_{k,\nu,i}\}$ are null sequences and $p^+ > 1$ we have

$$\delta_k := \left(\frac{2p^+}{\alpha_i}C\lambda\left(\beta_{k,1,i} + \max\{\beta_{k,\nu,i}^{\nu^+}, \beta_{k,\nu,i}^{\nu^-}\}\right)\right)^{\frac{1}{p^+-1}} \to 0 \quad \text{as } k \to \infty.$$

Hence, there exists $k_0 \in \mathbb{N}$ such that $\delta_k < 1$ for all $k \ge k_0$. Thus, for all $u \in Z_{k,i}$ with $\|u\|_{E_i} = \delta_k, k \ge k_0$, we have

$$\mathcal{E}_{i,\lambda}(u) \ge \frac{\alpha_i}{2p^+}\delta_k^{p^+} > 0.$$

This shows (B_1).

Verification of (B_2). First, we show that for any $M > 0$ we can find $C_M > 0$ such that

$$F(x,t) \geq M\,|t|^{p^-} - C_M\,|t|^{\nu(x)} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega. \tag{3.3}$$

By (f_5), there is $0 < \varepsilon < 1$ such that

$$F(x,t) \geq M\,|t|^{p^-} \quad \text{for all } |t| < \varepsilon \text{ and all } x \in \Omega. \tag{3.4}$$

By (f_1), there is $C > 0$ such that (3.2) hold. Hence, if $|t| \geq \varepsilon$, then

$$|t| \leq \varepsilon^{1-\nu(x)}\,|t|^{\nu(x)} \leq \varepsilon^{1-\nu^+}\,|t|^{\nu(x)} \quad \text{and} \quad |t|^{p^-} \leq \varepsilon^{p^- - \nu^+}\,|t|^{\nu(x)}.$$

Thus

$$\begin{aligned} F(x,t) &\geq -C\left(|t| + |t|^{\nu(x)}\right) \geq -C\left(\varepsilon^{1-\nu^+} + 1\right)|t|^{\nu(x)} = -C_1\,|t|^{\nu(x)} \\ &= -C_1\,|t|^{\nu(x)} - M\,|t|^{p^-} + M\,|t|^{p^-} \\ &\geq -C_1\,|t|^{\nu(x)} - M\varepsilon^{p^- - \nu^+}\,|t|^{\nu(x)} + M\,|t|^{p^-} \\ &= -C_M\,|t|^{\nu(x)} + M\,|t|^{p^-}, \end{aligned}$$

where $C_1 = C\left(\varepsilon^{1-\nu^+} + 1\right)$ and $C_M = M\varepsilon^{p^- - \nu^+} + C_1$. This and (3.4) give us (3.3).

Since $Y_{k,i}$ is finite dimensional, all the norms are equivalent, so we can find $0 < \theta_k < 1$ such that for $u \in Y_{k,i}$ with $\|u\|_{E_i} \leq \theta_k$ we have $\|u\|_{\nu(\cdot)} < 1$, and so, by Lemma 2.1 and Lemma 2.3

$$\begin{aligned} \mathcal{E}_{i,\lambda}(u) &\leq \frac{\beta_i}{p^-}\,\|u\|_{E_i}^{p^-} - \lambda \int_\Omega F(x,u)dx \\ &\leq \frac{\beta_i}{p^-}\,\|u\|_{E_i}^{p^-} - \lambda M \int_\Omega |u(x)|^{p^-}\,dx + \lambda C_M \int_\Omega |u(x)|^{\nu(x)}\,dx \\ &\leq \frac{\beta_i}{p^-}\,\|u\|_{E_i}^{p^-} - \lambda M\,\|u\|_{p^-}^{p^-} + \lambda C_M\,\|u\|_{\nu(\cdot)}^{\nu^-} \\ &\leq \frac{\beta_i}{p^-}\,\|u\|_{E_i}^{p^-} - \lambda M\tilde{c}\,\|u\|_{E_i}^{p^-} + \lambda C_M\tilde{C}\,\|u\|_{E_i}^{\nu^-} \\ &= \left(\frac{\beta_i}{p^-} - \lambda M\tilde{c}\right)\|u\|_{E_i}^{p^-} + \lambda C_M\tilde{C}\,\|u\|_{E_i}^{\nu^-} \end{aligned}$$

where constants $\tilde{c}, \tilde{C}$ arise from the equivalence of norms. Now, choosing $M > \frac{\beta_i}{\lambda\tilde{c}p^-}$, we can find $0 < \gamma_k < \min\{\theta_k, \delta_k\}$ such that $J(u) < 0$ for all $u \in Y_{k,i}$ with $\|u\|_{E_i} = \gamma_k$. This shows (B_2).

Verification of (B_3). First, we note that $d_k < 0$ for all $k \geq k_0$, since $Y_{k,i} \cap Z_{k,i} \neq \{\emptyset\}$, $0 < \gamma_k < \delta_k$ and $J(u) < 0$ for all $u \in Y_{k,i}$ with $\|u\|_{E_i} = \gamma_k$.

Using (3.1) and (3.2), we have for any $0 \leq t \leq \delta_k < 1$ and $w \in Z_k$ with $\|w\|_{E_i} = 1$

$$\begin{aligned}
\mathcal{E}_{i,\lambda}(tw) &\geq -\lambda\mathcal{B}_i(tw) \geq -\lambda C \int_\Omega |tw|\,dx - \lambda C \int_\Omega |tw|^{\nu(x)}\,dx \\
&\geq -\delta_k\lambda C \int_\Omega |w|\,dx - \delta_k^{\nu^-}\lambda C \int_\Omega |w|^{\nu(x)}\,dx \\
&\geq -\delta_k\lambda C \|w\|_1 - \delta_k^{\nu^-}\lambda C \max\left\{\|w\|_{\nu(\cdot)}^{\nu^-}, \|w\|_{\nu(\cdot)}^{\nu^+}\right\} \\
&\geq -\delta_k\lambda C\beta_{k,1,i}\|w\|_{E_i} - \delta_k^{\nu^-}\lambda C \max\left\{\beta_{k,\nu,i}^{v^-}\|w\|_{E_i}^{\nu^-}, \beta_{k,\nu,i}^{v^+}\|w\|_{E_i}^{\nu^+}\right\} \\
&= -\delta_k\lambda C\beta_{k,1,i} - \delta_k^{\nu^-}\lambda C \max\left\{\beta_{k,\nu,i}^{v^-}, \beta_{k,\nu,i}^{v^+}\right\},
\end{aligned}$$

which gives

$$-\delta_k\lambda C\beta_{k,1,i} - \delta_k^{\nu^-}\lambda C \max\left\{\beta_{k,\nu,i}^{v^-}, \beta_{k,\nu,i}^{v^+}\right\} \leq d_k < 0.$$

As $\{\delta_k\}, \{\beta_{k,1,i}\}, \{\beta_{k,\nu,i}\}$ are null sequences, (B_3) follows.

Verification of (B_4). The proof is the same as in [9, Lemma 3.12] for $i = 1$ and in [12, Lemma 2.13] for $i = 2$. This shows (B_4), and the proof is complete. □

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