

A NOTE ON HAUSDORFF CONVERGENCE OF PSEUDOSPECTRA

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Communicated by P.A. Cojuhari

Abstract. For a bounded linear operator on a Banach space, we study approximation of the spectrum and pseudospectra in the Hausdorff distance. We give sufficient and necessary conditions in terms of pointwise convergence of appropriate spectral quantities.

Keywords: resolvent, spectrum, pseudospectrum, Hausdorff convergence.

Mathematics Subject Classification: 47A10, 47A25.

1. INTRODUCTION

Given a bounded linear operator A on a Banach space X , we denote its *spectrum* and *pseudospectra* [14], respectively, by

$$\text{spec } A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

and

$$\text{spec}_\varepsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| > \frac{1}{\varepsilon}\}, \quad \varepsilon > 0, \quad (1.1)$$

where we identify $\|B^{-1}\| := \infty > \frac{1}{\varepsilon}$ if B is not invertible, so that $\text{spec } A \subseteq \text{spec}_\varepsilon A$ for all $\varepsilon > 0$.

Fairly convenient access to the norm of the inverse is given by the so-called *lower norm*, the number

$$\nu(A) := \inf_{\|x\|=1} \|Ax\|. \quad (1.2)$$

Indeed, putting $\mu(A) := \min\{\nu(A), \nu(A^*)\}$, we have

$$\|A^{-1}\| = 1/\mu(A), \quad (1.3)$$

where A^* is the adjoint on the dual space X^* and equation (1.3) takes the form $\infty = 1/0$ if and only if A is not invertible.

One big advantage of this approach is that, in case $X = \ell^p(\mathbb{Z}^d, Y)$ with $p \in [1, \infty]$, $d \in \mathbb{N}$ and a Banach space Y , $\nu(A)$ can be approximated by the same infimum (1.2) with $x \in X$ restricted to elements with finite support of given diameter D . We can even quantify the approximation error against D , see [1] and [10] (as well as [8] for a corresponding result on the norm).

By means of (1.3), we can rewrite spectrum and pseudospectra as follows:

$$\text{spec } A = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}$$

and

$$\text{spec}_\varepsilon A = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\}, \quad \varepsilon > 0.$$

In other words, $\text{spec } A$ is the level set of the function $f : \mathbb{C} \rightarrow [0, \infty)$ with

$$f(\lambda) := \mu(A - \lambda I) \tag{1.4}$$

for the level zero, and $\text{spec}_\varepsilon A$ is the sublevel set of f for the level $\varepsilon > 0$.

For a function $g : \mathbb{C} \rightarrow [0, \infty)$ and $\varepsilon > 0$, let

$$\text{sub}_\varepsilon(g) := \{\lambda \in \mathbb{C} : g(\lambda) < \varepsilon\}$$

denote the *sublevel set of g for the level ε* .

In general, pointwise convergence $g_n \rightarrow g$ of functions $\mathbb{C} \rightarrow [0, \infty)$ need not coincide with Hausdorff convergence of their sublevel sets:

Example 1.1. Suppose we have g and g_n such that a) $g_n \rightarrow g$ as well as b) $\text{sub}_\varepsilon(g_n) \xrightarrow{H} \text{sub}_\varepsilon(g)$ hold for all $\varepsilon > 0$. Increasing $g(\lambda)$ to a certain level $\varepsilon > 0$ in a point λ , where g was continuous and below ε before, changes the state of a), while it does not affect $\text{clos}(\text{sub}_\varepsilon(g))$ and hence b).

So let us look at continuous examples from here on.

Example 1.2. For $g_n(\lambda) := \frac{|\lambda|}{n} \rightarrow 0 =: g(\lambda)$, the Hausdorff distance of the sublevel sets

$$\text{sub}_\varepsilon(g_n) = n\varepsilon\mathbb{D} \quad \text{and} \quad \text{sub}_\varepsilon(g) = \mathbb{C}$$

remains infinite, where \mathbb{D} denotes the open unit disk in \mathbb{C} .

Of course, this problem was due to the unboundedness of $\text{sub}_\varepsilon(g)$. So let us further focus on functions that go to infinity at infinity, so that all sublevel sets are bounded.

Example 1.3 (locally constant). Let $g(\lambda) := h(|\lambda|)$ and $g_n(\lambda) := h_n(|\lambda|)$ for $n \in \mathbb{N}$, where

$$h(x) := \max\{\min\{|x|, 1\}, |x| - 1\}, \quad h_1(x) := \frac{1}{4}x^2, \quad h_n := h + \frac{1}{n}(h_1 - h) \rightarrow h$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, unlike any g_n , g is locally constant in $2\mathbb{D} \setminus \mathbb{D}$. Consequently,

$$\text{sub}_1(g_n) \equiv 2\mathbb{D} \xrightarrow{H} \mathbb{D} = \text{sub}_1(g) \quad \text{but} \quad g_n \rightarrow g.$$

Example 1.4 (increasingly oscillating). Let $g_n(\lambda) := h_n(|\lambda|)$ for $n \in \mathbb{N}$, where

$$h_n(x) := \begin{cases} |\sin(n\pi x)|, & x \in [0, 1], \\ x - 1, & x > 1. \end{cases}$$

Then $h_n(x) < \varepsilon$ for all $\varepsilon > 0$ and

$$x \in \frac{1}{n}\mathbb{Z} \cap [0, 1] \xrightarrow{\text{H}} [0, 1] \quad \text{as } n \rightarrow \infty.$$

It follows that $\text{sub}_\varepsilon(g_n) \xrightarrow{\text{H}} (1 + \varepsilon)\mathbb{D}$ for all $\varepsilon > 0$, while g_n does not converge pointwise at all.

For a sequence of bounded operators A_n on X and their corresponding functions $f_n : \mathbb{C} \rightarrow [0, \infty)$ with

$$f_n(\lambda) := \mu(A_n - \lambda I), \quad n \in \mathbb{N}, \tag{1.5}$$

we show equivalence of pointwise convergence $f_n \rightarrow f$ and Hausdorff convergence of their sublevel sets, i.e. of the corresponding pseudospectra,

$$f_n \rightarrow f \iff \forall \varepsilon > 0 : \text{spec}_\varepsilon A_n \xrightarrow{\text{H}} \text{spec}_\varepsilon A.$$

This result is not surprising (and similar arguments have been used e.g. in [2] in a more specific situation) but there are some little details that deserve to be written down as this separate note.

In [5], the approximation of the lower norm of $H(b) - \lambda I$ for a (generalized) discrete Schrödinger operator $H(b)$ and $\lambda \in \mathbb{C}$, is established via successive exhaustion of the set of finite subwords of the potential $b \in \ell^\infty(\mathbb{Z})$. Together with our paper here, this yields Hausdorff approximation of the pseudospectrum of $H(b)$, also see [9].

2. LIPSCHITZ CONTINUITY AND NON-CONSTANCY OF μ

Our functions ν and μ , and hence f and f_n , have two properties that rule out effects as in Examples 1.1–1.4: Lipschitz continuity and the fact that their level sets have no interior points, i.e. μ is not constant on any open set.

The first property is straightforward, but the latter is a very nontrivial subject [3, 6, 11–13], and it actually limits the choice of our Banach space X as shown in Lemma 2.2.

Lemma 2.1. *For all bounded operators B, C on X , one has*

$$|\nu(B) - \nu(C)| \leq \|B - C\|,$$

so that also $\mu(B) = \min\{\nu(B), \nu(B^)\}$ is Lipschitz continuous with Lipschitz constant 1. The same follows for the functions f from (1.4) and f_n with $n \in \mathbb{N}$ from (1.5).*

This result is absolutely standard but we give the (short) proof, for the reader’s convenience.

Proof of Lemma 2.1. For all $x \in X$ with $\|x\| = 1$, one has

$$\|B - C\| \geq \|Bx - Cx\| \geq \|Bx\| - \|Cx\| \geq \nu(B) - \|Cx\|.$$

Now pass to the infimum in $\|Cx\|$ to get $\|B - C\| \geq \nu(B) - \nu(C)$. Finally, swap B and C . \square

It will become crucial to understand when the resolvent norm of a bounded operator cannot be constant on an open subset. This is a surprisingly rich and deep problem. As it turns out it is connected to a geometric property, the complex uniform convexity, of the underlying Banach space (see [11, Definition 2.4 (ii)]).

Lemma 2.2 (Globevnik [6], Shargorodsky *et al.* [3, 11–13]). *Let X be a Banach space which satisfies at least one of the following properties:*

- (a) $\dim(X) < \infty$,
- (b) X is complex uniform convex,
- (c) its dual X^* is complex uniform convex.

Then, for every bounded operator A on X , the resolvent norm,

$$\lambda \mapsto \|(A - \lambda I)^{-1}\| = 1/\mu(A - \lambda I),$$

cannot be locally constant on any open set in \mathbb{C} , and, consequently,

$$\forall \varepsilon > 0 : \text{clos}(\text{spec}_\varepsilon A) = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\}.$$

For example, every Hilbert space is subject to the condition (b) above, and every space $X = \ell^p(\mathbb{Z}^d, Y)$ with $p \in [1, \infty]$ and $d \in \mathbb{N}$ falls into this category as soon as Y does [4].

3. SET SEQUENCES AND HAUSDORFF CONVERGENCE

Let (S_n) be a sequence of bounded sets in \mathbb{C} and recall the following notations (see e.g. [7, §3.1.2]):

$\liminf S_n =$ the set of all limits of sequences (s_n) with $s_n \in S_n$,

$\limsup S_n =$ the set of all partial limits of sequences (s_n) with $s_n \in S_n$.

Both limiting sets are closed. Moreover, let us write $S_n \rightarrow S$ if

$$\liminf S_n = \limsup S_n = S.$$

It holds that $S_n \rightarrow S$ if and only if $\text{clos } S_n \rightarrow S$, where, as we know, automatically $S = \text{clos } S$ holds.

Here is an apparently different approach to set convergence: For $z \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, set $\text{dist}(z, S) := \inf_{s \in S} |z - s|$. The *Hausdorff distance* of two bounded sets $S, T \subseteq \mathbb{C}$ is defined via

$$d_H(S, T) := \max \left\{ \sup_{s \in S} \text{dist}(s, T), \sup_{t \in T} \text{dist}(t, S) \right\}.$$

Although d_H is a metric on the set of all compact subsets of \mathbb{C} , it is just a pseudometric on the set of all bounded subsets of \mathbb{C} : it enjoys symmetry and triangle inequality, but not definiteness. Indeed, one has $d_H(S, T) = 0$ if and only if $\text{clos } S = \text{clos } T$ since

$$d_H(S, T) = d_H(\text{clos } S, T) = d_H(S, \text{clos } T) = d_H(\text{clos } S, \text{clos } T).$$

Let us still write $S_n \xrightarrow{H} S$ if $d_H(S_n, S) \rightarrow 0$, also for merely bounded sets S_n, S , knowing that the limit S in $S_n \xrightarrow{H} S$ is not unique: one has $S_n \xrightarrow{H} S$ and $S_n \xrightarrow{H} T$ if and only if $d_H(S, T) = 0$, i.e. $\text{clos } S = \text{clos } T$.

Both notions of set convergence are connected, via the Hausdorff theorem:

$$S_n \xrightarrow{H} S \iff S_n \rightarrow \text{clos } S. \tag{3.1}$$

Lemma 3.1. *Let S_n and T_n be bounded subsets of \mathbb{C} with $S_n \rightarrow S$ and $T_n \rightarrow T$. In addition, suppose $S_n \setminus T_n \neq \emptyset$. Then:*

(a) *in general, it does not follow that*

$$S_n \setminus T_n \rightarrow S \setminus T,$$

(b) *however, it always holds that*

$$\liminf(S_n \setminus T_n) \supseteq S \setminus T.$$

Proof. (a) Consider

$$S_n := [0, 1] \rightarrow [0, 1] =: S \quad \text{and} \quad T_n := \frac{1}{n}\mathbb{Z} \cap [0, 1] \rightarrow [0, 1] =: T.$$

Then

$$S_n \setminus T_n \rightarrow [0, 1] \neq \emptyset = S \setminus T.$$

(b) Let $x \in S \setminus T$. Since $x \in S$, there is a sequence (x_n) with $x_n \in S_n$ such that $x_n \rightarrow x$. We show that $x_n \notin T_n$, eventually. Suppose $x_n \in T_n$ for infinitely many $n \in \mathbb{N}$. Then there is a strictly monotonic sequence (n_k) in \mathbb{N} with $x_{n_k} \in T_{n_k}$. But then

$$x = \lim_n x_n = \lim_k x_{n_k} \in \limsup_n T_n = T,$$

which contradicts $x \in S \setminus T$. Consequently, just finitely many elements of the sequence (x_n) can be in T_n . Replacing these by elements from $S_n \setminus T_n$ does not change the limit x . So $x \in \liminf(S_n \setminus T_n)$. □

4. EQUIVALENCE OF POINTWISE CONVERGENCE $f_n \rightarrow f$ AND HAUSDORFF CONVERGENCE OF THE PSEUDOSPECTRA

Here is our main theorem. Note that we do not require any convergence of A_n to A .

Theorem 4.1. *Let X be a Banach space with the properties from Lemma 2.2 and let A and $A_n, n \in \mathbb{N}$, be bounded linear operators on X . Then the following are equivalent for the functions and sets introduced in (1.1), (1.4) and (1.5):*

- (i) $f_n \rightarrow f$ pointwise,
- (ii) for all $\varepsilon > 0$, one has $\text{spec}_\varepsilon A_n \xrightarrow{\text{H}} \text{spec}_\varepsilon A$.

Proof. (i) \Rightarrow (ii). Assume (i) and take $\varepsilon > 0$. For $f(\lambda) < \varepsilon$, (i) implies $f_n(\lambda) < \varepsilon$ for sufficiently large n . So it follows that

$$\text{spec}_\varepsilon A \subseteq \liminf \text{spec}_\varepsilon A_n \subseteq \limsup \text{spec}_\varepsilon A_n.$$

Now let $\lambda \in \limsup \text{spec}_\varepsilon A_n$, i.e. $\lambda = \lim \lambda_{n_k}$ with $\lambda_{n_k} \in \text{spec}_\varepsilon A_{n_k}$, so that $f_{n_k}(\lambda_{n_k}) < \varepsilon$. Then

$$|f(\lambda) - f_{n_k}(\lambda_{n_k})| \leq \underbrace{|f(\lambda) - f_{n_k}(\lambda)|}_{\rightarrow 0 \text{ by (i)}} + \underbrace{|f_{n_k}(\lambda) - f_{n_k}(\lambda_{n_k})|}_{\leq |\lambda - \lambda_{n_k}| \rightarrow 0} \rightarrow 0.$$

Consequently, $f(\lambda) \leq \varepsilon$ and hence $\lambda \in \text{clos}(\text{spec}_\varepsilon A)$, by Lemma 2.2. We get

$$\text{spec}_\varepsilon A \subseteq \liminf \text{spec}_\varepsilon A_n \subseteq \limsup \text{spec}_\varepsilon A_n \subseteq \text{clos}(\text{spec}_\varepsilon A).$$

Passing to the closure everywhere in this chain of inclusions, just changes $\text{spec}_\varepsilon A$ at the very left into $\text{clos}(\text{spec}_\varepsilon A)$, and we have $\text{spec}_\varepsilon A_n \rightarrow \text{clos}(\text{spec}_\varepsilon A)$. Hence, by (3.1), we obtain (ii).

(ii) \Rightarrow (i). Take $\lambda \in \mathbb{C}$ and put $\varepsilon := f(\lambda)$.

Case 1. $\varepsilon = 0$. Take an arbitrary $\delta > 0$. Then, by (ii), $\lambda \in \text{spec}_\delta A \xleftarrow{\text{H}} \text{spec}_\delta A_n$. So, by (3.1), there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \in \text{spec}_\delta A_n = f_n^{-1}([0, \delta])$ and $\lambda_n \rightarrow \lambda$.

Case 2. $\varepsilon > 0$. Take an arbitrary $\delta \in (0, \varepsilon)$. By (ii), we have

$$S_n := \text{spec}_{\varepsilon+\delta} A_n \xrightarrow{\text{H}} \text{spec}_{\varepsilon+\delta} A =: S, \quad \text{i.e. } S_n \rightarrow \text{clos } S, \text{ by (3.1),}$$

and

$$T_n := \text{spec}_{\varepsilon-\delta} A_n \xrightarrow{\text{H}} \text{spec}_{\varepsilon-\delta} A =: T, \quad \text{i.e. } T_n \rightarrow \text{clos } T, \text{ by (3.1).}$$

By Lemma 3.1 (b),

$$\lambda \in \text{clos}(\text{spec}_{\varepsilon+\delta} A) \setminus \text{clos}(\text{spec}_{\varepsilon-\delta} A) \subseteq \liminf \left(\text{spec}_{\varepsilon+\delta} A_n \setminus \text{spec}_{\varepsilon-\delta} A_n \right),$$

in short:

$$\lambda \in f^{-1}((\varepsilon - \delta, \varepsilon + \delta]) \subseteq \liminf f_n^{-1}([\varepsilon - \delta, \varepsilon + \delta]).$$

So there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \in f_n^{-1}([\varepsilon - \delta, \varepsilon + \delta])$ and $\lambda_n \rightarrow \lambda$.

In both cases, we conclude

$$|f(\lambda) - f_n(\lambda)| \leq \underbrace{|f(\lambda) - f_n(\lambda_n)|}_{\leq \delta} + \underbrace{|f_n(\lambda_n) - f_n(\lambda)|}_{\leq |\lambda_n - \lambda| \rightarrow 0} < 2\delta$$

for all sufficiently large n , and hence $f_n(\lambda) \rightarrow f(\lambda)$ as $n \rightarrow \infty$, i.e. (i) holds. □

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. If X is a Hilbert space and the operators A and A_n , $n \in \mathbb{N}$, are normal then (i) and (ii) are also equivalent to (iii) $\text{spec } A_n \xrightarrow{H} \text{spec } A$.*

Proof. For normal operators, the ε -pseudospectrum is exactly the ε -neighborhood of the spectrum, e.g. [14]. But $B_\varepsilon(S_n) \xrightarrow{H} B_\varepsilon(S)$ for all $\varepsilon > 0$ implies $S_n \xrightarrow{H} S$. □

Remark 4.3.

- (a) The pointwise convergence $f_n \rightarrow f$ is uniform on compact subsets of \mathbb{C} . (Take an $\frac{\varepsilon}{3}$ -net for the compact set and use the uniform Lipschitz continuity of the f_n).
- (b) It is well-known [14] that $\text{spec}_\varepsilon A \subseteq r\mathbb{D}$ with $r = \|A\| + \varepsilon$. So if $(A_n)_{n \in \mathbb{N}}$ is a bounded sequence then $\text{spec}_\varepsilon B \subset r\mathbb{D}$ for all $B \in \{A, A_n : n \in \mathbb{N}\}$ with $r = \max\{\|A\|, \sup \|A_n\|\} + \varepsilon$. By (a), the convergence $f_n \rightarrow f$ is uniform on $\text{clos}(r\mathbb{D})$.

Remark 4.4. Sometimes (especially in earlier works), pseudospectra are defined in terms of non-strict inequality:

$$\text{Spec}_\varepsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\}, \quad \varepsilon > 0.$$

One benefit is to get compact pseudospectra, in which case d_H is a metric and \xrightarrow{H} has a unique limit. By Lemma 2.2, $\text{Spec}_\varepsilon A = \text{clos}(\text{spec}_\varepsilon A)$ for all $\varepsilon > 0$. But since $S_n \xrightarrow{H} S$ if and only if $\text{clos}(S_n) \xrightarrow{H} \text{clos}(S)$, one could add this further equivalent statement to Theorem 4.1:

- (iv) $\forall \varepsilon > 0 : \text{Spec}_\varepsilon A_n \xrightarrow{H} \text{Spec}_\varepsilon A$.

Acknowledgements

The authors thank Fabian Gabel and Riko Ukena from TU Hamburg for helpful comments and discussions.

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Received: October 13, 2022.

Revised: November 1, 2022.

Accepted: November 5, 2022.