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EXISTENCE OF POSITIVE RADIAL SOLUTIONS TO A p-LAPLACIAN KIRCHHOFF TYPE PROBLEM ON THE EXTERIOR OF A BALL

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Abstract. In this paper the authors study the existence of positive radial solutions to the Kirchhoff type problem involving the p-Laplacian

$$-\Big(a+b\int\limits_{\Omega_e}|\nabla u|^p dx\Big)\Delta_p u=\lambda f\left(|x|,u\right),\ x\in\Omega_e,\quad u=0 \text{ on } \partial\Omega_e,$$

where $\lambda>0$ is a parameter, $\Omega_e=\{x\in\mathbb{R}^N:|x|>r_0\}$, $r_0>0$, $N>p>1$, Δ_p is the p-Laplacian operator, and $f\in C([r_0,+\infty)\times[0,+\infty),\mathbb{R})$ is a non-decreasing function with respect to its second variable. By using the Mountain Pass Theorem, they prove the existence of positive radial solutions for small values of λ.

Keywords: Kirchhoff problem, p-Laplacian, positive radial solution, variational methods.

Mathematics Subject Classification: 35A01, 35A15, 35B38, 35D30, 35J92.

1. INTRODUCTION

The aim of this work is to prove the existence of positive radial solutions on the exterior of a ball to the Kirchhoff type problem

$$\begin{cases}-\Big(a+b\int\limits_{\Omega_e}|\nabla u|^p dx\Big)\Delta_p u=\lambda f\left(|x|,u\right), & x\in\Omega_e,\\ u(x)=0, & |x|=r_0,\\ u(x)\to 0, & |x|\to\infty,\end{cases}\tag{1.1}$$

where a and b are positive constants, $\lambda>0$ is a parameter, $\Omega_e=\{x\in\mathbb{R}^N:|x|>r_0\}$, $r_0>0$, $N>p>1$, Δ_p is the p-Laplacian operator ($\Delta_p u=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$), and $f:[r_0,+\infty)\times[0,+\infty)\to\mathbb{R}$ is continuous and is non-decreasing in its second variable.

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Note that Kirchhoff type problems are nonlinear, and as such present several interesting challenges; see, for instance, the recent work in [1, 2, 12–15, 18, 21, 24] for various issues and applications. Additional work on p-Laplacian problems can be found in [6–10, 17, 20, 23] and other related results in [16, 22].

In [7, 8, 17, 23], the equations considered are of the form

$$\Delta_p u = \lambda f(u) \quad \text{in } \Omega$$

with Dirichlet boundary conditions, where Ω is a bounded domain in $\mathbb{R}^N$. Concerning the existence of positive radial solutions to a class of p-Laplacian problems on the exterior of a ball, we mention the papers [9] for $p = 2$ and [20] for any $p > 1$. In these articles the equation is of the form $-\Delta_p u = \lambda K(|x|)f(u)$ and the authors appeal to the Mountain Pass Theorem (MPT).

Notice that our problem (1.1) can be written as

$$\begin{cases} -M\Big(\int\limits_{\Omega_e} |\nabla u|^p dx\Big)\Delta_p u = \lambda f\left(|x|, u\right), & x \in \Omega_e, \\ u(x) = 0, & |x| = r_0, \\ u(x) \to 0, & |x| \to \infty, \end{cases}$$

where $M(\zeta) = a + b\zeta$.

In the case where $p = 2$ and M is any positive function defined on $\mathbb{R}^+$ (with some additional conditions), problems of the type

$$\begin{cases} -M\Big(\int\limits_{\Omega} |\nabla u|^2 dx\Big)\Delta u = f\left(x, u\right), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω a bounded domain in $\mathbb{R}^N$, have physical motivations. For example, the Kirchhoff operator $M(\int_\Omega |\nabla u|^2 dx)\Delta u$ appears in nonlinear vibration equations such as

$$\begin{cases} u_{tt} - M\Big(\int\limits_{\Omega} |\nabla u|^2 dx\Big)\Delta u = f\left(x, u\right), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x). \end{cases}$$

Such equations generalize to higher dimensions the equation studied by Kirchhoff [19],

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0$$

as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings.

Kirchhoff type problems have been treated in many papers. For example, in [2], by using truncations and the MPT, the authors proved the existence of solutions to the problem

$$\begin{cases} -M\Big(\int\limits_{\Omega} |\nabla u|^2 dx\Big)\Delta u = f\,(x,u)\,, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. In [18], He *et al.* considered a similar problem where Ω is a bounded domain in $\mathbb{R}^3$ or is all of $\mathbb{R}^3$, and instead of $f(t,u)$, they had $f(u)+h$ with $h \geq 0$ and $h \in L^2(\Omega)$. In [24], Wang et al. also took Ω to also be a bounded and smooth domain in $\mathbb{R}^N$ and used the MPT to prove the existence of solutions to the problem

$$\begin{cases} -M\Big(\int\limits_{\Omega} |\nabla u|^p dx\Big)\Delta_p u = \lambda f(x,u) + |u|^{p^\star-2}u, & \text{in } \Omega,\\ \qquad\qquad\qquad\qquad\qquad\qquad u=0, & \text{on } \partial\Omega, \end{cases}$$

for all λ greater than some $\lambda^\star > 0$, where $p^\star = \frac{Np}{N-p}$. An important feature of that study is that M could be zero at zero. Additional recent results on Kirchhoff type problems can be found in [1, 12–15, 21].

Extending the ideas in [9, 20], instead of $\Delta_p u$, we consider a Kirchhoff type operator and generalize the term $K(|x|)f(u)$ to $f(|x|,u)$, where $f : [r_0,+\infty)\times[0,+\infty)\to\mathbb{R}$ is continuous, non-decreasing in its second variable, and satisfies:

(F1) there exist continuous functions A, $B : [r_0,+\infty)\to(0,+\infty)$ with $q > 2p-1$ and $\mu \in \left(0, \frac{N-p}{p-1}\right)$ such that

$$A(\xi)(t^q-1) \leq f(\xi,t) \leq B(\xi)(t^q+1) \quad \text{for all } (\xi,t)\in[r_0,+\infty)\times[0,+\infty)$$

where $A(\xi)$, $B(\xi) \leq \frac{1}{\xi^{N+\mu}}$ for $\xi \gg 1$,

(F2) for all $\xi \in [r_0,+\infty)$, $f(\xi,0) < 0$,

(F3) (Ambrosetti-Rabinowitz condition) There exists $\theta > 2p$ such that, for all sufficiently large t,

$$tf(\xi,t) > \theta F(\xi,t) \quad \text{for all } \xi \geq r_0,$$

where $F(\xi,t) = \int_0^t f(\xi,\sigma)d\sigma$.

Applying the change of variables $r = |x|$ and $s = \left(\frac{r}{r_0}\right)^{\frac{p-N}{p-1}}$ transforms (1.1) into the boundary-value problem (see, for example, [6])

$$\begin{cases} -\Big(a+\alpha\int\limits_0^1 |u'|^p d\sigma\Big)(\phi_p(u'))' = \lambda h(s) f(r_0 s^{\frac{p-1}{p-N}}, u(s)), & s\in(0,1),\\ u(0)=u(1)=0, \end{cases} \tag{1.2}$$

where

$$\alpha = bN\omega_N r_0^{N-p}\left(\frac{N-p}{p-1}\right)^{p-1}, \quad \phi_p(\zeta) = |\zeta|^{p-2}\zeta, \quad h(s) = \left(r_0\frac{(p-1)}{(N-p)}\right)^p s^{\frac{-p(N-1)}{N-p}},$$

and ω_N is the volume of the unit ball in $\mathbb{R}^N$.

Remark 1.1. If in (F1) we assume that $\mu \geq \frac{N-p}{p-1}$, this would imply that the functions defined by $h(s)B(r_0 s^{\frac{p-1}{p-N}})$ or $h(s)A(r_0 s^{\frac{p-1}{p-N}})$ are dominated in neighborhoods of zero by a continuous function on $[0,1]$, and in fact, we would have a simpler situation. But $\mu \in \left(0, \frac{N-p}{p-1}\right)$ implies the singularity of these functions at $s = 0$, but they would still belong to $L^1(0,1)$.

Remark 1.2. Consider the function

$$f(\xi,t) = \frac{1}{2\xi^{\frac{7}{2}}}(2t^4 - 1).$$

Then f satisfies all of the above conditions for $N = 3$, $p = 2$, $q = 4$, $\mu = \frac{1}{2}$, and $\theta = \frac{9}{2}$.

As a second example, we have the following.

Remark 1.3. Take $N = 4$, $p = 2$, and $q = 4 > 2p - 1 = 3$. We need $\mu \in (0,2)$ so choose $\mu = 1$. Let

$$A(\xi) = \frac{2 + \sin\xi}{\xi^5} \quad \text{and} \quad B(\xi) = \frac{(4 - \cos\xi)(\xi^2 + 1)}{\xi^7}.$$

Then

$$f(\xi,t) = \frac{3(\xi^2+1)}{\xi^7}\frac{(e^{t^2} - 2)(t^4 + 1)}{e^{t^2}}$$

satisfies all of the above conditions for some $\theta > 4$.

Next, we define what is meant by a solution of our problem.

Definition 1.4. We say that $u \in W_0^{1,p}(0,1)$ is a weak solution of problem (1.2) if

$$(a + \alpha\|u\|_{1,p}^p)\int_0^1 |u'(s)|^{p-2}u'(s)v'(s)ds = \lambda\int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u(s))v(s)ds$$

for all $v \in W_0^{1,p}(0,1)$.

We will establish the following theorem, which is our main result in this paper.

Theorem 1.5. *Assume that* (F1)–(F3) *hold. Then* (1.2) *admits a positive weak solution for* $\lambda \approx 0$.

2. PRELIMINARIES

In order to apply variational techniques such as the MPT, we extend the function f to $[r_0,+\infty)\times\mathbb{R}$ by setting $f(\xi,t)=f(\xi,0)$ for $(\xi,t)\in[r_0,+\infty)\times(-\infty,0)$. We also need the Banach spaces $W_0^{1,p}(0,1)$, $C[0,1]$, and $L^r(0,1)$ equipped their respective norms $\|\cdot\|_{1,p}$, $\|\cdot\|_\infty$, and $\|\cdot\|_r$. We recall that $W_0^{1,p}(0,1)$ is compactly embedded in $C[0,1]$, and this implies that $\|u\|_\infty\le k\|u\|_{1,p}$ for every u in $W_0^{1,p}(0,1)$, where k is a fixed positive constant (see [5]).

Remark 2.1. Let

$$D=\{(\xi,t)\in[r_0,+\infty)\times\mathbb{R}: f(\xi,t)\ge 0\}$$

and

$$D^c=\{(\xi,t)\in[r_0,+\infty)\times\mathbb{R}: f(\xi,t)< 0\}.$$

On D, we have

$$|f(\xi,t)|=f(\xi,t)\le B(\xi)(t^q+1),$$

and on D^c,

$$|f(\xi,t)|\le A(\xi).$$

Hence, for all $(\xi,t)\in[r_0,+\infty)\times\mathbb{R}$,

$$|f(\xi,t)|\le \max(A(\xi),B(\xi))(|t|^q+1),$$

and for every compact interval $I\subset\mathbb{R}$, there exists a constant M_I such that

$$|f(\xi,t)|\le M_I\max(A(\xi),B(\xi))\quad\text{for all } \xi\ge r_0 \text{ and all } t\in I.$$

Remark 2.2. If f satisfies (F1) and (F3), then:

(F4) There exists a continuous function $\theta_1:[r_0,+\infty)\to(0,+\infty)$ and a constant $C>0$ such that

$$\theta_1(\xi)\le\frac{C}{\xi^{N+\mu}}$$

and

$$tf(\xi,t)>\theta F(\xi,t)-\theta_1(\xi)\quad\text{for all } (\xi,t)\in[r_0,+\infty)\times[0,+\infty).$$

Remark 2.3. We note that (F1) implies that there exist continuous functions A_1, $B_1:[r_0,+\infty)\to(0,+\infty)$ and a positive constant C_1 such that

$$F(\xi,t)\le B_1(\xi)(|t|^{q+1}+1) \text{ for all } (\xi,t)\in[r_0,+\infty)\times\mathbb{R},$$

and

$$F(\xi,t)\ge A_1(\xi)(t^{q+1}-C_1) \text{ for all } (\xi,t)\in[r_0,+\infty)\times[0,+\infty).$$

Furthermore, $A_1(\xi)$, $B_1(\xi)\le\frac{1}{\xi^{N+\mu}}$ for $\xi\gg 1$, where μ is given in (F1). Notice that the second inequality above follows from the fact that $\frac{t^{q+1}}{q+1}-t\ge\frac{t^{q+1}}{2(q+1)}$ for all $t\ge(2(q+1))^{\frac{1}{q}}$.

Lemma 2.4. *Let* $J : W_0^{1,p}(0,1) \to \mathbb{R}$ *be defined by*

$$J(u) = \frac{1}{p}\hat{M}(\|u\|_{1,p}^p) - \lambda K(u)$$

where

$$K(u) = \int_0^1 h(s)F(r_0 s^{\frac{p-1}{p-N}}, u(s))ds$$

and

$$\hat{M}(t) = \int_0^t M(\sigma)d\sigma \quad \textit{with} \quad M(t) = a + \alpha t.$$

Then J *is well defined, continuously differentiable, and for all* $v \in W_0^{1,p}(0,1)$, *its Gâteaux derivative is given by*

$$J'(u)(v) = M(\|u\|_{1,p}^p)\int_0^1 |u'(s)|^{p-2}u'(s)v'(s)ds - \lambda \int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u(s))v(s)ds.$$

Proof. It is clear that $\hat{M}(\|u\|_{1,p}^p)$ is finite, and since $W_0^{1,p}(0,1) \hookrightarrow C[0,1]$, by Remark 2.1, for $I = [-\|u\|_\infty, \|u\|_\infty]$, there exists $M_I > 0$ such that

$$\left|f(r_0 s^{\frac{p-1}{p-N}}, \sigma)\right| \le M_I \max(A(r_0 s^{\frac{p-1}{p-N}}), B(r_0 s^{\frac{p-1}{p-N}})) \quad \text{for all } \sigma \in I.$$

Therefore,

$$\int_0^1 |h(s)||F(r_0 s^{\frac{p-1}{p-N}}, u(s))|ds \le M_I \|u\|_\infty \int_0^1 \max(A(r_0 s^{\frac{p-1}{p-N}}), B(r_0 s^{\frac{p-1}{p-N}}))|h(s)|ds < \infty.$$

The functional J is continuous. Moreover, if we set $L(u) = \|u\|_{1,p}^p$, then $\hat{M} \circ L$ is differentiable, and for all $v \in W_0^{1,p}(0,1)$,

$$\frac{1}{p}(\hat{M} \circ L)'(u)(v) = M(\|u\|_{1,p}^p)\int_0^1 |u'(s)|^{p-2}u'(s)v'(s)ds.$$

On the other hand, from the continuity of f, we see that K is Gâteaux differentiable and its Gâteaux derivative is continuous. Hence, K is continuously differentiable and

$$K'(u)(v) = \int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u(s))v(s)ds \quad \text{for all } v \in W_0^{1,p}(0,1).$$

Therefore, J is continuously differentiable, and for all $v \in W_0^{1,p}(0,1)$,

$$J'(u)(v) = M(\|u\|_{1,p}^p) \int_0^1 |u'(s)|^{p-2} u'(s) v'(s) - \lambda \int_0^1 h(s) f(r_0 s^{\frac{p-1}{p-N}}, u(s)) v(s) ds$$

as we wished to show. □

In order to prove our main result, Theorem 1.5 above, we will apply the Mountain Pass Theorem stated below.

Theorem 2.5 (Mountain Pass Theorem [3])**.** *Let X be a Banach space and let $J \in C^1(X;\mathbb{R})$ satisfy:*

(i) (*Palais–Smale condition*) *any sequence $(u_n) \subset X$ such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence,*
(ii) $J(0) = 0$,
(iii) *there exist ν, $R > 0$ such that $J(u) \geq \nu$ for all u with $\|u\|_X = R$,*
(iv) *there exists $e \in X$ such that $\|e\|_X > R$ and $J(e) < 0$.*

In addition, let

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$$

and

$$\hat{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Then $\hat{c}$ is a critical value of the functional J.

3. PROOF OF THEOREM 1.5

In this section, we construct the proof of our main result. We begin by recalling that proving Theorem 1.5 is equivalent to proving that the functional J defined above admits a positive critical point for $\lambda \approx 0$ (see[4]). As was seen in Lemma 2.4, the functional J is in $C^1(W_0^{1,p}(0,1), \mathbb{R})$, so we need to prove that J satisfies the conditions of the MPT.

In the following, for all $s \in (0,1]$, we denote by $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{A}_1(s)$, $\tilde{B}_1(s)$, and $\tilde{\theta}_1(s)$ the quantities $A(r_0 s^{\frac{p-1}{p-N}})$, $B(r_0 s^{\frac{p-1}{p-N}})$, $A_1(r_0 s^{\frac{p-1}{p-N}})$, $B_1(r_0 s^{\frac{p-1}{p-N}})$, and $\theta_1(r_0 s^{\frac{p-1}{p-N}})$, respectively.

3.1. THE PALAIS–SMALE CONDITION

In order to show that our functional J satisfies the Palais–Smale condition, we first recall the following proposition.

Proposition 3.1 ([11])**.** *Let $\psi : W^{1,p}(0,1) \to [0,+\infty)$ be defined by*

$$\psi(u) = \frac{1}{p} \int_0^1 |u'(s)|^p ds.$$

Then ψ' exists and

$$\langle \psi'(u), v\rangle = \int_0^1 |u'(s)|^{p-2}u'(s)v'(s)ds.$$

In addition, if $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle \psi'(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ strongly in $W^{1,p}(0,1)$.

Lemma 3.2. *The functional J satisfies the Palais–Smale condition.*

Proof. Let $(u_n)_n \subset W_0^{1,p}(0,1)$ such that $(J(u_n))_n$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. First, we will prove that $(u_n)_n$ is bounded in $W_0^{1,p}(0,1)$. Assume to contrary that $(u_n)_n$ is such that $J'(u_n) \to 0$ as $n \to \infty$, there exists $M > 0$ such that $|J(u_n)| \le M$ for all $n \ge 1$, but $\|u_n\|_{1,p} \to \infty$ as $n \to \infty$.

We consider the quantity

$$\frac{\theta J(u_n) - \langle J'(u_n), u_n\rangle}{\|u_n\|_{1,p}},$$

where $\theta > 2p$ is chosen as in (F3). Since $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\theta J(u_n) - \langle J'(u_n), u_n\rangle}{\|u_n\|_{1,p}} = 0.$$

However, we have

$$\begin{aligned}
\theta J(u_n) - \langle J'(u_n), u_n\rangle &= a\Big(\frac{\theta}{p} - 1\Big)\|u_n\|_{1,p}^p + \alpha\Big(\frac{\theta}{2p} - 1\Big)\|u_n\|_{1,p}^{2p} \\
&\quad - \lambda \int_0^1 h(s)\Big(\theta F(r_0 s^{\frac{p-1}{p-N}}, u_n(s)) - f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))u_n(s)ds\Big) \\
&= a\Big(\frac{\theta}{p} - 1\Big)\|u_n\|_{1,p}^p + \alpha\Big(\frac{\theta}{2p} - 1\Big)\|u_n\|_{1,p}^{2p} - \lambda(I_1 + I_2),
\end{aligned}$$

where

$$I_1 = \int_{\{u_n \ge 0\}} h(s)\Big(\theta F(r_0 s^{\frac{p-1}{p-N}}, u_n(s)) - f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))u_n(s)\Big)\,ds$$

and

$$I_2 = \int_{\{u_n < 0\}} h(s)\Big(\theta F(r_0 s^{\frac{p-1}{p-N}}, u_n(s)) - f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))u_n(s)\Big)\,ds,$$

with

$$\{u_n \ge 0\} = \{s \in [0,1] : u_n(s) \ge 0\}$$

and

$$\{u_n < 0\} = \{s \in [0,1] : u_n(s) < 0\}.$$

Using (F4), we can write

$$I_1 \le \int\limits_{\{u_n \ge 0\}} h(s)\theta_1(r_0 s^{\frac{p-1}{p-N}})ds \le \int\limits_{[0,1]} h(s)\theta_1(r_0 s^{\frac{p-1}{p-N}})ds \le \|h\tilde{\theta_1}\|_1,$$

and on $\{u_n < 0\}$,

$$\begin{aligned}\theta F(r_0 s^{\frac{p-1}{p-N}}, u_n(s)) - f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))u_n(s) &= (\theta-1)f(r_0 s^{\frac{p-1}{p-N}}, 0)u_n(s)\\ &\le (\theta-1)A(r_0 s^{\frac{p-1}{p-N}})\|u_n\|_\infty\\ &\le k(\theta-1)A(r_0 s^{\frac{p-1}{p-N}})\|u_n\|_{1,p},\end{aligned}$$

so that

$$\begin{aligned}\frac{\theta J(u_n) - \langle J'(u_n), u_n\rangle}{\|u_n\|_{1,p}} &\ge a\Big(\frac{\theta}{p}-1\Big)\|u_n\|_{1,p}^{p-1} + \alpha\Big(\frac{\theta}{2p}-1\Big)\|u_n\|_{1,p}^{2p-1}\\ &\quad - \lambda\frac{\|h\tilde{\theta_1}\|_{L^1}}{\|u_n\|_{1,p}} - k\lambda(\theta-1)\|h\tilde{A}\|_1.\end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain a contradiction. Thus, (u_n) is bounded in $W_0^{1,p}(0,1)$ and this implies that there exists a subsequence, again calling it (u_n), that converges weakly in $W_0^{1,p}(0,1)$ and strongly in $C[0,1]$.

We want to show that $u_n \to u$ strongly in $W_0^{1,p}(0,1)$. Since for all $v \in W_0^{1,p}(0,1)$,

$$J'(u_n)(v) = M(\|u_n\|_{1,p}^p)\int\limits_0^1 |u_n^{'}(s)|^{p-2}u_n^{'}(s)v'(s)ds - \lambda\int\limits_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))v(s)ds,$$

we have

$$\Big|\int\limits_0^1 |u_n'|^{p-2}u_n'(u_n'-u')\Big| \le \frac{|J'(u_n)(u_n-u)| + \lambda|\int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))(u_n-u)ds|}{a}.$$

Since $J'(u_n) \to 0$ and (u_n) is bounded in $W_0^{1,p}(0,1)$, we have $J'(u_n)(u_n-u) \to 0$ as $n \to +\infty$. On the other hand, since $u_n \to u$ strongly in $C[0,1]$ and since (u_n) is bounded in the same space, we have (see Remark 2.1)

$$\Big|\int\limits_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u_n(s))(u_n(s)-u(s))ds\Big| \le M_I\|u_n-u\|_\infty\|h\max(\tilde{A},\tilde{B})\|_1 \to 0, \quad (3.1)$$

where $I = [-M, M]$ is such that u, $u_n \in [-M, M]$ for $s \in [0,1]$. This implies that

$$\Big|\int\limits_0^1 |u_n^{'}(s)|^{p-2}u_n^{'}(s)(u_n^{'}(s)-u'(s))ds\Big| \to 0$$

as $n \to \infty$. By Proposition 3.1, $u_n \to u$ strongly in $W_0^{1,p}(0,1)$, which completes the proof of the lemma. □

3.2. THE GEOMETRY OF J

We begin by pointing out that $J(0) = 0$. Next, we prove two lemmas that will be needed to complete the proof that J satisfies the Mountain Pass Theorem.

Lemma 3.3. *For any positive function w in $W_0^{1,p}(0,1)$ satisfying $\|w\|_{1,p} = 1$, we have $\lim_{\sigma\to+\infty} J(\sigma w) = -\infty$ for any $\sigma > 0$.*

Proof. Let $w \in W_0^{1,p}(0,1)$ be such that w is positive with $\|w\|_{1,p} = 1$, and let $\sigma > 0$ be a parameter. We have

$$J(\sigma w) = \frac{1}{p}\hat{M}(\|\sigma w\|_{1,p}^p) - \lambda \int_0^1 h(s)F(r_0 s^{\frac{p-1}{p-N}}, \sigma w(s))ds$$

with

$$\hat{M}(\|\sigma w\|_{1,p}^p) = a\sigma^p + \frac{\alpha}{2}\sigma^{2p}.$$

By (F1) and an integration,

$$\int_0^1 h(s)F(r_0 s^{\frac{p-1}{p-N}}, \sigma w(s))ds \geq \int_0^1 h(s)A(r_0 s^{\frac{p-1}{p-N}}) \left(\frac{(\sigma w(s))^{q+1}}{q+1} - \sigma w(s)\right) ds. \quad (3.2)$$

Then,

$$J(\sigma w) \leq \frac{a}{p}\sigma^p + \frac{\alpha}{2p}\sigma^{2p} - \lambda\frac{\sigma^{q+1}}{q+1}\int_0^1 h(s)A(r_0 s^{\frac{p-1}{p-N}})(w(s))^{q+1}ds$$
$$+ \lambda\sigma \int_0^1 h(s)A(r_0 s^{\frac{p-1}{p-N}})w(s)ds.$$

But

$$0 < \int_0^1 h(s)A(r_0 s^{\frac{p-1}{p-N}})(w(s))^{q+1}ds \leq k^{q+1}\|h\tilde{A}\|_1 \|w\|_{1,p}^{q+1} < \infty$$

and

$$0 < \int_0^1 h(s)A(r_0 s^{\frac{p-1}{p-N}})w(s)ds \leq k\|h\tilde{A}\|_1 \|w\|_{1,p} < \infty,$$

so $\lim_{\sigma\to+\infty} J(\sigma w) = -\infty$ since $q > 2p - 1$. This proves the lemma. $\square$

Lemma 3.4. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and $u \in W_0^{1,p}(0,1)$ be such that $\|u\|_{1,p} = \lambda^{\frac{-1}{2(q+1-2p)}}$, we have $J(u) \geq \frac{\alpha}{4p}\lambda^{\frac{-p}{q+1-2p}}$.*

Proof. Let $\lambda > 0$ and $u \in W_0^{1,p}(0,1)$ be such that

$$\|u\|_{1,p} = \lambda^{\frac{-1}{2(q+1-2p)}}.$$

We have

$$\begin{aligned} J(u) &\geq \frac{a}{p}\lambda^{\frac{-p}{2(q+1-2p)}} + \frac{\alpha}{2p}\lambda^{\frac{-p}{q+1-2p}} - \lambda\int_0^1 h(s)B_1(r_0 s^{\frac{p-1}{p-N}})(|u(s)|^{q+1}+1)ds \\ &\geq \frac{a}{p}\lambda^{\frac{-p}{2(q+1-2p)}} + \frac{\alpha}{2p}\lambda^{\frac{-p}{q+1-2p}} - \lambda\|u\|_\infty^{q+1}\|h\tilde{B}_1\|_1 - \lambda\|h\tilde{B}_1\|_1 \\ &\geq \frac{a}{p}\lambda^{\frac{-p}{2(q+1-2p)}} + \frac{\alpha}{2p}\lambda^{\frac{-p}{q+1-2p}} - k\lambda\lambda^{\frac{-(q+1)}{2(q+1-2p)}} - \lambda\|h\tilde{B}_1\|_1 \\ &\geq \lambda^{\frac{-p}{q+1-2p}}\left(\frac{\alpha}{2p} + \frac{a}{p}\lambda^{\frac{p}{2(q+1-2p)}} - k\|h\tilde{B}_1\|_1\lambda^{\frac{1}{2}} - \|h\tilde{B}_1\|_1\lambda^{\frac{q+1-p}{q+1-2p}}\right). \end{aligned}$$

Since

$$\lim_{\lambda\to 0}\frac{a}{p}\lambda^{\frac{p}{2(q+1-2p)}} - k\|h\tilde{B}_1\|_1\lambda^{\frac{1}{2}} - \|h\tilde{B}_1\|_1\lambda^{\frac{q+1-p}{q+1-2p}} = 0,$$

there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$,

$$J(u) \geq \frac{\alpha}{4p}\lambda^{\frac{-p}{q+1-2p}},$$

which proves the lemma. □

From Lemma 3.3 and Lemma 3.4, we can deduce that conditions (iii) and (iv) of Theorem 2.5 are satisfied. We have then proved that the functional J admits a critical value for $\lambda \approx 0$. We need to show that this critical point is positive.

3.3. POSITIVITY OF THE MPT SOLUTION

Let $r = \frac{1}{q+1-2p}$. We start with two lemmas.

Lemma 3.5. *Let u_λ be a mountain pass solution to* (1.1). *Then there exists $M_0 > 0$ and $\lambda_1 > 0$ such that*

$$\|u_\lambda\|_\infty \geq M_0\lambda^{-r\frac{p}{q+1}}$$

for all $\lambda \in (0, \lambda_1)$.

Proof. Since u_λ is a mountain pass solution, we have

$$\begin{aligned}\lambda\int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))u_\lambda(s)ds &= a\|u_\lambda\|_{1,p}^p + \alpha\|u_\lambda\|_{1,p}^{2p} \\ &= pJ(u_\lambda) + \frac{\alpha}{2}\|u_\lambda\|_{1,p}^{2p} + p\lambda\int_0^1 h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda)ds \\ &\geq pJ(u_\lambda) + p\lambda\int_{\{u_\lambda<0\}} h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds \\ &\quad + p\lambda\int_{\{u_\lambda\geq 0\}} h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds \\ &\geq pJ(u_\lambda) + p\lambda\int_{\{u_\lambda\geq 0\}} h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds.\end{aligned}$$

In view of Remark 2.3 and the fact that u_λ satisfies

$$J(u_\lambda) \geq \frac{\alpha}{4p}\lambda^{-rp} \quad \text{for all } \lambda \in (0, \lambda_0),$$

we see that

$$\begin{aligned}\lambda\int_0^1 h(s)f(r_0 s^{\frac{p-1}{p-N}}, u_\lambda)u_\lambda ds &\geq \frac{\alpha}{4}\lambda^{-rp} + p\lambda\int_{\{u_\lambda\geq 0\}} h(s)A_1(r_0 s^{\frac{p-1}{p-N}})u_\lambda^{q+1}ds \\ &\quad - C_1 p\lambda\int_{\{u_\lambda\geq 0\}} h(s)A_1(r_0 s^{\frac{p-1}{p-N}})ds \\ &\geq \frac{\alpha}{4}\lambda^{-rp} - C_1 p\lambda\|h\tilde{A}_1\|_1 \geq \frac{\alpha}{8}\lambda^{-rp},\end{aligned}$$

for all $\lambda \in \left(0, \min\left(\lambda_0,\ \left(\frac{\alpha}{8p\|h\tilde{A}_1\|_1}\right)^{\frac{1}{rp+1}}\right)\right)$. By Remark 2.1, for all

$$\lambda \in \left(0, \min\left(\lambda_0,\ \left(\frac{\alpha}{8p\|h\tilde{A}_1\|_1}\right)^{\frac{1}{rp+1}}, 1\right)\right),$$

we have

$$\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty \geq \frac{\alpha}{8\|h\max(\tilde{A},\tilde{B})\|_1}\lambda^{-rp}.$$

Then, for all

$$\lambda \in \left(0, \min\left(\lambda_0,\ \left(\frac{\alpha}{8p\|h\tilde{A}_1\|_1}\right)^{\frac{1}{rp+1}}, 1, \left(\frac{\alpha}{16\|h\max(\tilde{A},\tilde{B})\|_1}\right)^{\frac{1}{rp}}\right)\right),$$

we have $\|u_\lambda\|_\infty \geq 1$, so

$$\|u_\lambda\|_\infty^{q+1} \geq \frac{\alpha}{16\|h\max(\tilde{A},\tilde{B})\|_1}\lambda^{-rp},$$

or

$$\|u_\lambda\|_\infty \geq \left(\frac{\alpha}{16\|h\max(\tilde{A},\tilde{B})\|_1}\right)^{\frac{1}{q+1}} \lambda^{-r\frac{p}{q+1}}.$$

Taking

$$M_0 = \left(\frac{\alpha}{16\|h\max(\tilde{A},\tilde{B})\|_1}\right)^{\frac{1}{q+1}}$$

and

$$\lambda_1 = \min\left(\lambda_0, \left(\frac{\alpha}{8p\|h\tilde{A}_1\|_1}\right)^{\frac{1}{rp+1}}, 1, \left(\frac{\alpha}{16\|h\max(\tilde{A},\tilde{B})\|_1}\right)^{\frac{1}{rp}}\right)$$

completes the proof of the lemma. □

Lemma 3.6. *Let u_λ be a mountain pass solution of* (1.2). *Then there exists $C_0 > 0$ and $\lambda_2 > 0$ such that*

$$\|u_\lambda\|_{1,p} \leq C_0\lambda^{-r},$$

for all $\lambda \in (0, \lambda_2)$.

Proof. Since u_λ is a solution of (1.2), by Remark 2.2, we have

$$\begin{aligned}
a\|u_\lambda\|_{1,p}^p + \frac{\alpha}{2}\|u_\lambda\|_{1,p}^{2p} &= pJ(u_\lambda) + p\lambda \int_0^1 h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds \\
&= pJ(u_\lambda) + p\lambda \int_{\{u_\lambda<0\}} h(s)u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, 0)ds \\
&\quad + p\lambda \int_{\{u_\lambda\geq 0\}} h(s)F(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds \\
&\leq pJ(u_\lambda) + p\lambda \int_{\{u_\lambda<0\}} h(s)u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, 0)ds \\
&\quad + p\lambda \int_{\{u_\lambda\geq 0\}} \frac{h(s)}{\theta}\Big(u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s)) + \theta_1(r_0 s^{\frac{p-1}{p-N}})\Big) ds \\
&\leq pJ(u_\lambda) + p\lambda \int_{\{u_\lambda<0\}} h(s)u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, 0)ds \\
&\quad + \frac{p}{\theta}\lambda \int_0^1 h(s)\Big(u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s)) + \theta_1(r_0 s^{\frac{p-1}{p-N}})\Big) ds \\
&\quad - \frac{p}{\theta}\lambda \int_{\{u_\lambda<0\}} h(s)\Big(u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, 0) + \theta_1(r_0 s^{\frac{p-1}{p-N}})\Big) ds \\
&\leq pJ(u_\lambda) + \frac{p}{\theta}\lambda \int_0^1 h(s)u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, u_\lambda(s))ds \\
&\quad + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 + p\lambda\Big(1-\frac{1}{\theta}\Big) \int_{\{u_\lambda<0\}} h(s)u_\lambda(s) f(r_0 s^{\frac{p-1}{p-N}}, 0)ds \\
&\leq pJ(u_\lambda) + \frac{p}{\theta}(a\|u_\lambda\|_{1,p}^p + \alpha\|u_\lambda\|_{1,p}^{2p}) \\
&\quad + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 + p\lambda k\|u_\lambda\|_{1,p}\|h\max(\tilde{A},\tilde{B})\|_1.
\end{aligned}$$

Then,

$$a\Big(1-\frac{p}{\theta}\Big)\|u_\lambda\|_{1,p}^p + \Big(\frac{\alpha}{2}-\frac{p\alpha}{\theta}\Big)\|u_\lambda\|_{1,p}^{2p} \leq pJ(u_\lambda) + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 + p\lambda k'\|u_\lambda\|_{1,p}, \quad (3.3)$$

where $k' = k\|h\max(\tilde{A},\tilde{B})\|_1$. On the other hand, since u_λ is a mountain pass solution, we have $J(u_\lambda) \leq \max_{\sigma\geq 0} J(\sigma w)$ where $w>0$ is such that $\|w\|_{1,p}=1$, and so in view of Remark 2.3,

$$J(u_\lambda) \le \max_{\sigma \ge 0} \frac{a}{p}\sigma^p + \frac{\alpha}{2p}\sigma^{2p} - \frac{\lambda}{q+1} D_1 \sigma^{q+1} + C_1\lambda \|h\tilde{A}_1\|_1,$$

where

$$0 < D_1 := \int_0^1 hA_1(r_0 s^{\frac{p-1}{p-N}}) w(s)^{q+1} ds \le k^{q+1} \|h\tilde{A}_1\|_1 \|w\|_{1,p}^{q+1} < \infty.$$

Let

$$P(\sigma) = \frac{\alpha}{2p}\sigma^{2p} + \frac{a}{p}\sigma^p - \frac{C\lambda}{q+1}\sigma^{q+1} + C_1\lambda\|h\tilde{A}_1\|_1,$$
$$P_1(\sigma) = \Big(\frac{a}{p} + \frac{\alpha}{2p}\Big)\sigma^p - \frac{C\lambda}{q+1}\sigma^{q+1} + C_1\lambda\|h\tilde{A}_1\|_1$$

and

$$P_2(\sigma) = \Big(\frac{a}{p} + \frac{\alpha}{2p}\Big)\sigma^{2p} - \frac{C\lambda}{q+1}\sigma^{q+1} + C_1\lambda\|h\tilde{A}_1\|_1.$$

On $[0,1]$, $P(\sigma) \le P_1(\sigma)$ and on $(1,+\infty)$, $P(\sigma) \le P_2(\sigma)$. Also, $P_1(\sigma)$ is maximized for $\sigma_1 = \tilde{K}_1^{\frac{1}{q+1-p}} \lambda^{\frac{-1}{q+1-p}}$ and $P_2(\sigma)$ is maximized for $\sigma_2 = \tilde{K}_2^r \lambda^{-r}$, where $\tilde{K}_1 = \frac{2a+\alpha}{2C}$ and $\tilde{K}_2 = \frac{2a+\alpha}{C}$. Note that if $\lambda \le 1$, then $\lambda \le \lambda^{-2pr}$, $\lambda \le \lambda^{\frac{-p}{q+1-p}}$ and $\lambda^{\frac{-p}{q+1-p}} \le \lambda^{-2pr}$. Therefore,

$$\begin{aligned}
pP_1(\sigma) + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 &\le \Big(a + \frac{\alpha}{2}\Big)\tilde{K}_1^{\frac{p}{q+1-p}}\lambda^{\frac{-p}{q+1-p}} + \lambda p\left(C_1\|h\tilde{A}_1\|_1 + \frac{\|h\tilde{\theta}_1\|_1}{\theta}\right)\\
&\le \lambda^{\frac{-p}{q+1-p}}\left(\Big(a + \frac{\alpha}{2}\Big)\tilde{K}_1^{\frac{p}{q+1-p}} + p\Big(C_1\|h\tilde{A}_1\|_1 + \frac{\|h\tilde{\theta}_1\|_1}{\theta}\Big)\right)\\
&\le \lambda^{-2pr}\left(\Big(a + \frac{\alpha}{2}\Big)\tilde{K}_1^{\frac{p}{q+1-p}} + p\Big(C_1\|h\tilde{A}_1\|_1 + \frac{\|h\tilde{\theta}_1\|_1}{\theta}\Big)\right)\\
&= \tilde{C}_1\lambda^{-2pr}
\end{aligned}$$

and

$$\begin{aligned}
pP_2(\sigma) + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 &\le \Big(a + \frac{\alpha}{2}\Big)\tilde{K}_2^{2pr}\lambda^{-2pr} + \lambda p\left(C_1\|h\tilde{A}_1\|_1 + \frac{\|h\tilde{\theta}_1\|_1}{\theta}\right)\\
&\le \lambda^{-2pr}\left(\Big(a + \frac{\alpha}{2}\Big)\tilde{K}_2^{2pr} + p\Big(C_1\|h\tilde{A}_1\|_1 + \frac{\|h\tilde{\theta}_1\|_1}{\theta}\Big)\right)\\
&= \tilde{C}_2\lambda^{-2pr}.
\end{aligned}$$

Setting $\tilde{C}_3 = \max(\tilde{C}_1, \tilde{C}_2)$ gives

$$pJ(u_\lambda) + \frac{p}{\theta}\lambda\|h\tilde{\theta}_1\|_1 \le \tilde{C}_3\lambda^{-2pr},$$

and from (3.3), we have

$$\left(\frac{\alpha}{2}-\frac{p\alpha}{\theta}\right)\|u_\lambda\|_{1,p}^{2p} \le \tilde{C}_3\lambda^{-2pr} + p\lambda k'\|u_\lambda\|_{1,p}.$$

By Lemma 3.5, for all $\lambda \in (0,\lambda_1)$,

$$\|u_\lambda\|_{1,p} \ge \frac{1}{k}\|u_\lambda\|_\infty \ge \frac{M_0}{k}\lambda^{-r\frac{p}{q+1}}.$$

Then for all $\lambda \in \left(0, \min(\lambda_1, \big(\frac{M_0}{k}\big)^{\frac{q+1}{rp}}\right)$, we have $\|u_\lambda\|_{1,p} \ge 1$, so

$$\left(\frac{\alpha}{2}-\frac{p\alpha}{\theta}\right)\|u_\lambda\|_{1,p}^{2p} \le \tilde{C}_3\lambda^{-2pr} + p\lambda k'\|u_\lambda\|_{1,p}^{2p}.$$

This implies

$$\left(\frac{\alpha}{2}-\frac{p\alpha}{\theta}-p\lambda k'\right)\|u_\lambda\|_{1,p}^{2p} \le \tilde{C}_3\lambda^{-2pr}.$$

Hence, for all $\lambda \in \left(0, \min\left(\lambda_1, \big(\frac{M_0}{k}\big)^{\frac{q+1}{rp}}, \frac{\alpha(\theta-2p)}{4\theta p k'}\right)\right)$, we have

$$\frac{\alpha(\theta-2p)}{4\theta}\|u_\lambda\|_{1,p}^{2p} \le \tilde{C}_3\lambda^{-2pr}.$$

Taking $C_0 = \frac{4\theta\tilde{C}_3}{\alpha(\theta-p)}$ and $\lambda_2 = \min(\lambda_1, \big(\frac{M_0}{k}\big)^{\frac{q+1}{rp}}, \frac{\alpha(\theta-2p)}{4\theta p k'})$, we see that the lemma is proved. □

To prove the positivity of the mountain pass solution, assume to the contrary, that there exists a sequence $\{(\lambda_i, u_{\lambda_i})\}_{i=1}^\infty \subset (0,1)\times C([0,1])$ of mountain pass solutions to (1.2) such that $\lambda_i \to 0$ as $i \to \infty$ and $m(\{x \in (0,1) : u_{\lambda_i}(x) \le 0\}) > 0$. Let $w_i = \frac{u_{\lambda_i}}{\|u_{\lambda_i}\|_\infty}$. Since

$$-(\phi_p(u'_{\lambda_i}))' = \frac{\lambda_i h(s) f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha\|u_{\lambda_i}\|_{1,p}^p},$$

we have

$$-(\phi_p(w'_i))' = \frac{\lambda_i h(s) f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha\|u_{\lambda_i}\|_{1,p}^p}\|u_{\lambda_i}\|_\infty^{1-p}.$$

From Remark 2.1 and Lemmas 3.5 and 3.6, we obtain

$$\begin{aligned}\left|\frac{\lambda_i f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha\|u_{\lambda_i}\|_{1,p}^p}\|u_{\lambda_i}\|_\infty^{1-p}\right| &\le \left(\frac{\lambda_i\|u_{\lambda_i}\|_\infty^{q+1-p}}{\alpha\|u_{\lambda_i}\|_{1,p}^p} + \frac{\lambda_i}{a}M_0^{1-p}\lambda_i^{\frac{-rp(1-p)}{q+1}}\right)\max(\tilde{A},\tilde{B})\\ &\le \left(\frac{\lambda_i}{\alpha}k^{q+1-p}\|u_{\lambda_i}\|_{1,p}^{\frac{1}{r}} + \frac{M_0^{1-p}}{a}\right)\max(\tilde{A},\tilde{B})\\ &\le \left(\frac{\lambda_i}{\alpha}k^{q+1-p}C_0^{\frac{1}{r}}\lambda_i^{-1} + \frac{M_0^{1-p}}{a}\right)\max(\tilde{A},\tilde{B})\\ &\le D_2\max(\tilde{A}(s),\tilde{B}(s)),\end{aligned}$$

where

$$D_2 = \frac{k^{q+1-p} C_0^{\frac{1}{r}}}{\alpha} + \frac{M_0^{1-p}}{a}.$$

So for all $s \in (0,1)$, the sequence $\left\{ \frac{\lambda_i f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} \right\}$ is bounded. Thus, there exists a subsequence (named the same) that converges to a limit $z_1(s)$. Moreover, $z_1(s) \geq 0$ since

$$z_1(s) = \lim_{i\to\infty} \frac{\lambda_i f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} \geq \lim_{i\to\infty} \frac{\lambda_i f(r_0 s^{\frac{p-1}{p-N}}, 0)}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} = 0.$$

Hence, for all $s \in (0,1)$, the sequence $\left\{ \frac{\lambda_i h(s) f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} \right\}$ converges to $z(s) = h(s) z_1(s) \geq 0$.

Let $s_i \in (0,1)$ be a maximum of w_i. Then,

$$\phi_p(w_i'(s)) = \int_s^{s_i} (-\phi_p(w_i'(\sigma)))' d\sigma = \int_s^{s_i} \frac{\lambda_i h(\sigma) f(r_0 \sigma^{\frac{p-1}{p-N}}, u_{\lambda_i}(\sigma))}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} d\sigma.$$

From (3.4),

$$|w_i'(s)|^{p-1} = |\phi_p(w_i'(s))| \leq \int_s^{s_i} C \max(\tilde{A}(\sigma), \tilde{B}(\sigma)) h(\sigma) d\sigma \leq C \|\max(\tilde{A}, \tilde{B}) h\|_1,$$

so $|w_i'(s)| \leq \|\max(\tilde{A}, \tilde{B}) h\|_1^{\frac{1}{p-1}}$ for all $s \in [0,1]$. By the Arzelà-Ascoli theorem, there exists $w \in C([0,1])$ such that $w_i \to w$ in $C([0,1])$.

Since (s_i) is bounded, there exists a subsequence (again denote by (s_i)) that converges to some s_0. Again by (3.4), we have

$$\left| \frac{\lambda_i f(r_0 s^{\frac{p-1}{p-N}}, u_{\lambda_i})}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \right| \|u_{\lambda_i}\|_\infty^{1-p} \leq C \max(\tilde{A}(s), \tilde{B}(s)) h(s).$$

Since $\max(\tilde{A}, \tilde{B}) h \in L^1(0,1)$, by the Lebesgue dominated convergence theorem,

$$\int_s^{s_i} \frac{\lambda_i h(\sigma) f(r_0 \sigma^{\frac{p-1}{p-N}}, u_{\lambda_i}(\sigma))}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} d\sigma \to \int_s^{s_0} z(\sigma) d\sigma.$$

Therefore,

$$\phi_p^{-1} \left(\int_s^{s_i} \frac{\lambda_i h(\sigma) f(r_0 \sigma^{\frac{p-1}{p-N}}, u_{\lambda_i}(\sigma))}{a+\alpha \|u_{\lambda_i}\|_{1,p}^p} \|u_{\lambda_i}\|_\infty^{1-p} d\sigma \right) \to \phi_p^{-1} \left(\int_s^{s_0} z(\sigma) d\sigma \right),$$

so we get

$$\int_0^\tau \phi_p^{-1}\left(\int_s^{s_i} \frac{\lambda_i h(\sigma) f(r_0\sigma^{\frac{p-1}{p-N}}, u_{\lambda_i}(\sigma))}{a+\alpha\|u_{\lambda_i}\|_{1,p}^p}\|u_{\lambda_i}\|_\infty^{1-p} d\sigma\right) ds \to \int_0^\tau \phi_p^{-1}\left(\int_s^{s_0} z(\sigma)d\sigma\right) ds.$$

We see that

$$w_i(\tau) \to \int_0^\tau \phi_p^{-1}\left(\int_s^{s_0} z(\sigma)d\sigma\right) ds = w(\sigma),$$

and so

$$w_i^{'}(\tau) = \phi_p^{-1}\left(\int_\tau^{s_i} \frac{\lambda_i h(\sigma) f(r_0\sigma^{\frac{p-1}{p-N}}, u_{\lambda_i}(\sigma))}{a+\alpha\|u_{\lambda_i}\|_{1,p}^p}\|u_{\lambda_i}\|_\infty^{1-p} d\sigma\right)$$

converges to $\phi_p^{-1}\left(\int_\tau^{s_0} z(\sigma)d\sigma\right) = w'(\tau)$ for all $\tau \in [0,1]$. Hence, $-(\phi_p(w'))' = z \geq 0$ with $w(0) = 0 = w(1)$. Since $\|w\|_\infty = 1$, clearly $w \neq 0$. Then, since w is concave, $w > 0$ in $(0,1)$, $w'(0) > 0$, and $w'(1) < 0$. Because $w_i \to w$ in $C([0,1])$, we conclude that $w_i(s) > 0$ for all $s \in (0,1)$ for i sufficiently large. Hence, $u_{\lambda_i}(s) > 0$ for all $s \in (0,1)$ for i sufficiently large. This contradicts $m(\{x \in (0,1) : u_{\lambda_i}(x) \leq 0\}) > 0$ for all sufficiently large i.

Thus, the mountain pass solution is positive, and this completes the proof of Theorem 1.5.

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