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ON THE BLOWING UP SOLUTIONS OF THE 4-D GENERAL q-KURAMOTO–SIVASHINSKY EQUATION WITH EXPONENTIALLY "DOMINATED" NONLINEARITY AND SINGULAR WEIGHT

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^4$ with smooth boundary and let $x^1, x^2, \dots, x^m$ be m-points in Ω. We are concerned with the problem

$$\Delta^2 u - H(x,u,D^k u) = \rho^4 \prod_{i=1}^{n} |x-p_i|^{4\alpha_i} f(x) g(u),$$

where the principal term is the bi-Laplacian operator, $H(x,u,D^k u)$ is a functional which grows with respect to Du at most like $|Du|^q$, $1 \le q \le 4$, $f : \Omega \to [0,+\infty[$ is a smooth function satisfying $f(p_i) > 0$ for any $i = 1, \dots, n$, α_i are positives numbers and $g : \mathbb{R} \to [0,+\infty[$ satisfy $|g(u)| \le ce^u$. In this paper, we give sufficient conditions for existence of a family of positive weak solutions $(u_\rho)_{\rho>0}$ in Ω under Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. The solutions we constructed are singular as the parameters ρ tends to 0, when the set of concentration $S = \{x^1, \dots, x^m\} \subset \Omega$ and the set $\Lambda := \{p_1, \dots, p_n\} \subset \Omega$ are not necessarily disjoint. The proof is mainly based on nonlinear domain decomposition method.

Keywords: singular limits, Green's function, nonlinearity, gradient, nonlinear domain decomposition method.

Mathematics Subject Classification: 35J40, 35J08, 35J60, 35J75, 29M27.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper, we consider the following generalized stationary q-Kuramoto–Sivashinsky problem

$$\begin{cases} \Delta^2 u - H(x,u,D^k u) = \rho^4 V(x) g(u) & \text{in } \Omega \subset \mathbb{R}^4, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$H(x,u,D^k u) = \gamma \Delta u + \lambda |\nabla u|^q, \quad q \in [1,4], \lambda, \gamma > 0,$$

V is the singular potential given by

$$V(x)=\prod_{i=1}^{n}|x-p_i|^{4\alpha_i}f(x),$$

$f:\Omega\to[0,+\infty[$ is a smooth function such that $f(p_i)>0$ for any $i=1,\dots,n$, α_i are positives numbers and $g:\mathbb{R}\to[0,+\infty[$ is a function satisfying $g(u)\le ce^u$, where $c>0$.

Before introducing our main results, let us denote by $G(x,\cdot)$ the Green function of the bi-Laplacian operator with Navier boundary condition, namely the solution of

$$\begin{cases}\Delta^2 G(x,\cdot)=64\pi^2\delta_x & \text{in } \Omega,\\ G(x,\cdot)=\Delta\, G(x,\cdot)=0 & \text{on } \partial\Omega.\end{cases} \tag{1.2}$$

We denote by

$$R(x,y):=G(x,y)+8\,\log|x-y| \tag{1.3}$$

the regular part of G and we define the functional

$$W(x^1,\dots,x^m):=\sum_{j=1}^{m}R(x^j,x^j)+\sum_{j\neq\ell}G(x^j,x^\ell)+2\sum_{j=1}^{m}\log V(x^j) \tag{1.4}$$

for m-points $x^1,x^2,\dots,x^m$ in Ω.

Given $u:\Omega\to\mathbb{R}$, we define the set of its blow-up points by

$$\begin{aligned} S&:=\{x\in\Omega:\exists(x_n)\text{ s.t } x_n\to x \text{ and } u(x_n)\to+\infty\}\\ &:=\{x^1,\dots,x^m\}.\end{aligned} \tag{1.5}$$

Denote by $\Lambda:=\{p_1,\dots,p_n\}$ the set of singular source given by Dirac masses δ_{p_i}.

We will ask the following question: Is it possible to construct a family

$$\left\{u^{\alpha_i}_{\rho,\lambda,\gamma}\right\}_{0<\rho<\rho_0,0<\lambda<\lambda_0,0<\gamma<\gamma_0}$$

of solutions of (1.1) in any domain Ω in $\mathbb{R}^4$, for a (dominated) exponential nonlinearity, which converges to some nontrivial singular function outside some singular set $\{x^j\}_{j\le 1\le m}\subset\Omega$, as ρ, λ and γ tend to 0, when the set of concentration points $x^j\in\Omega$, $j=1,\dots,m$ and the set of zeros of V are not necessarily disjoint.

In the present paper in Theorem 1.3, we give a positive answer to this question. We construct the bubbling solution to problem (1.1) which blows up at the origin, in radial case, for an exponential nonlinearity ($g(u)=e^u$) and which admits an interesting asymptotic behavior which is strongly related not only to the values of the parameters λ,γ,ρ and q, but also to the geometry of the domain Ω. Indeed, let $\Omega=B_r(p_{i_0})\subset\mathbb{R}^4$, the ball of radius $r>1$ centered at the point $p_{i_0}\in\mathbb{N}$ and we suppose that there exists i_0 such that $\alpha_{i_0}\neq 0$, or $x\in\Omega\cap\Lambda=\{p_{i_0}\}$. Then we are interested in positive solutions of the following problem:

$$\begin{cases}\Delta^2 u-\gamma\Delta u-\lambda|\nabla u|^q=\rho^4|x-p_{i_0}|^{4\alpha_{i_0}}f(x)e^u & \text{in } \Omega=B_r(p_{i_0})\subset\mathbb{R}^4,\\ u=\Delta u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.6}$$

when the parameters ρ, λ and γ tend to 0 with different speeds and $q \in [1,4]$. $f : \Omega \to [0,+\infty[$ is a smooth function such that $f(p_{i_0}) > 0$ and α_{i_0} is a positive number. From now on, for the sake of convenience, we assume that $\alpha_{i_0} := \alpha$ and $p_{i_0} := p = 0$, and we denote by ε the smallest positive parameter which verifies

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Remark that $\rho \sim \varepsilon$ as $\varepsilon \to 0$.

For $(\gamma, \lambda) = (0,0)$, problem (1.6) becomes

$$\begin{cases} \Delta^2 u = \rho^4 |x - p_{i_0}|^{4\alpha} f(x)\, e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

when the parameter ρ tends to 0 (see, for example, [4]).

In dimension 4, in [9], Wei studied the behavior of solutions to the following nonlinear eigenvalue problem for the biharmonic operator Δ^2 in $\mathbb{R}^4$. More precisely, he considered the following problem

$$\begin{cases} \Delta^2 u = \lambda\, f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and u^* the limits solution given by

$$\begin{cases} \Delta^2 u^* = 64\pi^2 \sum_{i=1}^{m} \delta_{x^i} & \text{in } \Omega, \\ u^* = \Delta u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

The author proved the following result:

Theorem 1.1 ([9]). *Let Ω be a smooth bounded domain in $\mathbb{R}^4$ and f be a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \quad \text{and} \quad e^{-u} \int_0^u f(s)ds \quad \text{tend to } 1 \text{ as } u \to +\infty. \tag{1.10}$$

For a solution u_λ of (1.8), denote by $\Sigma_\lambda = \lambda \int_\Omega f(u_\lambda)dx$. There are three possibilities.

(i) $\Sigma_\lambda \to 0$. *Then* $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ *as* $\lambda \to 0$.
(ii) $\Sigma_\lambda \to +\infty$. *Then* $u_\lambda \to +\infty$ *as* $\lambda \to 0$.
(iii) $\Sigma_\lambda \to 64\pi^2 m$ *for some positive integer m. Then the limiting function* $u^* = \lim_{\lambda\to 0} u_\lambda$ *has m blow-up points,* $\{x^1, \ldots, x^m\}$, *where* $u_\lambda(x^i) \to +\infty$ *as* $\lambda \to 0$. *Moreover,* $(x^1, \ldots, x^m)$ *is a critical point of W given by (1.4) with $V = 1$.*

In [3], the authors studied existence and qualitative properties of positives solutions to the boundary-value problem

$$\begin{cases} \Delta^2 u = \rho^4\, V(x) e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $V \in C^2(\Omega)$ is a non-negative function not identically zero, Ω a bounded open regular domain in $\mathbb{R}^4$ and ρ is a small and positive parameter. Recently, the existence of other branches of solutions as ρ tends to 0 was studied in [1]. The authors constructed a non-minimal solutions with singular limit as the parameter ρ tends to 0. Their results, which give the converse of the case (iii) given in the last theorem, can be stated as follows.

Theorem 1.2 ([1])**.** *Assume that* $(x^1, \dots, x^m)$ *is a nondegenerate critical point of* W *given by* (1.4) *with* $V = 1$*, then there exist* $\rho_0 > 0$ *and* $(u_\rho)_{\rho \in (0,\rho_0)}$ *a one parameter family of solutions of* (1.11)*, such that*

$$\lim_{\rho \to 0} u_\rho = \sum_{j=1}^{m} G(x^j, \cdot)$$

in $\mathcal{C}^{4,\alpha}_{loc}(\Omega - \{x^1, \dots, x^m\})$.

The result we have is the following.

Theorem 1.3. *Let* $\alpha \in (0,1)$ *and* $\Omega = B_r \subset \mathbb{R}^4$ *be the ball of radius* $r > 1$ *centered at the origin. Denote by* $\sigma := \max(\lambda, \gamma)$ *and* $r_{\sigma,\varepsilon} := \max(\sqrt{\sigma}, \varepsilon^{1/(\alpha+2)})$*. Then the following assertions hold.*

(1) (i) *If* $q \in [1,4)$ *and* $\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon}) = \alpha$ *which satisfies the condition:*

$$\forall \delta \in \Big(0, \min(1, 4-q)\Big) \,:\, \alpha\, \varepsilon^{-\delta/(\alpha+1)} r^{\delta}_{\sigma,\varepsilon} \to 0 \quad as \quad \alpha \to 0, \qquad (A^{\alpha}_{\sigma,\varepsilon})$$

then there exist $\alpha_0 > 0$ *and a family of solutions* $\{u^{\delta}_{\alpha}\}_{\alpha < \alpha_0}$ *of* (1.6) *such that*

$$\lim_{\alpha \to 0} u^{\delta}_{\alpha} = G$$

in $C^{4,\beta}_{loc}(\Omega \backslash \{0\})$ *for* $\beta \in (0,1)$.

(ii) *If* $q \in [1,4)$ *and* $\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon}) = \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon}$ *for* $\delta \in (0,1)$*, then there exist* $\sigma_0 > 0, \varepsilon_0 > 0$ *and a family of solutions* $(u_{\sigma,\varepsilon})_{\sigma<\sigma_0, \varepsilon<\varepsilon_0}$ *of* (1.6) *such that*

$$\lim_{(\sigma,\varepsilon) \to (0,0)} u_{\sigma,\varepsilon} = (1+\alpha) G$$

in $C^{4,\beta}_{loc}(\Omega \backslash \{0\})$ *for* $\beta \in (0,1)$.

(2) (i) *If* $q = 4$ *and* $\max(\alpha, r^2_{\sigma,\varepsilon}) = \alpha$ *which satisfies the condition* $(A^{\alpha}_{\sigma,\varepsilon})$*, for all* $\delta \in (0,1)$ *then there exist* $\alpha_0 > 0, \sigma_0 > 0, \varepsilon_0 > 0$ *and a family of solutions* $\{u_\alpha\}_{\alpha<\alpha_0}$ *of* (1.6) *such that*

$$\lim_{\alpha \to 0} u_\alpha = G$$

in $C^{4,\beta}_{loc}(\Omega \backslash \{0\})$ *for* $\beta \in (0,1)$.

(ii) *If* $q = 4$ *and* $\max(\alpha, r^2_{\sigma,\varepsilon}) = r^2_{\sigma,\varepsilon}$ *which satisfies the condition*

$$\forall\delta \in (0,1) \,:\, \varepsilon^{-\delta/(\alpha+1)} r^{2+\delta}_{\sigma,\varepsilon} \to 0 \quad \textit{as} \quad (\sigma,\varepsilon) \to (0,0), \qquad (A_{\sigma,\varepsilon})$$

then there exist $\sigma_0 > 0, \varepsilon_0 > 0$ *and a family of solutions* $\left\{u_{\sigma,\varepsilon}\right\}_{0<\sigma<\sigma_0, 0<\varepsilon<\varepsilon_0}$ *of* (1.6) *such that*

$$\lim_{\substack{\sigma\to 0\\ \varepsilon\to 0}} u_{\sigma,\varepsilon} = (1+\alpha)G$$

in $C^{4,\beta}_{loc}(\Omega\backslash\{0\})$ *for* $\beta \in (0,1)$.

Remark 1.4. We remark that the origin is a critical point of Robin's function, because of the symmetry of the domain.

Remark 1.5. Observe that $\alpha\,\varepsilon^{-1/(\alpha+1)} = \mathcal{O}(1)$ verifies the condition $(A^{\alpha}_{\sigma,\varepsilon})$ and if $r_{\sigma,\varepsilon} := \sqrt{\sigma} = \mathcal{O}(\varepsilon^{1/(\alpha+1)})$, then the condition $(A_{\sigma,\varepsilon})$ holds.

Remark 1.6. In the case where the set of concentration points S and the set Λ of zeros of V are disjoint, or $\alpha_i = 0$, or $\Lambda = \emptyset$, Theorem 1.3 gives a direct extension of the result in [2] by taking $f = \text{const} > 0$ and a smooth bounded domain $\Omega \subset \mathbb{R}^4$.

Remark 1.7. We can establish equivalent results for the dominated nonlinearity model for $g(u) = (e^u + e^{su})$, $s \in (0,1)$.

(i) When the set of concentration points $x^j \in \Omega$, $j = 1, \ldots, m$ and the set of zeros of $V(x) = \prod_{i=1}^{n} |x - p_i|^{4\alpha_i} f(x)$ are not necessarily disjoint we get the same conclusion as in Theorem 1.3 in radial cases.

(ii) In the case where the set of concentration points x^j and the set of zeros of V are disjoint a.e $S \cap \Lambda = \emptyset$, or $\alpha_i = 0$, or $\Lambda = \emptyset$, we have the analogues results as in [2] for the dominated nonlinearity problem for any regular and bounded domain Ω of $\mathbb{R}^4$.

The proof of Theorem 1.3 relies on efficient method to solve such singularly perturbed problems in the context of partial differential equations. This method based on some nonlinear domain decomposition has already been used successfully in a geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kähler metrics, etc.) and has been used in many other papers, see for instance [1, 2, 4] and references therein.

We briefly describe the plan of the paper. In Section 2, we discuss the nonlinear interior problem, where we study some linearized operator about radially symmetric solution and to estimate the error. In Section 3, we study the nonlinear exterior problem and to estimate the error. Both sections strongly use the b-operator which has been developed by Melrose [6] in the context of weighted Sobolev spaces and by Mazzeo [5] in the context of weighted Hölder spaces (see also [7]). Finally, in Section 4, we gather the results of previous section to show how elements of these infinite dimensional families can be connected to produce solutions of (1.6) described in Theorem 1.3. This last section borrows ideas from applied mathematics were domain decomposition methods are of common use.

2. THE NONLINEAR INTERIOR PROBLEM AND LINEAR FOURTH ORDER ELLIPTIC OPERATOR ON $\mathbb{R}^4$

For all $\tau > 0$ and $\alpha > 0$, we define the function

$$v_{\varepsilon,\tau,\alpha}(x) := \log \frac{(1+\varepsilon^2)^4 \tau^4 (4\alpha^2+8\alpha+6)(\alpha+1)^2}{6\,(\varepsilon^2+\tau^2\,|x|^{2(1+\alpha)})^4} \tag{2.1}$$

satisfying the equation

$$\Delta^2 v_{\varepsilon,\tau,\alpha} - \rho^4\,|x|^{4\alpha}\,e^{v_{\varepsilon,\tau,\alpha}} = -\frac{64\alpha(\alpha+2)(\alpha+1)^2\,\tau^2\varepsilon^2 r^{2(\alpha-1)}}{(\varepsilon^2+\tau^2 r^{2(1+\alpha)})^4}\left(\tau^4\,r^{4(\alpha+1)}+\varepsilon^4\right) \tag{2.2}$$

in $\mathbb{R}^4$. Denote by

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+|x|^2)^4},$$

the linearization of $\Delta^2 v - 24\,e^v = 0$ about the approximative local solution $v_{1,1,0}(:= v_{\varepsilon=1,\tau=1,\alpha=0})$ and denote by

$$\mathbb{L}_\alpha := \Delta^2 - \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(\alpha+1)})^4}, \quad \text{where} \quad C_\alpha = 64(4\alpha^2+8\alpha+6)(\alpha+1)^2, \tag{2.3}$$

the linear fourth order elliptic operator which corresponds to the linearization of $\Delta^2 u - 24\,|x|^{4\alpha}\,e^u = 0$ about the approximative local solution $v_{1,1,\alpha}(:= v_{\varepsilon=1,\tau=1,\alpha})$. This operator can be written as

$$\mathbb{L}_\alpha := \mathbb{L} + K_\alpha(x),$$

where $K_\alpha(x)$ is given by

$$K_\alpha(x) = \frac{384}{(1+|x|^2)^4} - \frac{64\,(4\alpha^2+8\alpha+6)(\alpha+1)^2\,|x|^{4\alpha}}{(1+|x|^{2(\alpha+1)})^4} \tag{2.4}$$

satisfying the inequality

$$|K_\alpha(x)| \le c\frac{1+|\log r|}{(1+r^2)^4}\,\alpha \tag{2.5}$$

for α small enough. For $k \in \mathbb{N}$, $\beta \in (0,1)$ and $\delta \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\beta}_{loc}(\mathbb{R}^4)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\beta}(\bar{B}_1)} + \sup_{r\ge 1}\left((1+r^2)^{-\delta/2}\|w(r\,\cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_1 - B_{1/2})}\right),$$

is finite. We define the subspace of radial functions in $\mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4)$ by

$$\mathcal{C}^{k,\beta}_{rad,\delta}(\mathbb{R}^4) = \{f \in \mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4) : f(x) = f(|x|) \text{ for all } x \in \mathbb{R}^4\}.$$

First, we recall the surjectivity result of $\mathbb{L}$ given in [1].

Proposition 2.1 ([1]). *For $\delta > 0$ and $\delta \notin \mathbb{Z}$, $\mathbb{L} : \mathcal{C}^{k,\beta}_{rad,\delta}(\mathbb{R}^4) \to \mathcal{C}^{0,\beta}_{rad,\delta-4}(\mathbb{R}^4)$ is surjective.*

Denote by $\mathcal{G}_\delta$ the right inverse of $\mathbb{L}$. By a perturbation argument, we obtain the following result.

Proposition 2.2. *There exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ if $\delta > 0$ and $\delta \notin \mathbb{N}$, then $\mathbb{L}_\alpha : \mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4) \to \mathcal{C}^{0,\beta}_{rad,\delta-4}(\mathbb{R}^4)$ is surjective.*

Denote by $\mathcal{G}_{\delta,\alpha}$ the right inverse of $\mathbb{L}_\alpha$, given by $\mathcal{G}_{\delta,\alpha} = \mathcal{G}_\delta + \mathcal{O}(\alpha)$. Starting from this section we will use the following notations:

$$R_{\sigma,\varepsilon} := (\tau/\varepsilon)^{1/(\alpha+1)} r_{\sigma,\varepsilon} \quad \text{and} \quad r_{\sigma,\varepsilon} := \max(\sqrt{\sigma}, \varepsilon^{1/(\alpha+2)}) \tag{2.6}$$

for $\sigma := \max(\lambda, \gamma)$. Now, we are interested to study the equation of type

$$\Delta^2 v - \gamma \Delta v - \lambda |\nabla v|^q - \rho^4\, V(|x|)\, e^v = 0 \tag{2.7}$$

in $B_{r_{\sigma,\varepsilon}}$. As the first step, we will treat the case where 0 is a zero of the potential V, so we can write $V(x) = |x|^{4\alpha} f(|x|)$, where f is a smooth function such that $f(0) > 0$. This is equivalent to find a solution w of

$$\Delta^2 w - \gamma \left(\frac{\varepsilon}{\tau}\right)^{2/(\alpha+1)} \Delta w - \lambda \left(\frac{\varepsilon}{\tau}\right)^{\frac{4-q}{(\alpha+1)}} |\nabla w|^q - 24|x|^{4\alpha} f((\varepsilon/\tau)^{1/(\alpha+1)}|x|) e^w = 0 \tag{2.8}$$

in $B_{R_{\sigma,\varepsilon}}$. We define

$$\mathbf{v} := v_{1,1,\alpha} + h - \log(f(0)) + H^i(\cdot/R_{\sigma,\varepsilon}) \tag{2.9}$$

where $H^i (:= H^i_{\eta,\theta})$ represents a lower order correction term (see [4, Lemma 1]) and a priori estimate for H^i is established, that satisfies orthogonality conditions with respect to ϕ_0 using maximum principle. Almost every H^i is small, so that the boundary condition zero is satisfied, that is, $\Delta^2 H^i = 0$ in B_1, $H^i_{|\partial B_1} = \eta$, $\Delta H^i_{|\partial B_1} = \theta$, with

$$\|H^i\|_{\mathcal{C}^{4,\beta}_2(\bar{B}_1^*)} \le c_\kappa |\eta|, \tag{2.10}$$

where, as already mentioned, B_1 denotes the unit ball in $\mathbb{R}^4$ and for $k \in \mathbb{N}$, $\beta \in (0,1)$ and $\delta \in \mathbb{R}$, the Hölder weighted space $\mathcal{C}^{k,\beta}_\delta(\bar{B}_1^*)$ is defined as the space of functions in $\mathcal{C}^{k,\beta}_{loc}(\bar{B}_1^*)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\beta}_\delta(\bar{B}_1^*)} = \sup_{r \le 1/2} r^{-\delta} \|u(r\,\cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_2 - B_1)},$$

is finite. Here $B_1^* = B_1 - \{0\}$. We will look for solutions to (2.8) of the form $\mathbf{v} + v$, where v is a function which is small when compared with $\mathbf{v}$. By Proposition 2.2 and taking into account that H^i is bi-harmonic, this is equivalent to solve

$$v = \mathcal{G}_{\delta,\alpha} \circ \mathcal{E}_{R_{\varepsilon,\alpha}}(\mathcal{N}(v) + \mathcal{F}_1(v) + \mathcal{F}_2(v)) := \mathfrak{F}(v) \tag{2.11}$$

in $B_{R_{\sigma,\varepsilon}}$, where $\xi_r : \mathcal{C}^{0,\beta}_{rad,\delta}(B_r) \to \mathcal{C}^{0,\beta}_{rad,\delta}(\mathbb{R}^4)$ is the extension operator defined by $\xi_r(f) = f$ in B_r, $\xi_r\,(f)(x) = \frac{2r-|x|}{r}\, f\left(r\,\frac{x}{|x|}\right)$ in $B_{2r} - B_r$ and $\xi_r(f) = 0$ in $\mathbb{R}^4 - B_{2r}$, such that

$$\|\xi_r(w)\|_{\mathcal{C}^{0,\beta}_{\delta}(\mathbb{R}^4)} \le c\|w\|_{\mathcal{C}^{0,\beta}_{\delta}(\bar{B}_\sigma)}, \tag{2.12}$$

$$\begin{aligned} \mathcal{N}(v) \;&= \frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4} e^{H^i(\cdot/R_{\sigma,\varepsilon})+h+v}\Big(\frac{f((\varepsilon/\tau)^{1/(\alpha+1)}|x|)}{f(0)} - 1\Big) \\ &+\frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\; e^h\,\Big(e^{H^i(\cdot/R_{\sigma,\varepsilon})+v} - v - 1\Big) \\ &+\frac{C_\alpha\,|x|^{4\alpha}}{(1+|x|^{2(1+\alpha)})^4}\;\big(e^h - 1\big)\, v, \end{aligned} \tag{2.13}$$

$$\mathcal{F}_1(v) = \gamma\left(\frac{\varepsilon}{\tau}\right)^{2/(\alpha+1)}\Delta\Big(H^i(\cdot/R_{\sigma,\varepsilon}) + v\Big) \tag{2.14}$$

and

$$\begin{aligned} \mathcal{F}_2(v) = \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{\frac{4-q}{(\alpha+1)}}&\Big|\nabla\Big(v_{1,1,\alpha} + h + H^i(\cdot/R_{\sigma,\varepsilon}) + v\Big)\Big|^q \\ -\lambda\Big(\frac{\varepsilon}{\tau}\Big)^{\frac{4-q}{(\alpha+1)}}&\Big|\nabla(v_{1,1,\alpha} + h)\Big|^q. \end{aligned} \tag{2.15}$$

We fix $\delta \in \Big(0, \min(1, 4-q)\Big)$ for $q \in [1,4)$ and $\delta \in (0,1)$ for $q = 4$.

Given $\kappa > 0$ (whose value will be fixed later on). Suppose that the parameter η which appears in (2.10) satisfies

$$|\eta| \le \kappa r^2_{\sigma,\varepsilon}. \tag{2.16}$$

Then, the following results hold.

Proposition 2.3. *Let $q \in [1,4)$, $\delta \in (0, \min(1, 4-q))$, $\alpha \in (0,1)$ and $r_{\sigma,\varepsilon} := \max(\sqrt{\sigma}, \varepsilon^{1/(\alpha+2)})$, where $\sigma := \max(\lambda, \gamma)$. Given $x \longmapsto R(x)$, defined by (1.3), the regular part of Green's function associated to the operator Δ^2 and the Dirichlet boundary condition. Let $\kappa > 0$, $\sigma_\kappa > 0$ and $\varepsilon_\kappa > 0$ (depending on κ) such that for all $\alpha \in (0, \alpha_\kappa)$, $\sigma \in (0, \sigma_\kappa)$, $\varepsilon \in (0, \varepsilon_\kappa)$, and the constant $\tau > 0$ satisfy*

$$\frac{1}{\log r_{\sigma,\varepsilon}}\Big[4\log(\tau/\tau^*)\Big] = -(1+\alpha)R(0) - \log(f(0)) + \mathcal{O}\Big(\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon})\Big) + \mathcal{O}(r^2_{\sigma,\varepsilon}),$$

where $\tau_* > 0$ is fixed. Then there exists a unique $v^\alpha_{\sigma,\varepsilon} (= \bar{v}_{\alpha,\sigma,\varepsilon,\tau,\eta})$ solution of (2.11) such that

$$\|v^\alpha_{\sigma,\varepsilon}\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \le 2c_\kappa r^2_{\sigma,\varepsilon}$$

and

$$\mathbf{v} + v^\alpha_{\sigma,\varepsilon} = v_{1,1,\alpha} + h - \log(f(0)) + H^i_{\eta,\theta}(\cdot/R_{\sigma,\varepsilon}) + \bar{v}_{\alpha,\sigma,\varepsilon,\tau,\eta}, \tag{2.17}$$

solves (2.8) in $B_{R_{\sigma,\varepsilon}}$. Furthermore:

(i) *if* $\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2) = \alpha$, *then there exist* $\alpha_0 > 0$ *and* $c > 0$ *such that for all* $\alpha \in (0, \alpha_0)$ *satisfying* $(A_{\sigma,\varepsilon}^\alpha)$, *we have*

$$\|h\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2c\, \alpha,$$

(ii) *if* $\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2) = \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2$, *then there exist* $\sigma_0 > 0$, $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all* $\sigma \in (0, \sigma_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$, *we have*

$$\|h\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2c\, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2.$$

Proposition 2.4. *Let* $q = 4$, $\delta \in (0,1)$, $\alpha \in (0,1)$, $r_{\sigma,\varepsilon} := \max\left(\sqrt{\sigma}, \varepsilon^{1/(\alpha+2)}\right)$ *where* $\sigma := \max(\lambda, \gamma)$. *Given* $x \longmapsto R(x)$, *defined by* (1.3), *the regular part of Green's function associated to the operator* Δ^2 *and the Dirichlet boundary condition. Let the constant* $\tau > 0$ *satisfy*

$$\frac{1}{\log r_{\sigma,\varepsilon}} \Big[4 \log(\tau/\tau^*)\Big] = -(1+\alpha) R(0) - \log(f(0)) + \mathcal{O}\Big(\max(\alpha, r_{\sigma,\varepsilon}^2)\Big) + \mathcal{O}(r_{\sigma,\varepsilon}^2),$$

where $\tau_* > 0$ *is fixed. Then there exists a unique* $v_{\sigma,\varepsilon}^\alpha (= \bar{v}_{\alpha,\sigma,\varepsilon,\tau,\eta})$ *solution of* (2.11) *we have*

$$\|v_{\sigma,\varepsilon}^\alpha\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)} \leq 2\, c_\kappa\, r_{\sigma,\varepsilon}^2$$

and

$$\mathbf{v} + v_{\sigma,\varepsilon}^\alpha = v_{1,1,\alpha} + \hbar - \log(f(0)) + H_{\eta,\theta}^i(\cdot/R_{\sigma,\varepsilon}) + \bar{v}_{\alpha,\sigma,\varepsilon,\tau,\eta}, \tag{2.18}$$

solves (2.8) *in* $B_{R_{\sigma,\varepsilon}}$. *Furthermore:*

(i) *if* $\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2) = \alpha$, *then there exist* $\alpha_0 > 0$ *and* $c > 0$ *such that for all* $\alpha \in (0, \alpha_0)$, *satisfying* $(A_{\sigma,\varepsilon}^\alpha)$, *we have*

$$\|\hbar\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2c\, \alpha,$$

(ii) *If* $\max(\alpha, r_{\sigma,\varepsilon}^2) = r_{\sigma,\varepsilon}^2$, *then there exist* $\sigma_0 > 0$, $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all* $\sigma \in (0, \sigma_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$, *satisfying* $(A_{\sigma,\varepsilon})$, *we have*

$$\|\hbar\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2c\, r_{\sigma,\varepsilon}^2.$$

Proof. We will obtain the desired existence results by means of a fixed point argument applied to $\mathcal{N}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ defined by (2.13), (2.14) and (2.15), respectively. Indeed, Let c_κ denote constants which only depend on κ (provided α, σ and ε are chosen small enough). We will prove that

$$\|\mathcal{N}(0)\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)} \leq c_\kappa\, r_{\sigma,\varepsilon}^2, \qquad \|\mathcal{F}_1(0)\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)} \leq c_\kappa\, r_{\sigma,\varepsilon}^2 \qquad \|\mathcal{F}_2(0)\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)} \leq c_\kappa\, r_{\sigma,\varepsilon}^2, \tag{2.19}$$

$$\begin{aligned} &\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)_{|\delta\in(0,\min(1,4-q)),\ q\in[1,4)}} \\ &\leq c_\kappa\, \max(\alpha, \varepsilon^{\delta/(\alpha+1)}\, r_{\sigma,\varepsilon}^2) \|v_2 - v_1\|_{\mathcal{C}^{4,\beta}_\delta(\mathbb{R}^4)}, \end{aligned} \tag{2.20}$$

$$\|\mathcal{N}(v_2)-\mathcal{N}(v_1)\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)_{|\delta\in(0,1),\ q=4}} \leq c_\kappa\ \max(\alpha,\ r^2_{\sigma,\varepsilon})\|v_2-v_1\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)}, \tag{2.21}$$

$$\|\mathcal{F}_1(v_2)-\mathcal{F}_1(v_1)\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \leq c_\kappa\, r^2_{\sigma,\varepsilon}\|v_2-v_1\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \tag{2.22}$$

and

$$\|\mathcal{F}_2(v_2)-\mathcal{F}_2(v_1)\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \leq c_\kappa\, r^2_{\sigma,\varepsilon}\ \|v_2-v_1\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \tag{2.23}$$

provided $v_1, v_2 \in \mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)$ satisfy $\|v_i\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \leq 2\, c_\kappa r^2_{\sigma,\varepsilon}$.

Let $q \in [1,4)$. From equation (2.10) and for the good estimate of η given by $|\eta| \leq c_\kappa\, r^2_{\sigma,\varepsilon}$, we have

$$\|H^i(r\ \cdot/R_{\sigma,\varepsilon})\|_{\mathcal{C}^{4,\beta}_{2}(\bar{B}_2-B_1)} \leq c_\kappa\, R^{-2}_{\sigma,\varepsilon}\,|\eta| \leq c_\kappa\,|\eta|\varepsilon^{2/(\alpha+1)}r^{-2}_{\sigma,\varepsilon} \leq c_\kappa\,\varepsilon^{2/(\alpha+1)},$$

For $|x| \leq R_{\sigma,\varepsilon}/2$, we have

$$|h(x)| \leq c_\kappa \varepsilon^{-\delta/(\alpha+1)}\ r^{\delta}_{\sigma,\varepsilon}\max(\alpha, \varepsilon^{\delta/(\alpha+1)}r^2_{\sigma,\varepsilon}) \to 0$$

as α or σ and ε tend to 0 (using the assumption $(A^{\alpha}_{\sigma,\varepsilon})$), then

$$\left\|(1+|\cdot|^{2(\alpha+1)})^{-4}\,|\cdot|^{4\alpha}\,e^{h}\ \left(e^{H^i(\cdot/R_{\sigma,\varepsilon})}-1\right)\right\|_{\mathcal{C}^{0,\beta}_{\delta-4}(\bar{B}_{R_{\sigma,\varepsilon}})} \leq c_\kappa\, r^2_{\sigma,\varepsilon}$$

and

$$\left\|(1+|\cdot|^{2(\alpha+1)})^{-4}|\cdot|^{4\alpha}\,e^{H^i(\cdot/R_{\varepsilon,\alpha})+h}\left(\frac{f((\varepsilon/\tau)^{1/(\alpha+1)}\cdot)}{f(0)}-1\right)\right\|_{\mathcal{C}^{0,\beta}_{\delta-4}(\bar{B}_{R_{\sigma,\varepsilon}})}$$
$$\leq c_\kappa\ \varepsilon^{1/(\alpha+1)} \leq c_\kappa\, r^2_{\sigma,\varepsilon}.$$

Then, the first estimate follows. For the second estimate, using the same arguments as below,

$$\|h_i\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2\, c_\kappa\ \max(\alpha,\ \varepsilon^{\delta/(\alpha+1)}r^2_{\sigma,\varepsilon})$$

and provided H^i satisfies the good estimate (2.10), then we deduce that

$$\begin{aligned}\|\mathcal{F}_1(0)\|_{\mathcal{C}^{0,\beta}_{\delta-4}(\bar{B}_{R_{\sigma,\varepsilon}})} &\leq \left\|\gamma\left(\frac{\varepsilon}{\tau}\right)^{2/(\alpha+1)}\Delta\Big(H^i(\cdot/R_{\sigma,\varepsilon})\Big)\right\|_{\mathcal{C}^{0,\beta}_{\delta-4}(\bar{B}_{R_{\sigma,\varepsilon}})}\\ &\leq c_\kappa\gamma r^{4-\delta}_{\sigma,\varepsilon}\varepsilon^{(\delta-2)/(\alpha+1)}\|H^i(\cdot/R_{\sigma,\varepsilon})\|_{\mathcal{C}^{4,\beta}_{2}(\bar{B}_{R_{\sigma,\varepsilon}})} \leq c_\kappa\,\varepsilon^{\delta/(\alpha+1)}r^2_{\sigma,\varepsilon}\end{aligned}$$

and

$$\|\mathcal{F}_2(0)\|_{\mathcal{C}^{0,\beta}_{\delta-4}(\bar{B}_{R_{\sigma,\varepsilon}})} \leq \begin{cases} c_\kappa\lambda\,\varepsilon^{\delta/(\alpha+1)} & \text{for } \varepsilon^{\delta/(\alpha+1)}r^2_{\sigma,\varepsilon} < \alpha \text{ using condition } (A^{\alpha}_{\sigma,\varepsilon}),\\ c_\kappa\lambda\,\varepsilon^{\delta/(\alpha+1)} & \text{for } \alpha < \varepsilon^{\delta/(\alpha+1)}r^2_{\sigma,\varepsilon}.\end{cases}$$

For the proof of the contraction mapping, we first recall an important result which plays an essential role in our construction (see details in [8] and some references therein).

Lemma 2.5 ([8]). *Given x and y two real numbers, $x > 0$, $q \geq 1$ and for any small $\eta \in \mathbb{R}$, there exists a positive constant C_η such that*

$$\Big||x+y|^q - x^q\Big| \leq (1+\eta)qx^{q-1}|y| + C_\eta |y|^q.$$

Indeed, making use, Lemma 2.5 and for h in $B(0, 2c_\kappa \max(\alpha, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2)$ of $\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)$, satisfying for each $x \in \bar{B}_{R_{\sigma,\varepsilon}}$, $|h(x)| \leq c_\kappa \max(\alpha, \varepsilon^{\delta/(\alpha+1)} r_{\sigma,\varepsilon}^2) \to 0$ as α or σ tends to 0, (under conditions $(A^\alpha_{\sigma,\varepsilon})$), we get the desired results.

Next, let $q = 4$. The proof is very similar. Making use of Lemma 2.5 and for $\hbar$ in $B(0, 2c_\kappa \max(\alpha, r_{\sigma,\varepsilon}^2))$ of $\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)$, satisfying for each $x \in \bar{B}_{R_{\sigma,\varepsilon}}$, $|\hbar(x)| \leq c_\kappa \max(\alpha, r_{\sigma,\varepsilon}^2) \to 0$ as α or σ tend to 0 (under conditions $(A^\alpha_{\sigma,\varepsilon})$ if $\max(\alpha, r_{\sigma,\varepsilon}^2) = \alpha$ or $(A_{\sigma,\varepsilon})$ if $\max(\alpha, r_{\sigma,\varepsilon}^2) = r_{\sigma,\varepsilon}^2$), we get the assertion. □

3. THE NONLINEAR EXTERIOR PROBLEM

Now we intend to find a solution of the equation

$$\Delta^2 v - \gamma \Delta v - \lambda |\nabla v|^q - \rho^4 |x|^{4\alpha} f(|x|) e^v = 0 \tag{3.1}$$

in $\bar{\Omega}^*_{r_{\sigma,\varepsilon}} := \bar{B}_{r_{\sigma,\varepsilon}} \backslash \{0\}$. First, we define

$$\tilde{\mathbf{v}} := (1 + \alpha + \tilde{\varrho})\, G + \chi_{r_0}\, H^e_{\tilde{\eta},\tilde{\theta}}(\cdot / r_{\sigma,\varepsilon}), \tag{3.2}$$

where $\tilde{\varrho}$ a small parameter, χ_{r_0} is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. Let $G(x)$ denote the unique solution of $\Delta^2 G = 64\pi^2 \delta_0$ in Ω, with $G = \Delta\, G = 0$ on $\partial\Omega$. Recall that the following decomposition holds

$$G(x) = -8 \log |x| + R(x),$$

where $x \longmapsto R(x)$ is a smooth function and $H^e (:= H^e_{\tilde{\eta},\tilde{\theta}})$ represents a lower order correction term (see [4, Lemma 2]), where a priori estimate for H^e is established, that satisfies orthogonality conditions with respect to ϕ_i for $i = 1, \ldots, 4$ using maximum principle. Almost every H^e is small, so that the boundary condition zero is satisfied, that is, $\Delta^2 H^e = 0$ in $\mathbb{R}^4 - B_1$, $H^e_{|\partial B_1} = \tilde{\eta}, \Delta H^e_{|\partial B_1} = \tilde{\theta}$ which decays at infinity, with

$$\|H^e_{\tilde{\eta},\tilde{\theta}}\|_{\mathcal{C}^{4,\beta}_{-1}(\mathbb{R}^4 - B_1)} \leq c|\tilde{\eta}|,$$

where, as already mentioned, $\mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4 - B_1)$ is the Hölder weighted spaces defined as the space of functions $w \in \mathcal{C}^{k,\beta}_{loc}(\mathbb{R}^4 - B_1)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\beta}_\delta(\mathbb{R}^4 - B_1)} = \sup_{r \geq 1} r^{-\delta} \|w(r\, \cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_2 - B_1)},$$

is finite. As in Section 2, by perturbation, writing $v = \tilde{\mathbf{v}} + \tilde{v}$. Then to solve (3.1) is equivalent to solve

$$\Delta^2 \tilde{v} = \rho^4 |x|^{4\alpha} f(|x|) e^{\tilde{\mathbf{v}}} e^{\tilde{v}} - \Delta^2 \tilde{\mathbf{v}} + \gamma \Delta(\tilde{\mathbf{v}} + \tilde{v}) + \lambda |\nabla(\tilde{\mathbf{v}} + \tilde{v})|^q. \tag{3.3}$$

We use the fact that the bi-Laplace operator in weighted is invertible (see [1]) and denote by $\tilde{\mathcal{G}}_{\delta,\alpha} : \mathcal{C}^{0,\beta}_{\delta-4}(\bar{\Omega}^*) \to \mathcal{C}^{4,\beta}_{\delta}(\bar{\Omega}^*)$ the right inverse of the Δ^2 with $\bar{\Omega}^* = \bar{\Omega} - \{0\}$. Applying the fixed point theorem for contraction mappings, we conclude that the following result hold:

Proposition 3.1. *Given $\kappa > 0$, $\delta \in (-1, 0)$, there exist $\sigma_\kappa > 0$, $\varepsilon_\kappa > 0$ (depending on κ), such that for all $\sigma \in (0, \sigma_\kappa)$, for all $\varepsilon \in (0, \varepsilon_\kappa)$ and for small parameter $\tilde{\eta}$, satisfying (2.16), there exists a unique $\tilde{v}(= \tilde{v}_{\sigma,\varepsilon,\tau,\tilde{\eta}})$ solution of (3.3) such that*

$$\|\tilde{v}\|_{\mathcal{C}^{4,\beta}_{\tilde{\delta}}(\bar{\Omega}^*)} \leq 2\, c_\kappa r^2_{\sigma,\varepsilon}$$

and

$$\tilde{\mathbf{v}} + \tilde{v} = (1 + \alpha + \tilde{\varrho})\, G + \chi_{r_0}\, H^e_{\tilde{\gamma},\tilde{\delta}}(\cdot/r_{\sigma,\varepsilon}) + \tilde{v}_{\sigma,\varepsilon,\tau,\tilde{\eta}} \tag{3.4}$$

is a solution of (3.1) in $\bar{\Omega}^*$.

4. THE NONLINEAR CAUCHY-DATA MATCHING

We gather the results of the previous sections, keeping the notations. According to Propositions 2.3 and Proposition 2.4, and the expansions given by (2.17) and (2.18), we can find in each $B_{r_{\sigma,\varepsilon}}$ a solution of

$$\Delta^2 v - \gamma \Delta v - \lambda |\nabla v|^q - \rho^4\, |x|^{4\alpha} f(|x|) e^v = 0, \tag{4.1}$$

which can be decomposed as

$$v_{int}(x) = v_{\varepsilon,\tau,\alpha}(x) + h\,(R_{\sigma,\varepsilon} x / r_{\sigma,\varepsilon}) - \log f(0) + H^i_{\eta,\theta}(x/R_{\sigma,\varepsilon}) + \bar{v}_{\alpha,\sigma,\varepsilon,\tau}(R_{\sigma,\varepsilon} x / r_{\sigma,\varepsilon})$$

in $B_{r_{\sigma,\varepsilon}}$, where the function $v^\alpha_{\sigma,\varepsilon} = \bar{v}_{\alpha,\sigma,\varepsilon,\tau,\eta}$ satisfies $\|v^\alpha_{\sigma,\varepsilon}\|_{\mathcal{C}^{4,\beta}_{\delta}(\mathbb{R}^4)} \leq 2\, c_\kappa r^2_{\sigma,\varepsilon}$,

$$\|h\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2\, c_\kappa\, \max(\alpha, \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon}),\ \text{for } q \in [1, 4) \text{ and } \delta \in \Big(0, \min(1, 4-q)\Big)$$

and for $h := \hbar$, we have

$$\|\hbar\|_{\mathcal{C}^{4,\beta}_{rad,\delta}(\mathbb{R}^4)} \leq 2\, c_\kappa\, \max(\alpha, r^2_{\sigma,\varepsilon}), \quad \text{for } q = 4 \text{ and } \delta \in (0, 1).$$

Similarly, we use Proposition 3.1 and the relation (3.4) to find in $\bar{\Omega}^*$ a solution v_{ext} of (4.1) which can be decomposed as

$$v_{ext} = (1 + \alpha + \tilde{\varrho})\, G + \chi_{r_0}\, H^e_{\tilde{\eta},\tilde{\theta}}(\cdot/r_{\sigma,\varepsilon}) + \tilde{v}_{\sigma,\varepsilon,\tau,\tilde{\eta}},$$

where $\tilde{v}_{\sigma,\varepsilon} \in \mathcal{C}^{4,\beta}_{\tilde{\delta}}(\bar{\Omega}^*)$ satisfies $\|\tilde{v}_{\sigma,\varepsilon}\|_{\mathcal{C}^{4,\beta}_{\tilde{\delta}}(\bar{\Omega}^*)} \leq 2c_\kappa\, r^2_{\sigma,\varepsilon}$ for $\tilde{\delta} = -\delta \in (-1, 0)$. Then

$$\begin{aligned}(v_{int} - v_{ext})(x) = &-4 \log \tau + 8\tilde{\varrho}\, \log|x| + H^i_{\eta,\theta}(x/r_{\sigma,\varepsilon}) \\ &- H^e_{\tilde{\eta},\tilde{\theta}}(x/r_{\sigma,\varepsilon}) - (1+\alpha) R(0) - \log(f(0)) \\ &+ \mathcal{O}(r^2_{\sigma,\varepsilon}) + \mathcal{O}\big(\max(\alpha, \varepsilon^{\delta/(\alpha+1)} r^2_{\sigma,\varepsilon})\big).\end{aligned} \tag{4.2}$$

Also, it remains to determine the parameters $\lambda, \gamma, \eta, \theta, \tilde{\eta}, \tilde{\theta}, \tilde{\varrho}$ in such a way that the function which is equal to v_{int} in $B_{r_{\sigma,\varepsilon}}$ and which is equal to v_{ext} in $\bar{\Omega}_{r_{\sigma,\varepsilon}}$ is smooth. This amounts to solve

$$v_{int} = v_{ext}, \quad \partial_r v_{int} = \partial_r v_{ext}, \quad \Delta v_{int} = \Delta v_{ext}, \quad \partial_r \Delta v_{int} = \partial_r \Delta v_{ext}, \tag{4.3}$$

near $\partial B_{r_{\sigma,\varepsilon}}$. Then the system we have to solve reads as follows:

$$(\lambda, \gamma, \eta, \theta, \tilde{\eta}, \tilde{\theta}, \tilde{\varrho}) = \mathcal{O}\Big(\max(\alpha, r_{\sigma,\varepsilon}^2)\Big). \tag{4.4}$$

The nonlinear mapping (4.4) sends the ball of radius $\kappa \max(\alpha, r_{\sigma,\varepsilon}^2)$ (for the natural product norm) into itself, provided κ is fixed large enough. Applying the Schauder fixed point theorem in the ball of radius $\kappa \max(\alpha, r_{\sigma,\varepsilon}^2)$ in the product space, where the entries live the existence of a solution of equation (1.6) and a function $w_{\sigma,\varepsilon}^{\alpha} \in \mathcal{C}^{4,\beta}$ (which is obtained by patching together the functions v_{int} and the function v_{ext}) solution of our equation and elliptic regularity theory implies that this solution is in fact smooth, and the sequence of solutions we have obtained satisfy the required properties. Namely, away from the point 0 the sequence $w_{\sigma,\varepsilon}^{\alpha}$ converges to $(1+\alpha)G$ if σ and ε tend to 0 or $w_{\sigma,\varepsilon}^{\alpha}$ converges to G if α tends to 0 (using conditions $(A_{\sigma,\varepsilon}^{\alpha})$ and $(A_{\sigma,\varepsilon})$). This will completes the proof of Theorem1.3.

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