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NONOSCILLATION OF DAMPED LINEAR DIFFERENTIAL EQUATIONS WITH A PROPORTIONAL DERIVATIVE CONTROLLER AND ITS APPLICATION TO WHITTAKER–HILL-TYPE AND MATHIEU-TYPE EQUATIONS

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Abstract. The proportional derivative (PD) controller of a differential operator is commonly referred to as the conformable derivative. In this paper, we derive a nonoscillation theorem for damped linear differential equations with a differential operator using the conformable derivative of control theory. The proof of the nonoscillation theorem utilizes the Riccati inequality corresponding to the equation considered. The provided nonoscillation theorem gives the nonoscillatory condition for a damped Euler-type differential equation with a PD controller. Moreover, the nonoscillation of the equation with a PD controller that can generalize Whittaker–Hill-type equations is also considered in this paper. The Whittaker–Hill-type equation considered in this study also includes the Mathieu-type equation. As a subtopic of this work, we consider the nonoscillation of Mathieu-type equations with a PD controller while making full use of numerical simulations.

Keywords: nonoscillation, proportional derivative controller, Riccati technique, Mathieu equation, Whittaker–Hill equation.

Mathematics Subject Classification: 34C10, 26A24, 34B30.

1. INTRODUCTION

In recent years, formulations of operators that can be interpreted as generalizations of derivatives and integrals have been actively introduced. For example, the fractional derivative may be a recent theme (see [1, 8, 12, 13, 15, 19]). More recently, a generalized derivative independent of the theory of fractional derivatives has been defined by Anderson and Ulness [6]. The definition is given below.

Definition 1.1. Let $\alpha \in [0,1]$ and functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that

$$\begin{cases} \kappa_0(0,t) = 0, \quad \kappa_1(0,t) = 1, \\ \kappa_0(1,t) = 1, \quad \kappa_1(1,t) = 0, \\ \kappa_0(\alpha,t) \neq 0, \quad \alpha \in (0,1], \quad \kappa_1(\alpha,t) \neq 0, \quad \alpha \in [0,1). \end{cases} \tag{1.1}$$

Define the differential operator D^α via

$$D^\alpha f(t) = \kappa_1(\alpha,t) f(t) + \kappa_0(\alpha,t) \frac{d}{dt} f(t). \tag{1.2}$$

Remark 1.2. From (1.1), we see that

$$D^0 f(t) = f(t) \quad \text{and} \quad D^1 f(t) = f'(t).$$

According to Anderson and Ulness [6], the impetus for the introduction of (1.2) was the proportional derivative (PD) controller formula used in control theory. For a controller output Y at time t, a PD controller with two tuning parameters follows the below algorithm:

$$Y(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t). \tag{1.3}$$

Here, κ_p is the proportional gain, κ_d is the derivative gain, and E is the input deviation. Note that (1.3), used in PD control systems, is applied to analyze robotics [2, 10]. Thus, (1.2) is called the *proportional derivative* or *conformable derivative* in control theory.

This study deals with the nonoscillation of the damped linear differential equation with a PD controller, as given below:

$$D^\alpha D^\alpha u + a(t) D^\alpha u + b(t) u = 0, \tag{1.4}$$

where $\alpha \in (0,1]$; $a, b : [0,\infty) \to \mathbb{R}$ are continuous on $\mathbb{R}$.

Eq. (1.4) is equivalent to the Sturm–Liouville type equation:

$$D^\alpha [r(t) D^\alpha u] + c(t) u = 0, \tag{1.5}$$

where $r(t) = e_{a+\kappa_1}(t, t_0) > 0$ for $t \geq t_0 \geq 0$ and $c(t) = r(t) b(t)$. Here, $e_{a+\kappa_1}(t, t_0)$ is an exponential function used in the proportional derivative (see Theorem 4.2 of Appendix for details). Anderson's result can be referred for the equivalence transformation [3, Theorem 3.1]. Recently, with respect to these fundamentals, a qualitative theory for Eq. (1.5) and Eq. (1.5) on the time scale has been developed [3–5, 9].

The nontrivial oscillatory and nonoscillatory solutions are defined as (1.4) and (1.5), respectively. For sufficiently large $t \geq t_0 \geq 0$, a nontrivial solution u of (1.4) (or (1.5)) is said to be *nonoscillatory* on $[t_0, \infty)$ if it is eventually either positive or negative. Conversely, the nontrivial solution u of (1.4) (or (1.5)) is said to be *oscillatory*. Sturm's comparison theorem has already been developed for (1.5) (see Anderson's result [3, Theorem 7.4]). Additionally, Sturm's separation theorem for linear dynamic equations on the time scale involving (1.5) has been established

[4, Theorem 8.3.6]. Accordingly, if a nontrivial solution of (1.4) (or (1.5)) is oscillatory or nonoscillatory, then all nontrivial solutions of (1.4) (or (1.5)) are oscillatory or nonoscillatory, respectively.

Let $\alpha = 1$. Then, Eq. (1.4) becomes the second-order linear differential equation

$$u'' + a(t)u' + b(t)u = 0. \tag{1.6}$$

There have been a large number of results on oscillation problems for (1.6). Among them, an excellent nonoscillation theorem for Eq. (1.6) is given by Zafer [24].

Theorem 1.3 ([24]). *Suppose there exists a differentiable function B such that $B'(t) = b(t)$. If*

$$(B(t) - a(t))B(t) \le 0 \tag{1.7}$$

for large t, then all nontrivial solutions of (1.6) *are nonoscillatory.*

The pioneering work on Theorem 1.3 was undertaken by Kwong and Wong [16, Theorem 2]. They assumed periodic functions for the coefficients a and b in Eq. (1.6). The results of Kwong and Wong [16, Theorem 2] corresponding to linear differential equations have been generalized by many subsequent researchers. In particular, Zafer [24] established a nonoscillation theorem for linear dynamic equations on time scales that removed the periodicity of the coefficient functions a and b. Theorem 1.3 is an excerpt from a small part of the results of Zafer's nonoscillation theorem. Several other researchers have extended the results of Kwong and Wong [16] for application to half-linear differential equations and impulsive differential equations [7, 11, 20–22].

The purpose of this study is to establish Theorem 1.3 corresponding to (1.4). The Riccati inequality corresponding to (1.4) is necessary to achieve the objective of this study. However, the Riccati inequality corresponding to (1.4) has not yet been published, and will be established in this paper. As an example of the new nonoscillation theorem obtained in this study, we consider an equation that generalizes the damped Euler-type differential equation

$$u'' + a(t)u' + \frac{1}{t^2}u = 0. \tag{1.8}$$

From Theorem 1.3, if $a \le -1/t$, then all nontrivial solutions of (1.8) are nonoscillatory. We generalize this to correspond to (1.4).

As another example, consider the following linear differential equation

$$u'' + \left(-\frac{1}{8} + \frac{1}{2}\cos t + \frac{1}{8}\cos(2t)\right)u = 0. \tag{1.9}$$

Equation (1.9) is a special case of the Whittaker–Hill-type equation [17, 23]. In particular, the solution u of (1.9) is $e^{\frac{1}{2}\cos t}$. This can be easily verified and the solution of (1.9) is clear. Hence, the solution u of (1.9) is nonoscillatory. In this study, we discuss the nonoscillation of the generalized equation (1.9) with a PD controller.

In Section 2, this paper introduces the nonoscillation theorem corresponding to (1.4), which extends Theorem 1.3. To prove the nonoscillation theorem, we introduce the Riccati inequality corresponding to (1.4). Moreover, we give the nonoscillation condition for a damped Euler-type equation with a PD controller that generalizes Eq. (1.8). In Section 3, we discuss the nonoscillation of a generalized Whittaker–Hill-type equation using a new nonoscillation theorem. As a reminder, the Whittaker–Hill-type equation (1.8) has no damping coefficient a. In order to change the result of (1.4) to the undamped equation (1.4), we clarify the equivalence of (1.4) and the undamped equation (1.4). From the equivalence, we discuss the nonoscillation of the Whittaker–Hill-type equation with a PD controller. In particular, the Whittaker–Hill-type equation includes the Mathieu-type equation. The Mathieu equation is a well-known mathematical model that appears in the fields of science and engineering. In Section 3, we consider the nonoscillation of the Mathieu-type equation with a PD controller, especially using numerical simulation. Section 4 contains a brief conclusion of this paper and a future open question. As an appendix to this paper, the last section of this paper summarizes the arithmetic formulas for the proportional derivatives used in this paper.

2. RICCATI TYPE PROPORTIONAL DERIVATIVE INEQUALITY AND NONOSCILLATION THEOREM

Consider the Riccati type proportional derivative inequality

$$D^\alpha z \geq b(t) - a(t)z + z(z + \kappa_1(\alpha, t)). \tag{2.1}$$

We have the following result:

Theorem 2.1. *The following conditions are equivalent.*

(i) *All nontrivial solutions to* (1.4) *are nonoscillatory.*
(ii) *There exists a differentiable function z that satisfies* (2.1) *for large t.*

Proof. To prove that (i) implies (ii), we need to find a solution z that satisfies (2.1). Assuming that a solution u of Eq. (1.4) is nonoscillatory, define

$$z(t) = -\frac{D^\alpha u(t)}{u(t)},$$

then, by using

$$z(t)+\kappa_1(\alpha,t) = -\frac{D^\alpha u(t)}{u(t)}+\kappa_1(\alpha,t) = -\frac{1}{u(t)}(D^\alpha u(t)-\kappa_1(\alpha,t)u(t)) = -\kappa_0(\alpha,t)\frac{u'(t)}{u(t)}$$

and

$$u'(t) = -\frac{(z(t)+\kappa_1(\alpha,t))u(t)}{\kappa_0(\alpha,t)},$$

we obtain

$$\begin{aligned}
&D^\alpha D^\alpha u(t) + a(t)D^\alpha u(t) + b(t)u(t)\\
&= D^\alpha[-z(t)u(t)] - a(t)z(t)u(t) + b(t)u(t)\\
&= -D^\alpha z(t)u(t) - z(t)\big(\kappa_1(\alpha,t)u(t) + \kappa_0(\alpha,t)u'(t)\big)\\
&\quad + \kappa_1(\alpha,t)z(t)u(t) - a(t)z(t)u(t) + b(t)u(t)\\
&= \Big(-D^\alpha z(t) + b(t) - a(t)z(t) + z(t)\big(z(t)+\kappa_1(\alpha,t)\big)\Big)u(t) = 0.
\end{aligned}$$

Hence, z is a solution of the inequality (2.1).

To prove that (ii) implies (i), we show that a nontrivial solution to Eq. (1.4) is nonoscillatory. Let z be a solution of (2.1). Define

$$d(t) := D^\alpha z(t) + a(t)z(t) - b(t) - z(t)(z(t) + \kappa_1(\alpha,t)) \geq 0.$$

We consider the equation

$$D^\alpha D^\alpha u + a(t)D^\alpha u + (b(t) + d(t))u = 0. \tag{2.2}$$

Using formulas for the proportional derivatives, we observe that Eq. (2.2) has the nonoscillatory solution $u(t) = e_{-z}(t,t_0)$. Now (2.2) may be written as follows:

$$D^\alpha[e_{a+\kappa_1}(t,t_0)D^\alpha u] + e_{a+\kappa_1}(t,t_0)(b(t)+d(t))u = 0, \tag{2.3}$$

(see Anderson's result [3, Theorem 3.1]). For (2.3), upon comparing the Strum–Liouville type equation (1.5) by Sturm's comparison theorem [3, Corollary 7.5], Eq. (1.5) must be nonoscillatory. In other words, we find that all nontrivial solutions to (1.4) are nonoscillatory. □

From Theorem 2.1, we obtain the following nonoscillation theorem for (1.4).

Theorem 2.2. *Assuming there exists a differentiable function B such that $D^\alpha B(t){=}b(t)$. If*

$$(B(t) - a(t) + \kappa_1(\alpha,t))B(t) \leq 0 \tag{2.4}$$

for large t, then all nontrivial solutions of (1.4) *are nonoscillatory.*

Proof. If $D^\alpha B(t)$ is added to both sides of (2.4), then

$$D^\alpha B(t) \geq b(t) - a(t)B(t) + B(t)(B(t) + \kappa_1(\alpha,t))$$

is obtained. Hence, $B(t)$ satisfies (2.1). From Theorem 2.1, the proof of Theorem 2.2 is complete. □

Remark 2.3. For $\alpha = 1$, Theorem 2.2 satisfies Theorem 1.3. Hence, Theorem 2.2 is a generalization of Theorem 1.3.

Example 2.4. As an example of Theorem 2.2, consider a damped Euler-type equation with a PD controller

$$D^\alpha D^\alpha u + a(t) D^\alpha u + \frac{\kappa_0(\alpha,t)e_0(t,t_0)}{t^2} u = 0, \quad t \geq t_0 > 0, \tag{2.5}$$

where

$$e_0(t,t_0) = e^{-\int_{t_0}^{t} \frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau}.$$

Note that if $\alpha = 1$, then Eq. (2.5) becomes Eq. (1.8). For (2.5), the following results can be obtained using Theorem 2.2.

If

$$a(t) \leq -\frac{e_0(t,t_0)}{t} + \kappa_1(\alpha,t), \tag{2.6}$$

then all nontrivial solutions of (2.5) are nonoscillatory.

Let $B(t) = -e_0(t,t_0)/t$. Then, using $D^\alpha e_0(t,t_0) = 0$ and the product rule of proportional derivative (see Theorem 5.1 of Appendix for details), we see that

$$D^\alpha B(t) = \frac{\kappa_0(\alpha,t)e_0(t,t_0)}{t^2}.$$

From (2.6), we have

$$\begin{aligned}(B(t) - a(t) + \kappa_1(\alpha,t))B(t) &= \frac{e_0^2(t,t_0)}{t^2} - \frac{\kappa_1(\alpha,t)e_0(t,t_0)}{t} + \frac{a(t)e_0(t,t_0)}{t} \\ &\leq \frac{e_0^2(t,t_0)}{t^2} - \frac{\kappa_1(\alpha,t)e_0(t,t_0)}{t} \\ &\quad - \frac{e_0^2(t,t_0)}{t^2} + \frac{\kappa_1(\alpha,t)e_0(t,t_0)}{t} = 0.\end{aligned}$$

Thus, the condition (2.3) of Theorem 2.2 is satisfied.

Remark 2.5. In the case that $\alpha = 1$, condition (2.6) becomes $a \leq -1/t$. Hence, we have generalized the nonoscillatory condition in Eq. (1.8).

3. WHITTAKER–HILL-TYPE AND MATHIEU-TYPE EQUATIONS WITH A PD CONTROLLER

Consider the undamped linear differential equation with a PD controller

$$D^\alpha D^\alpha v + c(t)v = 0 \tag{3.1}$$

more general than (1.8). Here, $c : [0,\infty) \to \mathbb{R}$ is continuous on $\mathbb{R}$.

Eq. (3.1) and Eq. (1.4) have the following equivalence relation.

Theorem 3.1. *The following conditions are equivalent.*

(i) *Let a be a continuously differentiable function. All nontrivial solutions of* (1.4) *are nonoscillatory.*

(ii) *All nontrivial solutions of the undamped equation* (3.1) *are nonoscillatory, where*

$$\begin{aligned} c(t) &= b(t) - \frac{1}{2}D^{\alpha}a(t) - \frac{1}{4}a^2(t) + \frac{1}{2}a(t)\kappa_1(\alpha, t) \\ &= b(t) - \frac{\kappa_0(\alpha, t)}{2}a'(t) - \frac{1}{4}a^2(t). \end{aligned} \tag{3.2}$$

Proof. To prove that (i) implies (ii), let

$$u(t) = v(t)e_{-\frac{a}{2}+\kappa_1}(t, t_0).$$

Then, from

$$D^{\alpha}u(t) = \left(D^{\alpha}v(t) - \frac{a(t)v(t)}{2}\right)e_{-\frac{a}{2}+\kappa_1}(t, t_0)$$

and

$$\begin{aligned} D^{\alpha}D^{\alpha}u(t) &= \left(D^{\alpha}D^{\alpha}v(t) - a(t)D^{\alpha}v(t)\right)e_{-\frac{a}{2}+\kappa_1}(t, t_0) \\ &\quad + \left(\frac{1}{2}a(t)\kappa_1(\alpha, t) - \frac{1}{2}D^{\alpha}a(t) + \frac{1}{4}a^2(t)\right)v(t)e_{-\frac{a}{2}+\kappa_1}(t, t_0), \end{aligned}$$

we see that Eq. (1.4) becomes Eq. (3.1) with (3.2). Therefore, if a nontrivial solution of (1.4) is nonoscillatory, then a nontrivial solution of (3.1) with (3.2) is nonoscillatory. Here, from Sturm's separation theorem, if a nontrivial solution of (3.1) with (3.2) is nonoscillatory, then all nontrivial solutions of (3.1) with (3.2) are nonoscillatory.

For the proof of (ii) through (i), we put $v(t) = u(t)e_{\frac{a}{2}+\kappa_1}(t, t_0)$. From

$$D^{\alpha}v(t) = \left(D^{\alpha}u(t) + \frac{a(t)u(t)}{2}\right)e_{\frac{a}{2}+\kappa_1}(t, t_0)$$

and

$$\begin{aligned} D^{\alpha}D^{\alpha}v(t) &= \left(D^{\alpha}D^{\alpha}u(t) + a(t)D^{\alpha}u(t)\right)e_{\frac{a}{2}+\kappa_1}(t, t_0) \\ &\quad + \left(\frac{\kappa_0(\alpha, t)}{2}a'(t) + \frac{1}{4}a^2(t)\right)u(t)e_{\frac{a}{2}+\kappa_1}(t, t_0), \end{aligned}$$

we see that Eq. (3.1) with (3.2) becomes Eq. (1.4). Therefore, if a nontrivial solution of (3.1) with (3.2) is nonoscillatory, then all nontrivial solutions of (1.4) are nonoscillatory. □

Now consider a Whittaker–Hill-type equation with a PD controller

$$D^{\alpha}D^{\alpha}v + \left(-\delta(t) + \left(\gamma - \frac{1}{2}\right)e_0(t, t_0)\kappa_0(\alpha, t)\cos t + \frac{e_0^2(t, t_0)}{8}\cos(2t)\right)v = 0, \tag{3.3}$$

which is a generalization of (1.9). Here, γ is a real number; the function δ is

$$\delta(t) = \frac{e_0^2(t,t_0)}{8} + \frac{\kappa_1^2(\alpha,t)}{4} + \frac{\kappa_0(\alpha,t)\kappa_1'(\alpha,t)}{2}.$$

Note that Eq. (3.3) becomes Eq. (1.9) if $\alpha = 1$ and $\gamma = 1$. Moreover, in the case that $\gamma = 1/2$, Eq. (3.3) becomes the Mathieu-type equation with a PD controller

$$D^\alpha D^\alpha v + \left(-\delta(t) + \frac{e_0^2(t,t_0)}{8}\cos(2t)\right) v = 0. \tag{3.4}$$

Note that Mathieu's equation is one of the most famous equations applied in the fields of mechanics and electrical engineering [18].

Using Theorems 2.2 and 3.1, the following result can be obtained.

Theorem 3.2. *Assume that κ_1 is differentiable for $t \geq 0$. If $0 \leq \gamma \leq 1$, then all nontrivial solutions of* (3.3) *are nonoscillatory.*

Proof. Let us compare Eq. (3.3) with the damped differential equation with a PD controller:

$$D^\alpha D^\alpha u + (e_0(t,t_0)\sin t + \kappa_1(\alpha,t))D^\alpha u + \gamma e_0(t,t_0)\kappa_0(\alpha,t)\cos t\, u = 0. \tag{3.5}$$

If we compare the coefficient functions of (1.4) and (3.5), they are

$$a(t) = e_0(t,t_0)\sin t + \kappa_1(\alpha,t) \quad \text{and} \quad b(t) = \gamma e_0(t,t_0)\kappa_0(\alpha,t)\cos t.$$

Then, Eqs. (3.5) and (3.3) are equivalent. Indeed, from

$$a'(t) = -\frac{\kappa_1(\alpha,t)e_0(t,t_0)}{\kappa_0(\alpha,t)}\sin t + e_0(t,t_0)\cos t + \kappa_1'(\alpha,t)$$

and

$$a^2(t) = e_0^2(t,t_0)\sin^2 t + 2\kappa_1(\alpha,t)e_0(t,t_0)\sin t + \kappa_1^2(\alpha,t),$$

we see that

$$\begin{aligned}
b(t) - \frac{\kappa_0(\alpha,t)}{2}a'(t) - \frac{1}{4}a^2(t) &= \gamma e_0(t,t_0)\kappa_0(\alpha,t)\cos t + \frac{e_0(t,t_0)\kappa_1(\alpha,t)}{2}\sin t \\
&\quad - \frac{\kappa_0(\alpha,t)e_0(t,t_0)}{2}\cos t - \frac{\kappa_0(\alpha,t)\kappa_1'(\alpha,t)}{2} \\
&\quad - \frac{e_0^2(t,t_0)}{4}\sin^2 t - \frac{e_0(t,t_0)\kappa_1(\alpha,t)}{2}\sin t - \frac{\kappa_1^2(\alpha,t)}{4} \\
&= -\left(\frac{e_0^2(t,t_0)}{8} + \frac{\kappa_1^2(\alpha,t)}{4} + \frac{\kappa_0(\alpha,t)\kappa_1'(\alpha,t)}{2}\right) \\
&\quad + \left(\gamma - \frac{1}{2}\right)e_0(t,t_0)\kappa_0(\alpha,t)\cos t + \frac{e_0^2(\alpha,t)}{8}\cos 2t \\
&= -\delta(t) + \left(\gamma - \frac{1}{2}\right)e_0(t,t_0)\kappa_0(\alpha,t)\cos t \\
&\quad + \frac{e_0^2(\alpha,t)}{8}\cos 2t.
\end{aligned}$$

Hence, the relation (3.2) is satisfied. If all nontrivial solutions of (3.5) are nonoscillatory, then all nontrivial solutions of (3.3) are also nonoscillatory. We use Theorem 2.2 to show that all nontrivial solutions of (3.5) are nonoscillatory.

If $B(t) = \gamma e_0(t, t_0) \sin t$ for $t > 0$, then we can find that

$$\begin{aligned} D^\alpha B(t) &= \gamma(\kappa_1(\alpha, t) \sin t + \kappa_0 \cos t) e_0(t, t_0) - \gamma \kappa_1(\alpha, t) e_0(t, t_0) \sin t \\ &= \gamma \kappa_0(\alpha, t) e_0(t, t_0) \cos t. \end{aligned}$$

Furthermore, from $0 \leq \gamma \leq 1$, we obtain

$$\begin{aligned} (B(t) - a(t) + \kappa_1(\alpha, t))B(t) &= \Big(\gamma e_0(t, t_0) \sin t - \big(e_0(t, t_0) \sin t + \kappa_1(\alpha, t)\big) \\ &\quad + \kappa_1(\alpha, t)\Big) \gamma e_0(t, t_0) \sin t \\ &= (\gamma - 1)\, \gamma\, e_0^2(t, t_0) \sin^2 t \leq 0 \end{aligned}$$

Then, the above inequality satisfies (2.1). Hence, all nontrivial solutions of (3.5) are nonoscillatory. In other words, all nontrivial solutions of (3.3) are nonoscillatory. □

Remark 3.3. In the case that $\gamma = 1$ and $\alpha = 1$, from Theorem 3.2, we see that all nontrivial solutions of (1.9) are nonoscillatory. In another case that $\gamma = 1/2$, all nontrivial solutions of (3.4) are nonoscillatory.

Now, for the special case of (3.4), if we change the parameter α, the nonoscillatory solution of the special case of (3.4) may have bifurcated results. Using numerical simulations, we will discuss the nonoscillatory solutions to (3.4) when $\gamma = 1/2$. Here, let $w = D^\alpha v$,

$$\kappa_0(\alpha, t) = \sin\left(\frac{\alpha\pi}{2}\right) \quad \text{and} \quad \kappa_1(\alpha, t) = \cos\left(\frac{\alpha\pi}{2}\right).$$

Then, Eq. (3.4) becomes

$$\begin{aligned} D^\alpha v &= w, \\ D^\alpha w &= \left(\delta(t) - \frac{e_0^2(t, t_0)}{8} \cos(2t)\right) v. \end{aligned} \tag{3.6}$$

In Figure 1, we draw the positive orbits of (3.6) starting at point $(1, 1)$. In the cases where $\alpha = 1, 0.9, 0.8$, the positive orbits of (3.6) are divergent. Meanwhile, when $\alpha = 0.7, 0.6, 0.5$, the positive orbits of (3.6) are asymptotic to the origin. In Figure 2, similarly, the solution curves of (3.4) are divergent if $\alpha = 1, 0.9, 0.8$. Furthermore, the solution curves of (3.4) are asymptotic to the origin if $\alpha = 0.7, 0.6, 0.5$. From Figures 1 and 2, it can be confirmed that the solutions of (3.4) and (3.6) are nonoscillatory.

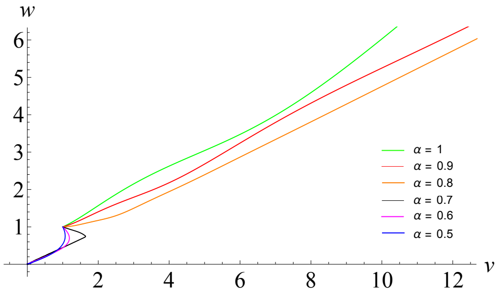

Fig. 1. The positive orbits of (3.6)

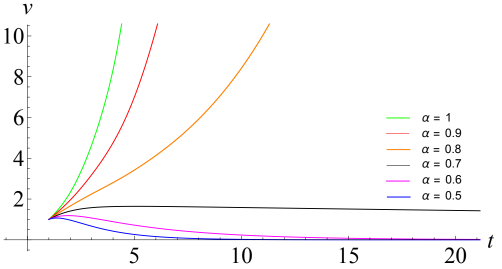

Fig. 2. The solution curves of (3.4)

4. CONCLUSION

In this study, we established the Riccati inequality corresponding to (1.4), which is necessary to derive the nonoscillation theorem. Moreover, the nonoscillatory solutions of Mathieu-type and Whittaker–Hill-type equations with a PD controller are discussed (see Section 3). According to Figures 1 and 2, the solution that does not oscillate is divided into divergent solutions and the solutions that converge to the origin. Depending on the value of parameter α, one question arises: what is the parameter condition for α that classifies diverging and converging nonoscillatory solutions? For example, we consider $\alpha = 0.705$ and $\alpha = 0.704$. In Figure 3, we draw the two positive orbits of (3.6) starting at point $(1, 1)$. In the case that $\alpha = 0.705$, a positive orbit of (3.6) is divergent. Meanwhile, in the case that $\alpha = 0.704$, a positive orbit of (3.6) is asymptotic to the origin. Thus, the following results can be predicted.

Conjecture 4.1. Suppose there exists α_* such that $0.704 < \alpha_* < 0.705$. If $\alpha < \alpha_*$, then all nontrivial solutions of (3.4) are asymptotic to the origin. If $\alpha > \alpha_*$, then all nontrivial solutions of (3.4) are divergent. Here,

$$\kappa_0(\alpha, t) = \sin\left(\frac{\alpha\pi}{2}\right) \quad \text{and} \quad \kappa_1(\alpha, t) = \cos\left(\frac{\alpha\pi}{2}\right).$$

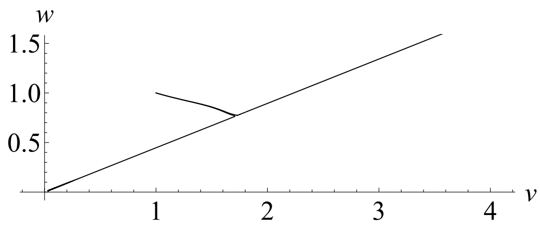

Fig. 3. The positive orbits of (3.6) for $\alpha = 0.705$ and $\alpha = 0.704$

In the future, the classification of nonoscillatory solutions of (3.4) may also be an interesting study.

5. APPENDIX

The fundamentals of the proportional derivative (1.2) defined by Anderson and Ulness [6] using (1.1) are summarized below.

Theorem 5.1 ([6]). *Let $\alpha \in (0, 1]$, the points $s, t \in \mathbb{R}$ with $s \le t$, and let the function $\phi : [s, t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy* (1.1)*, with ϕ/κ_0 and κ_1/κ_0 being Riemann integrable on $[s, t]$. Then, the exponential function with respect to D^α in* (1.2) *is defined as*

$$e_\phi(t, s) := e^{\int_s^t \frac{\phi(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \quad e_0(t, s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}$$

and

$$D^\alpha e_\phi(t, s) = \phi(t) e_\phi(t, s), \quad D^\alpha e_0(t, s) = 0.$$

Theorem 5.2 ([6]). *Let the proportional derivative D^α be given as* (1.2)*, where $\alpha \in [0, 1]$. Let the function $\phi : [s, t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy* (1.1)*. Assume the functions f and g are differentiable as needed. Then:*

(i) $D^\alpha[kf(t) + lg(t)] = kD^\alpha f(t) + lD^\alpha g(t)$ *for all* $k, l \in \mathbb{R}$,
(ii) $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t) - f(t)g(t)\kappa_1(\alpha, t)$,
(iii) $D^\alpha[f(t)/g(t)] = \dfrac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g^2(t)} + \dfrac{f(t)}{g(t)}\kappa_1(\alpha, t)$.

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