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NEW RESULTS ON IMBALANCE GRAPHIC GRAPHS

Sergiy Kozerenko and Andrii Serdiuk

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Abstract. An edge imbalance provides a local measure of how irregular a given graph is. In this paper, we study graphs with graphic imbalance sequences. We give a new proof of imbalance graphicness for trees and use the new idea to prove that the same holds for unicyclic graphs. We then show that antiregular graphs are imbalance graphic and consider the join operation on graphs as well as the double graph operation. Our main results are concerning imbalance graphicness of three classes of block graphs: block graphs having all cut vertices in a single block; block graphs in which the subgraph induced by the cut vertices is either a star or a path. In the end, we discuss open questions and conjectures regarding imbalance graphic graphs.

Keywords: edge imbalance, irregularity of a graph, imbalance sequence, graphic sequence.

Mathematics Subject Classification: 05C07, 05C70, 05C99.

1. INTRODUCTION

A regular graph is a graph having the same degree for every its vertex. One approach to measure how irregular a graph can be is to consider the multiset of its edge imbalances (the so-called imbalance sequence M_G of a graph G) [1]. Here, an imbalance of an edge is the absolute difference of the degrees of its vertices. Accordingly, the sum of all edge imbalances in a graph G is called its irregularity and denoted by $I(G)$. Various upper and lower bounds on $I(G)$ for general graphs and for special graph classes as well as the characterizations of graphs which attain extremal values of $I(G)$ can be found in [1, 4, 8, 15, 16, 18]. On the other hand, from a qualitative point of view, there are several interesting classes of "irregular" graphs: highly irregular [6] (graphs where for any vertex all its neighbors have distinct degrees) and stepwise irregular [9] (graphs in which the imbalance of every edge is one). For other aspects of irregularity in graphs we refer to the book [3].

Another approach in studying the multiset of edge imbalances was taken in [11], where the authors studied the graphicness of imbalance sequence. A graph G is called imbalance graphic provided its imbalance sequence M_G is graphic (which means that there is another graph H whose multiset of vertex degrees equals M_G). The imbalance graphicness of the following classes of graphs was established in [11]: trees, a particular case of split graphs (where the corresponding independent set consists of leaf vertices) as well as the so-called complete extensions of paths, cycles and complete graphs.

It was also conjectured in [11] that graphs having no edge with zero imbalance are necessarily imbalance graphic, which still is an open question. The second imbalance conjecture from [11] was motivated by the fact that the set of mean imbalances of imbalance non-graphic graphs is dense in $[0, 2]$. Hence, it was conjectured that the inequality $I(G) \geq 2|E(G)|$ implies that G is imbalance graphic. In this paper, we disprove this conjecture by constructing a counterexample with a given number of vertices $n \geq 9$. This line of research expanded in [10], where unary (subdivision and the complete extension of a graph) and binary (various graph products, the corona of graphs as well as the splice and the link of rooted graphs) operations on graphs which preserve imbalance graphicness were considered.

This paper is organized as follows. At first, we give basic definitions and results in the second section. In the third section we provide a new method of proving imbalance graphicness of trees and apply it to show that the same holds for unicyclic graphs. We then proceed by showing that antiregular graphs are imbalance graphic and consider the joins with complete as well as with empty graphs. In the end of the third section we prove that the double graph operation preserves the imbalance graphicness.

The fourth section of the paper is devoted to establishing the imbalance graphicness of block graphs (which are natural generalization of trees) from the following three classes: block graphs having all cut vertices in a single block; block graphs in which the subgraph induced by the cut vertices is either a star or a path. These results were announced at the 43rd Australiasian Combinatorics Conference [17].

The last section of the paper discusses open questions and conjectures regarding imbalance graphic graphs.

2. PRELIMINARIES

2.1. MAIN DEFINITIONS

For a set X denote by $\binom{X}{2}$ the collection of two-element subsets of X.

A *graph* G is an ordered pair (V, E), where $V = V(G)$ is the set of its *vertices* and $E = E(G) \subset \binom{V}{2}$ is the set of its *edges*. All graphs considered in this paper are finite. Frequently, an edge $\{u, v\} \in E(G)$ in a graph G will be denoted simply as uv. If there is an edge $e = uv \in E(G)$, then we say that the vertices u and v are *adjacent* and the edge e is *incident* with u, v. A graph is called *complete* provided every two of its vertices are adjacent. By K_n we denote the complete graph with n vertices.

The *neighborhood* of a vertex u in a graph G is the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of u is the number $d_G(u) = |N_G(u)|$. A vertex u is called a *leaf vertex* if $d_G(u) = 1$. The unique edge incident to a leaf vertex is called a *leaf edge*. An edge which is not leaf edge is called an *inner edge*. For a graph G by $\Delta(G)$ and $\delta(G)$ we denote the maximum and the minimum degree among vertices in G, respectively. A graph G is called *regular* provided $\Delta(G) = \delta(G)$.

In the course of this paper we will frequently use the following unary and binary operations on graphs. By $\overline{G}$ we denote the complement of a graph G. The complement of a complete graph is called an *empty graph*. The *line graph* $L(G)$ of a graph G has the vertex set $V(L(G)) = E(G)$ and the edge set $E(L(G)) = \{\{uv, vw\} : uv, vw \in E(G), u \neq w\}$. Further, by $G \cup H$ we denote the union of graphs G, H (note that $V(G \cup H) = V(G) \sqcup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$). For a graph G and a natural number $n \in \mathbb{N}$ we write nG for the union of n isomorphic copies of G. Another standard binary graph operation is the *join* $G + H$, where $V(G + H) = V(G) \sqcup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let $a_1, b_1, a_2, b_2 \in V(G)$ be four different vertices in a graph G such that $a_1b_1, a_2b_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$. The corresponding *2-switch operation* produces a graph which is obtained from G by deletion of the edges a_1b_1, a_2b_2 and addition of new edges a_1a_2, b_1b_2.

A graph is called *connected* if there is a path between every pair of its vertices (otherwise, the graph is *disconnected*). A *connected component* in a graph is its maximal connected subgraph. A *tree* is a connected graph without cycles. Prominent examples of trees are *paths* P_n (n-vertex trees with at most two leaf vertices) and *stars* $K_{1,n-1}$ (n-vertex trees with at most one non-leaf vertex). For example, the complete graph K_2 is a path P_2 and a star $K_{1,1}$, simultaneously. A graph which has exactly one cycle is called *unicyclic*.

For a set of vertices $A \subset V(G)$ by $G[A]$ we denote the subgraph of G induced by A. Also, we put $G - A = G[V(G) \backslash A]$ and $G - u = G - \{u\}$ for any vertex $u \in V(G)$. Similarly, for a set of edges $E' \subset E(G)$ by $G - E'$ we denote the spanning subgraph of G with $E(G - E') = E(G) \backslash E'$. And for a single edge $e \in E(G)$ we put $G - e = G - \{e\}$.

A vertex $u \in V(G)$ is called *simplicial* if its neighborhood $N_G(u)$ induces a complete subgraph in G. A vertex $u \in V(G)$ is called a *cut vertex* if the graph $G - u$ has more connected components than G (in particular, for connected graphs G the graph $G - u$ must be disconnected). An edge $e \in E(G)$ is a *bridge* provided $G - e$ has more connected components than G (in particular, for connected graphs G the graph $G - e$ must have exactly two connected components). A graph is called 2-*connected* if it does not have cut vertices. A maximal 2-connected subgraph in G is called its *block*. A *pendant block* in a graph G is its block which contains exactly one cut vertex of G. A graph is called a *block graph* provided all its blocks are complete subgraphs. Note that every vertex in a block graph is either a cut vertex or a simplicial vertex.

2.2. GRAPHIC MULTISETS AND IMBALANCE GRAPHIC GRAPHS

A *multiset* M is a pair (X, f), where X is a set and $f : X \to \mathbb{N}$ is the *multiplicity function* (which counts the number of appearances of elements $x \in X$ in M). We will also denote a multiset $M = (X, f)$ as $M = \{x[f(x)] : x \in X\}$.

For a multiset $M = (X, f)$ and a natural number $n \in \mathbb{N}$ we put $nM = (X, nf)$, where $(nf)(x) = nf(x)$ for all $x \in X$. Now let $M = (X, f)$ be a multiset of integers, i.e. $X \subset \mathbb{Z}$. By $M \bmod 0$ we denote the multiset $(X \backslash \{0\}, f|_{X\backslash\{0\}})$. For a number $n \in \mathbb{Z}$ by M^n we denote the multiset (nX, f), where $nX = \{nx : x \in X\}$.

A multiset M of non-negative integers is called *graphic* provided it is the multiset of vertex degrees of some graph G. Any such graph G is called a *realization* of M. There is a classical criterion for graphic multisets by Erdős and Gallai [7] which will be used many times throughout this paper.

Theorem 2.1 ([7])**.** *Let $d_1 \geq \ldots \geq d_n$ be $n \geq 2$ non-negative integers. The multiset $M = \{d_1, \ldots, d_n\}$ is graphic if and only if the sum $\sum_{i=1}^{n} d_i$ is even and for all $1 \leq k \leq n-1$ the following inequality holds:*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}.$$

The *imbalance* of an edge $uv \in E(G)$ is the value $\operatorname{imb}_G(uv) = |d_G(u) - d_G(v)|$. The *imbalance sequence* M_G of a graph G is the multiset of all edge imbalances in G. The sum of all edge imbalances in G is called the *irregularity* of G and denoted by $I(G)$. It is easy to see that $I(G) = 0$ if and only if every connected component of G is a regular graph.

A graph G is *imbalance graphic* provided its imbalance sequence M_G is graphic. Otherwise, a graph is called *imbalance non-graphic.* For any imbalance graphic graph G we assume by default that $V(H) = E(G)$ for every realization H of M_G.

A simple examination shows that every graph with $n \leq 5$ vertices is imbalance graphic. The next proposition contains basic properties of imbalance graphic graphs and will be used in this paper.

Proposition 2.2 ([11])**.** *Let G, G_1, G_2 be graphs. Then:*

1. *if G_1 and G_2 are imbalance graphic, then so is $G_1 \cup G_2$,*
2. *if G is imbalance graphic, then $G + K_1$ is also imbalance graphic,*
3. *if G has a constant edge imbalance, then G is imbalance graphic.*

3. NEW CLASSES OF IMBALANCE GRAPHIC GRAPHS

3.1. TREES AND UNICYCLIC GRAPHS

We start by presenting the new proof of imbalance graphicnesss of trees. The same idea would be used for unicyclic graphs. These two proofs are based on the next result and its corollary.

Proposition 3.1. *Let G_1, G_2 be two imbalance graphic graphs and $wx \in E(G_1)$, $yz \in E(G_2)$ be their edges with $d_{G_1}(w) \geq d_{G_1}(x)$, $d_{G_2}(y) \geq d_{G_2}(z)$ and $(d_{G_1}(w) - d_{G_2}(y)) \cdot (d_{G_2}(z) - d_{G_1}(x)) \geq 0$. Then the graph which is obtained from $G_1 \cup G_2$ by performing the 2-switch on the vertices w, x, y, z is also imbalance graphic.*

Proof. Put $G = G_1 \cup G_2$ and let G' be the graph which is obtained from G by 2-switching the vertices w, x, y, z. Since G_1 and G_2 are both imbalance graphic, there are realizations of their imbalance sequences. By H_1 we will denote the realization of M_{G_1} and by H_2 the realization of M_{G_2}. Let $A = N_{H_1}(wx) = \{a_1, \ldots, a_k\}$, $B = N_{H_2}(yz) = \{b_1, \ldots, b_m\}$ be the neighborhoods of wx and yz in H_1 and H_2, respectively. Clearly, G is imbalance graphic with $H = H_1 \cup H_2$ being a realization of M_G.

Without loss of generality, we can assume that $k \leq m$.

Firstly, we remove the vertices wx, yz from H and add new edges $\{a_i, b_i\}$, $1 \leq i \leq k$, to obtain the graph H'. Further, we will add two new vertices wy, xz to H' with the edges:

1. $\{b_i, wy\}$ for $k + 1 \leq i \leq k + \mathrm{imb}_{G'}(wy)$,
2. $\{b_i, xz\}$ for $k + \mathrm{imb}_{G'}(wy) + 1 \leq i \leq m$.

The obtained graph H'' is a realization of G'. Indeed, since

$$(d_{G_1}(w) - d_{G_2}(y)) \cdot (d_{G_2}(z) - d_{G_1}(x)) \geq 0,$$

we have

$$\begin{aligned} m - k &= \mathrm{imb}_{G_2}(yz) - \mathrm{imb}_{G_1}(wx) \\ &= d_{G_2}(y) - d_{G_2}(z) - (d_{G_1}(w)) - d_{G_1}(x)) \\ &= (d_{G_2}(y) - d_{G_1}(w)) + (d_{G_1}(x) - d_{G_2}(z)) \\ &= |d_{G_2}(y) - d_{G_1}(w)| + |d_{G_1}(x) - d_{G_2}(z)| \\ &= \mathrm{imb}_{G'}(wy) + \mathrm{imb}_{G'}(xz). \end{aligned}$$

□

Corollary 3.2. *Let G_1, G_2 be two imbalance graphic graphs and $xl_1 \in E(G_1)$, $yl_2 \in E(G_2)$ be their leaf edges with leaves l_1, l_2. Then the graph $((G_1 \cup G_2) - \{l_1, l_2\}) \cup \{xy\}$ (which is obtained from $G_1 \cup G_2$ by deleting the vertices l_1, l_2 and adding the edge xy) is imbalance graphic.*

Proof. Since $d_{G_1}(l_1) = d_{G_2}(l_2) = 1$, we can 2-switch the vertices x, l_1, y, l_2 in $G_1 \cup G_2$ and get an imbalance graphic graph H. Also, we know that H has two connected components H_1, H_2, where $V(H_2) = \{l_1, l_2\}$. Now recall that $\mathrm{imb}_H(l_1 l_2) = 0$, which implies that $((G_1 \cup G_2) - \{l_1, l_2\}) \cup \{xy\} = H_1$ is also imbalance graphic. □

Using Corollary 3.2, one can obtain a conceptually different and simpler proof of imbalance graphicness of trees, which was originally established in [11].

Corollary 3.3 ([11])**.** *Trees are imbalance graphic.*

Proof. The proof goes by induction on n. If $n \leq 5$, then trivially T is imbalance graphic. Now suppose that $n \geq 6$. If T is a star, then $M_T = \{(n-2)[n-1]\}$ is clearly graphic. Now we can assume that T is not a star. This means that there is an inner edge $xy \in E(T)$. We remove the edge xy from T and add to it two new vertices l_1, l_2 with the edges xl_1, yl_2. The obtained graph is a forest G consisting of two trees G_1 and G_2 (see Figure 1). Thus, G is imbalance graphic by the induction assumption and Proposition 2.2.

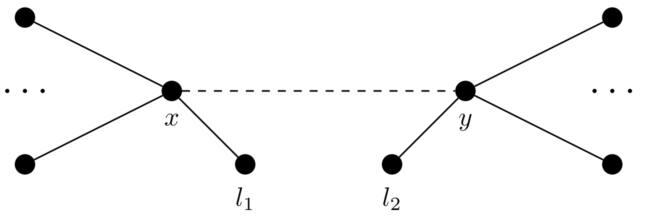


Fig. 1. T after removing the edge xy and adding new leaf vertices l_1, l_2

Since l_1 and l_2 are leaf vertices in G, we use Corollary 3.2 in order to show that the initial tree

$$T = ((G_1 \cup G_2) - \{l_1, l_2\}) \cup \{xy\}$$

is indeed imbalance graphic. □

Exploiting the same idea as in the proof of Corollary 3.3, we can prove the imbalance graphicness of unicyclic graphs.

Theorem 3.4. *Unicyclic graphs are imbalance graphic.*

Proof. It is sufficient to prove the theorem for connected unicyclic graphs. Thus, let G be a connected unicyclic graph with $n \geq 3$ vertices. In order to prove that G is imbalance graphic, we use induction on n. If $n = 3$, then the statement clearly holds. Hence, assume $n \geq 4$. If G is a cycle, then $M_G = \{0[n]\}$ is trivially graphic. Now suppose that G contains a bridge between two inner vertices. Then, using the same argument as in proof of Corollary 3.3, we obtain that G is imbalance graphic. Therefore, we further assume that G is not a cycle and that every bridge in G connects a leaf vertex with a vertex on the cycle.

Let $A = \{a_1, a_2, \ldots, a_k\} \subset V(G)$ be a maximal connected subset of vertices in G having $d_G(a) = \Delta(G)$ for all $a \in A$. It is clear that each element in A lies on a cycle in G. Now consider the function $f : A \to V(G)$, which for any $a \in A$ returns a leaf vertex from $N_G(a)$, i.e. $f(a) \in N_G(a)$ and $d_G(f(a)) = 1$ (see Figure 2). Put $H = G - f(A)$. By induction assumption, H is imbalance graphic. Let F be a realization of M_H (recall that we assume that $V(F) = E(H)$).

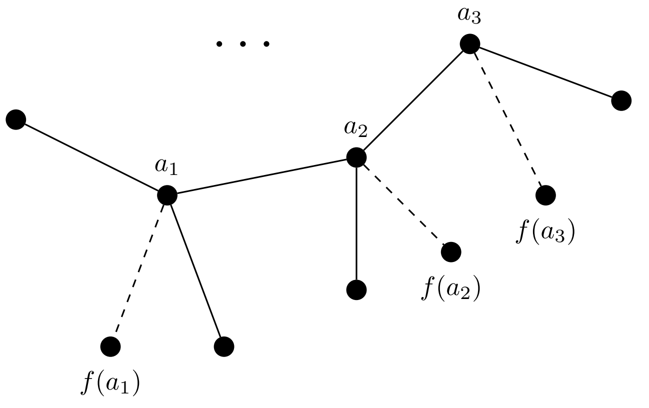


Fig. 2. An illustration for the induction step from the proof of Theorem 3.4

It is clear that for all $a \in A$ and $v \in N_H(a) \backslash A$ it holds $\mathrm{imb}_G(av) = \mathrm{imb}_H(av) + 1$. Moreover, for all $a \in A$ and $a' \in N_H(a) \cap A$ we have $\mathrm{imb}_H(aa') = \mathrm{imb}_G(aa')$.

Add to F new vertices $a_i f(a_i)$, $1 \leq i \leq k$ with new edges $\{a_i f(a_i), a_i v\}$, $1 \leq i \leq k$, $v \in N_H(a) \backslash A$. Further, add new edges $\{a_i f(a_i), a_j f(a_j)\}$ for $a_i a_j \in E(G)$, $1 \leq i, j \leq k$. The obtained graph is a realization of M_G. □

3.2. ANTIREGULAR GRAPHS, JOINS AND THE DOUBLE GRAPH

As is well known, there is no graph with $n \geq 2$ vertices which has pairwise different degrees of its vertices (this is a classical application of Dirichlet's principle as isolated and universal vertices cannot coexist simultaneously). A graph G is called *antiregular* [2], if there is exactly one pair of different vertices $u, v \in V(G)$ with $d_G(u) = d_G(v)$. It is known that for any $n \geq 2$ there are only two non-isomorphic antiregular graphs with n vertices and they are complementary (see [5]).

Proposition 3.5. *Antiregular graphs are imbalance graphic.*

Proof. The proof goes by induction on $n = |V(G)|$, $n \geq 2$. If $n = 2$, then the statement is obvious since K_2 and $\overline{K_2}$ are the only 2-vertex graphs. Recall that for every $n \geq 2$ there are exactly two antiregular graphs on n vertices, namely G_n and its complement $\overline{G_n}$. Also, $G_n = \overline{G_{n-1}} + K_1$ and $\overline{G_{n-1}} = G_{n-1} \cup K_1$. Since for every imbalance graphic graph G the graphs $G + K_1$ and $G \cup K_1$ are also imbalance graphic (see Proposition 2.2), the statement holds. □

Now we turn our attention to particular joins of graphs. At first, we show that the second statement from Proposition 2.2 does not hold for imbalance non-graphic graphs. Moreover, we prove an even stronger statement.

Proposition 3.6. *For any $k \geq 1$ there exists a graph G such that $G + K_k$ is imbalance non-graphic.*

Proof. At first, we deal with the case $k = 1$. For any number $m \geq 7$ consider the graph $G = (K_{m-1} \cup K_1) + \overline{K_2}$. We calculate the imbalances of edges in $G + K_1$. We have

$$M_{G+K_1} \bmod 0 = \{m-1[1], m-2[2], 1[m+1]\}.$$

To show that $M_{G+K_1} \bmod 0$ is not graphic, we use Erdős-Gallai criterion (Theorem 2.1) and consider the first three elements of $M_{G+K_1} \bmod 0$ (in a non-increasing order): $M' = \{m-1[1], m-2[2]\}$. Then

$$\sum_{d \in M'} d - |M'|(|M'|-1) - \sum_{d \in M_{G+K_1} \bmod 0 \setminus M'} \min\{d, |M'|\} = m - 1 + 2(m-2) - 6 - (m+1)$$
$$= 2m - 12 > 0.$$

Therefore, $M_{G+K_1} \bmod 0$ is non-graphic and hence so is M_{G+K_1}.

Now let $k \geq 2$. To prove the statement of the proposition, we construct an imbalance non-graphic graph H with exactly k universal vertices. Fix any number $m \in \mathbb{N}$ with $2m > k^2 + 7k$. Consider a complete graph $G' \simeq K_m$ and a subset $A \subset V(G')$ with $|A| = k-1$. Add to G' the new vertex v with edges uv for all $u \in A$ to obtain a graph G''. Finally, set $H = G'' + P$, where $P \simeq P_3$, $V(P) = \{x, y, z\}$ and $E(P) = \{xy, yz\}$ (see Figure 3).

At first, observe that H indeed has exactly k universal vertices, namely, the vertices from $A \cup \{y\}$.

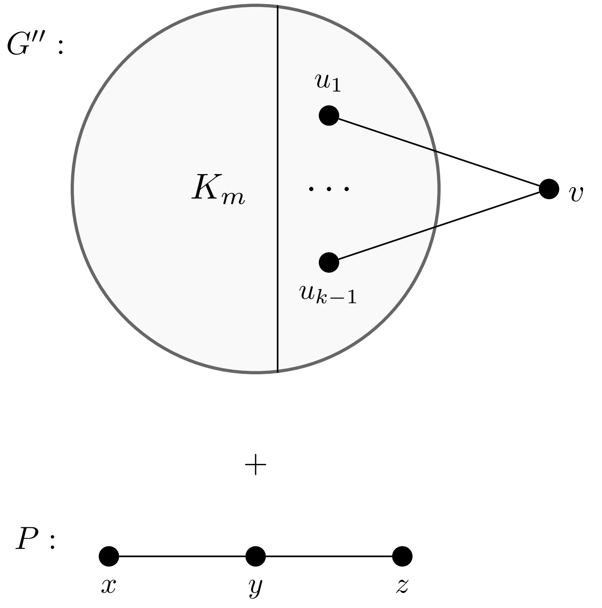


Fig. 3. The join $H = G'' + P$ from the proof of Proposition 3.6

Further, let us calculate the degrees of vertices in H. For all $t \in V(G')\backslash A$ we have $d_H(t) = d_H(x) = d_H(z) = m+2$. For all $u \in A$ it holds $d_H(u) = d_H(y) = m+3$. And clearly $d_H(v) = k+2$.

Now we calculate the imbalances of edges in H. For all $t \in V(G')\backslash A$ and $u \in A$ we have $\mathrm{imb}_H(tu) = 1$, $\mathrm{imb}_H(uv) = m-k+1$, $\mathrm{imb}_H(xt) = \mathrm{imb}_H(zt) = 0$, $\mathrm{imb}_H(xu) = \mathrm{imb}_H(zu) = 1$, $\mathrm{imb}_H(xv) = \mathrm{imb}_H(zv) = m-k$, $\mathrm{imb}_H(yt) = 1$, $\mathrm{imb}_H(yu) = 0$ and $\mathrm{imb}_H(yv) = m-k+1$. Thus

$$M_H \bmod 0 = \{m-k+1[k], m-k[2], 1[(m-k+1)k+2(k-1)]\}.$$

To show that M_H is not graphic, we again use Erdős-Gallai criterion and consider the first $k+2$ elements of M_H: $M' = \{m-k+1[k], m-k[2]\}$. Then

$$\begin{aligned}
&\sum_{d\in M'} d - |M'|(|M'|-1) - \sum_{d\in M_H\backslash M'} \min\{d, |M|\}\\
&= (m-k+1)k + 2(m-k) - (k+2)(k+1) - ((m-k+1)k + 2(k-1))\\
&= 2m - 2k - k^2 - 3k - 2 - 2k + 2 = 2m - k^2 - 7k > 0.
\end{aligned}$$

Therefore, M_H is non-graphic. Setting $G = H[V(H)\backslash(A \cup \{y\})]$, we conclude that $G + K_k \simeq H$ is imbalance non-graphic. □

Note that the construction of a graph H from the proof of Proposition 3.6 yields that the join of two imbalance graphic graphs is not necessarily imbalance graphic itself. Indeed, $P \simeq P_3$ is clearly imbalance graphic and

$$M_{G''} \bmod 0 = \{m-k+1[k-1], 1[(k-1)(m-k+1)]\}$$

has a realization $(k-1)K_{1,m-k+1}$. However, $H = G'' + P$ is imbalance non-graphic for $2m > k^2 + 7k$.

The next result establishes a positive result on imbalance graphicness for joins of arbitrary graphs with empty graphs of suitable order.

Theorem 3.7. *Let G be an n-vertex graph and $m \geq n - \max\{1, \delta(G)\}$. Then $G + \overline{K_m}$ is imbalance graphic.*

Proof. Put $H = G + \overline{K_m}$ and $V(H)\backslash V(G) = \{a_1, a_2, \ldots, a_m\}$. We arrange the edge set of H in the non-increasing order of their imbalances: $E(H) = \{e_1, e_2, \ldots, e_{|E(H)|}\}$. Since $m \geq n - \max\{1, \delta(G)\}$, then for all $1 \leq i \leq |E(H)|$ we have

$$\mathrm{imb}_H(e_i) \leq \Delta(G) - \max\{1, \delta(G)\} \leq m + \Delta(G) - n.$$

Thus, the imbalances of the first m edges from $E(H)$ are equal to $m + \Delta(G) - n$.

We use the Erdős–Gallai criterion. Hence, we must prove that for all $1 \leq k \leq |E(H)| - 1$ the following inequality holds:

$$\sum_{i=1}^{k} \mathrm{imb}_H(e_i) \leq k(k-1) + \sum_{i=k+1}^{|E(H)|} \min\{k, \mathrm{imb}_H(e_i)\}.$$

Note that $\Delta(G) - n \leq -1$. If $k > m$, then we have

$$\sum_{i=1}^{k} \mathrm{imb}_H(e_i) \leq k(m + \Delta(G) - n) \leq k(m-1) \leq k(k-1).$$

Now we show that the desired inequality holds for $k \leq m$ as well. At first, note that in this case

$$\sum_{i=1}^{k} \mathrm{imb}_H(e_i) = (m + \Delta(G) - n)k.$$

Further, for all $k+1 \leq i \leq m$ it also holds $\mathrm{imb}_H(e_i) = m + \Delta(G) - n$. We consider two subcases.

Subcase 1. $k > m + \Delta(G) - n$.

Here for all $k+1 \leq i \leq m$ we have

$$\min\{k, \mathrm{imb}_H(e_i)\} = \min\{k, m + \Delta(G) - n\} = m + \Delta(G) - n.$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{k} \mathrm{imb}_H(e_i) &= (m + \Delta(G) - n)k \leq (m + \Delta(G) - n)m \\
&= (m + \Delta(G) - n)(m-k) + (m + \Delta(G) - n)k \\
&\leq (m + \Delta(G) - n)(m-k) + (k-1)k \\
&\leq k(k-1) + \sum_{i=k+1}^{|E(H)|} \min\{k, \mathrm{imb}_H(e_i)\}.
\end{aligned}$$

Subcase 2. $k \leq m + \Delta(G) - n$.

Here for all $k+1 \leq i \leq m$ we have

$$\min\{k, \mathrm{imb}_H(e_i)\} = \min\{k, m + \Delta(G) - n\} = k.$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{k} \mathrm{imb}_H(e_i) &= (m + \Delta(G) - n)k \leq (m-1)k = (m - k + k - 1)k \\
&= (m-k)k + (k-1)k \leq k(k-1) + \sum_{i=k+1}^{|E(H)|} \min\{k, \mathrm{imb}_H(e_i)\}.
\end{aligned}$$

Hence, by Theorem 2.1, H is imbalance graphic. □

Note that, generally speaking, the statement of Theorem 3.7 does not hold for $m = n - \max\{1, \delta(G)\} - 1$. Indeed, consider the graph $G = K_3 \cup K_1$. It is clear that $n = |V(G)| = 4$ and $\delta(G) = 0$. However, for $m = n - \max\{1, \delta(G)\} - 1 = 2$ the graph

$G + \overline{K_2}$ is imbalance non-graphic. It is also an interesting question whether we can construct such a graph for an arbitrary number of vertices n.

To conclude this section, we consider another unary graph operation called the double graph. Namely, let G be a graph. Consider a graph G' which is an isomorphic copy of G with $V(G') = \{u' : u \in V(G)\}$. The *double graph* $\mathcal{D}[G]$ [14] of G is a graph with the vertex set $V(\mathcal{D}[G]) = V(G) \sqcup V(G')$ and the edge set $E(\mathcal{D}[G]) = E(G) \cup E(G') \cup \{uv' : uv \in E(G)\}$. In other words, to construct $\mathcal{D}[G]$ we take the union of G with its copy G' and add new edges uv' for every edge uv in G. An example of a double graph is given in Figure 4.

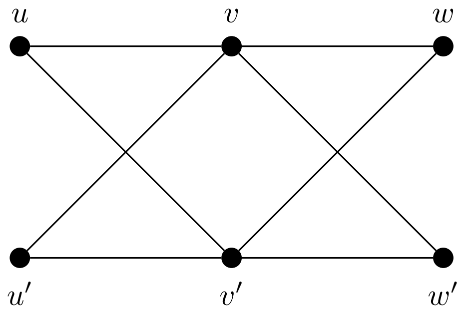


Fig. 4. The double graph $\mathcal{D}[P_3]$ of a 3-vertex path

Proposition 3.8. *The double graph of an imbalance graphic graph is also imbalance graphic.*

Proof. Let G be an imbalance graphic graph. From the construction of $\mathcal{D}[G]$ it follows that $d_{\mathcal{D}[G]}(u) = d_{\mathcal{D}[G]}(u') = 2d_G(u)$. Hence, we can calculate M_G directly: for all $uv \in E(G)$ we have $\operatorname{imb}_{\mathcal{D}[G]}(uv) = \operatorname{imb}_{\mathcal{D}[G]}(u'v') = \operatorname{imb}_{\mathcal{D}[G]}(uv') = 2\operatorname{imb}_G(uv)$. Therefore, $M_{\mathcal{D}[G]} = 4M_G^2$. This means that for a realization H of M_G the graph $2\mathcal{D}[H]$ is a realization of $M_{\mathcal{D}[G]}$. Hence, $\mathcal{D}[G]$ is imbalance graphic. $\square$

A closely related construction to the double graph of G is the so-called *bipartite double cover* of G, which is defined as the tensor product $G \times K_2$. One can observe that $G \times K_2 = \mathcal{D}[G] - (E(G) \cup E(G'))$. It is also easy tp see that $G \times K_2$ is imbalance graphic whenever G is (as $M_{G \times K_2} = 2M_G$).

4. IMBALANCE GRAPHIC BLOCK GRAPHS

Since every tree is a block graph, in light of Corollary 3.3 it is natural to ask whether all block graphs are imbalance graphic. To support the affirmative answer to the question, in this section we provide positive results about three particular classes of block graphs: block graphs having all cut vertices in a single block; block graphs in which the subgraph induced by the cut vertices is either a star or a path.

Theorem 4.1. *Block graphs having all cut vertices in a single block are imbalance graphic.*

Proof. Let G be a block graph with all the cut vertices lying in a single block. If G does not have cut vertices at all, then it is complete and hence trivially imbalance graphic. Now assume G contains pendant blocks. We use induction on $n = |V(G)|$ to prove that G is imbalance graphic. If $n \leq 5$, then G is trivially imbalance graphic. Assume that G contains $n \geq 6$ vertices. Let B be a block in G which contains all its cut vertices and let $B_1, B_2, \ldots, B_l$ be pendant blocks in G.

Put $m = \max_{v \in V(B)} d_G(v)$ and let $A = \{v_1, \ldots, v_k\}$ be the set of vertices from $V(B)$ whose degrees are equal to m. For any such vertex $v_i \in A$ fix an (arbitrary) simplicial vertex u_i from a pendant block which contains v_i (at least one such a block exists). For every $1 \leq i \leq k$ by B_i we denote the pendant block in G that contains the vertices v_i and u_i. Consider the graph G' obtained from G by deleting all such vertices u_i (see Figure 5).

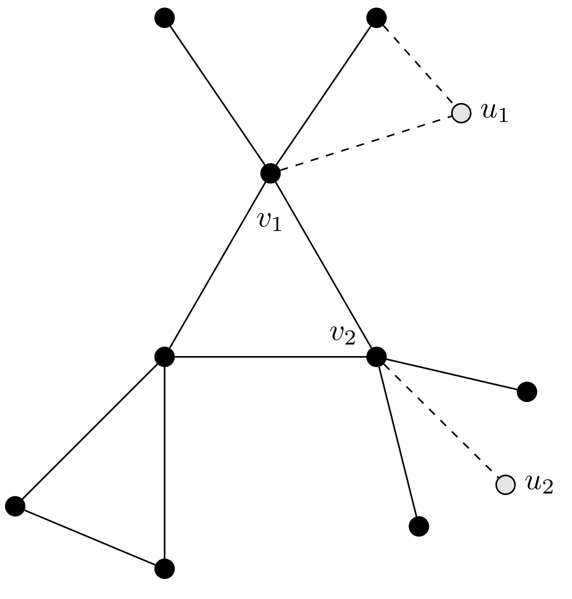


Fig. 5. The construction of the graph G' from the induction step

Clearly, G' is also a block graph having all its cut vertices in a single block. By induction assumption, G' is imbalance graphic. Let H' be a realization of $M_{G'}$ mod 0. It is clear that for all $1 \leq i \leq k$ and for all edges $e \in E(B_i - u_i)$ we have $\mathrm{imb}_G(e) = \mathrm{imb}_{G'}(e)$. Now consider an edge of the form $e = v_i w \in E(G')$ for some $1 \leq i \leq k$ and $w \in N_G(v_i) \backslash V(B_i)$. We have

$$\mathrm{imb}_G(e) = \begin{cases} \mathrm{imb}_{G'}(e), & \text{if } d_G(w) = m, \\ \mathrm{imb}_{G'}(e) + 1, & \text{if } d_G(w) < m. \end{cases}$$

Add k new vertices to H' of the form u_iv_i, $1 \le i \le k$ with new edges:

1. $\{u_iv_i, v_iw\}$, where $1 \le i \le k$, $w \notin V(B_i)$ and $d_G(w) < m$,
2. $\{u_iv_i, u_jv_j\}$, where $1 \le i, j \le k, i \ne j$.

The constructed graph is a realization of M_G mod 0 implying that M_G is graphic. □

Remark 4.2. If G is a block graph having one cut vertex, a realization of M_G can be constructed from the line graph $L(G)$ by the deletion of edges in $L(G)$ between the edges from a common block in G. An illustration of such a construction is given in Figure 6.

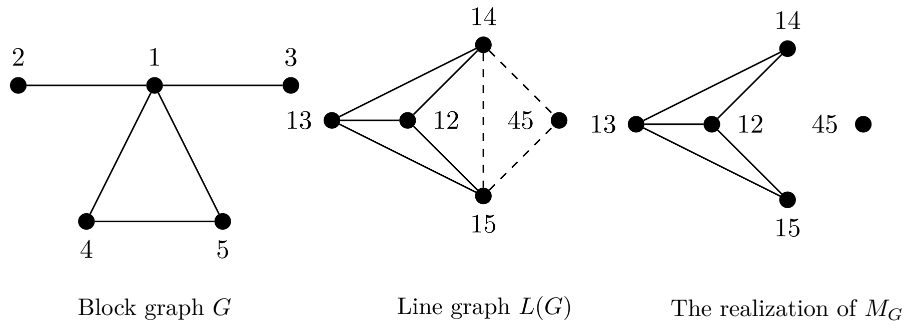


Fig. 6. Construction of a realization for M_G from $L(G)$

Theorem 4.3. *Block graphs in which cut vertices induce a star are imbalance graphic.*

Proof. Let G be a block graph in which cut vertices induce a star S, a be the center of S and $a_1, \dots, a_k$ be the leaves of S. Without loss of generality, we can assume that G is connected. We use induction on $n = |V(G)|$ to prove that G is imbalance graphic. If $n \le 5$, then G is imbalance graphic. Hence, suppose $n \ge 6$.

At first, assume there is a vertex $a_i \in V(G)$ with $d_G(a_i) > d_G(a)$. In this case, delete an arbitrary simplicial vertex u from the pendant block B, whose cut vertex is a_i (see Figure 7). It is clear that the obtained graph $G' = G - u$ is also a block graph in which cut vertices induce a star. By induction assumption, G' is imbalance graphic. Let H be a realization of $M_{G'}$ mod 0. Add to H the new vertex a_iu with the new edges of the form $\{a_iu, a_ib\}$ for every $b \in N_{G'}(a_i) \backslash V(B)$. The obtained graph is a realization of M_G mod 0.

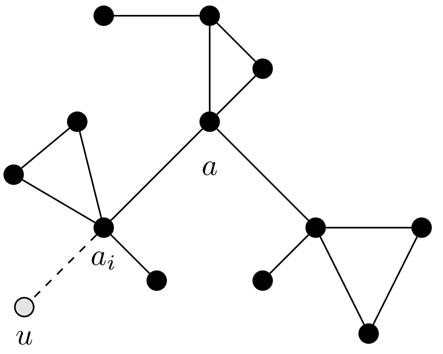


Fig. 7. The case where $d_G(a_i) > d_G(a)$ for some vertex a_i

Now consider the case when $d_G(a) \geq d_G(a_i)$ for all $1 \leq i \leq k$. Here, we have two subcases.

Subcase 1. There exists $1 \leq i \leq k$ with $N_G(a_i) \cap N_G(a) = \emptyset$.

In this subcase delete the bridge aa_i from G and add two new leaf edges al_1 and a_il_2 (see Figure 8).

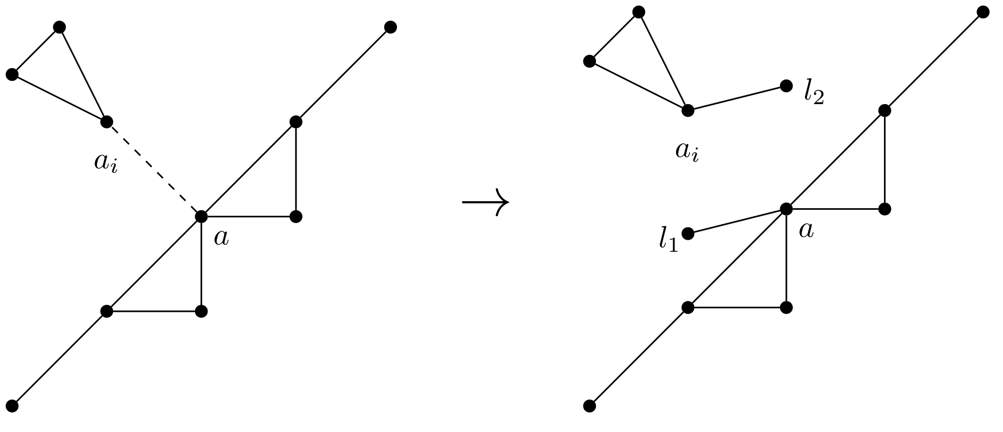


Fig. 8. Operation on G which produces two components G_1 and G_2.

The obtained graph has two connected components (as G is connected), one of which is a block graph G_1 with a unique cut vertex and the other one is a block graph G_2 in which cut vertices induce a star. By induction assumption and Theorem 4.1, both G_1 and G_2 are imbalance graphic. Let H be the (disjoint) union of realizations of graphic sequences M_{G_1} and M_{G_2}.

By Corollary 3.2, G is imbalance graphic.

Subcase 2. For all $1 \leq i \leq k$ it holds $N_G(a_i) \cap N_G(a) \neq \emptyset$.

Choose an arbitrary cut vertex a_i and a vertex $v \in N_G(a) \cap N_G(a_i)$. Delete the vertex v from G obtaining the imbalance graphic graph G' (see Figure 9). Let H be a realization of $M_{G'}$ mod 0. Add to H two new vertices va, va_i with the new edges of the form $\{va_i, a_iw\}$ for all $w \in N_G(a_i) \backslash N_G[a]$ and $\{va, aw\}$ for all $w \in N_G(a) \backslash N_G[a_i]$. The obtained graph is a realization of M_G mod 0.

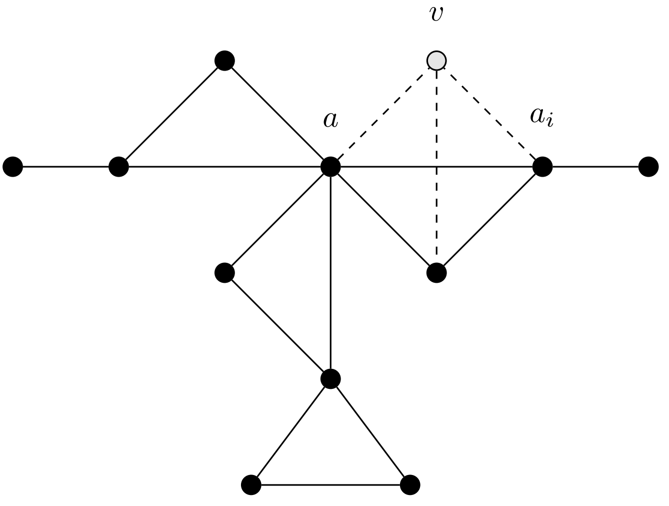


Fig. 9. Construction of the imbalance graphic graph G' from Subcase 2

□

Theorem 4.4. *Block graphs in which cut vertices induce a path are imbalance graphic.*

Proof. Let G be a block graph in which cut vertices induce a path. We use induction on $n = |V(G)|$ to prove that G is imbalance graphic. If $n \leq 5$, then G is imbalance graphic.

Let $a_1, \ldots, a_k$ be the cut vertices of G and $a_ia_j \in E(G)$ if and only if $|i - j| = 1$. If $k \leq 2$, then all cut vertices in G lie in a common block implying that by Proposition 4.1, G is imbalance graphic. Hence, we can assume that $k \geq 3$.

Denote by B the block in G which contains the vertices a_1 and a_2. If $|V(B)| = 2$, then B is a bridge in G and, therefore, using the same strategy as in the proof of Subcase 1 from Theorem 4.3, we can show that G is imbalance graphic. Thus, assume that B contains a simplicial vertex u. Clearly, $ua_1, ua_2 \in E(G)$.

Put $C_1 = N_G(a_1) \backslash B$ and $C_2 = (N_G(a_2) \backslash B) \backslash \{a_3\}$. Since a_1 is a cut vertex in G, then $C_1 \neq \emptyset$. Also, fix a vertex $w \in C_1$ (see Figure 10). Note that w is necessarily simplicial in G.

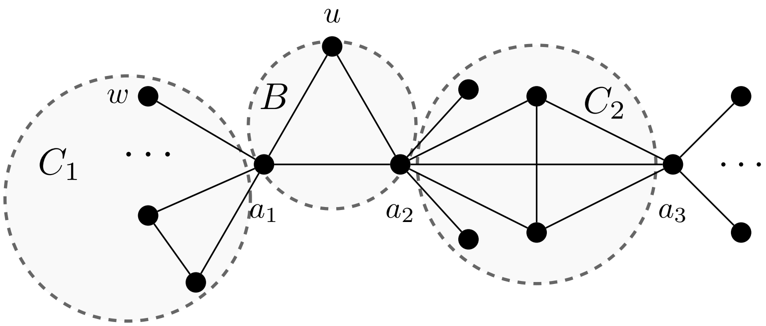


Fig. 10. Preparation for the induction step in the proof of Theorem 4.4.

It is clear that $G' = G - u$ is also a block graph in which cut vertices induce a path, thus by induction assumption G' is imbalance graphic. Let H be a realization of $M_{G'}$ mod 0. It is easy to see that for all $e = a_1c, c \in C_1$ it holds $\mathrm{imb}_G(e) = \mathrm{imb}_{G'}(e)+1$. Also, for all $e = a_2c, c \in C_2$ we have $\mathrm{imb}_G(e) = \mathrm{imb}_{G'}(e)+1$. Finally, $\mathrm{imb}_G(a_1u) = |C_1|$ and $\mathrm{imb}_G(a_2u) = |C_2| + 1$.

At first, add to H two new vertices a_1u and a_2u to obtain the new graph H'. Further we consider the next two cases.

Case 1. $d_G(a_2) > d_G(a_3)$.

In this case $\mathrm{imb}_G(a_2a_3) = \mathrm{imb}_{G'}(a_2a_3) + 1$. Adding to H' new edges of the form $\{a_1u, a_1c\}$, where $c \in C_1$, $\{a_2u, a_2c\}$, where $c \in C_2$, and the new edge $\{a_2u, a_2a_3\}$, we obtain the realization of M_G mod 0.

Case 2. $d_G(a_2) \leq d_G(a_3)$.

Since $d_G(a_2) \leq d_G(a_3)$, then $\mathrm{imb}_G(a_2a_3) = \mathrm{imb}_{G'}(a_2a_3) - 1$. At first, add to H' new edges $\{a_1u, a_1c\}, c \in C_1, c \neq w$. Further, add new edges $\{a_2u, a_2c\}, c \in C_2$.

Fix a vertex $x \in N_H(a_2a_3)$ and delete from H' the edge $\{a_2a_3, x\}$. If $x = a_2c$ for some $c \in C_2$, then we add the edges $\{a_1u, x\}$ and $\{a_2u, a_1w\}$. Otherwise, add the edges $\{a_1u, a_1w\}$ and $\{a_2u, x\}$. In both cases, the obtained graph is a realization of M_G mod 0. □

5. OPEN QUESTIONS AND CONJECTURES

As we saw in the previous section, some block graphs from specific classes are imbalance graphic. Also, all trees are imbalance graphic. In light of these results, we formulate our main conjecture:

Conjecture 5.1. *All block graphs are imbalance graphic.*

This conjecture was verified for all block graphs with ≤ 13 vertices using the library of chordal graphs created by McKay [12].

It is well known that line graphs of trees are exactly claw-free block graphs. Interestingly, every line graph of a tree up to 21 vertices is imbalance graphic (here and later we used the library `nautyTraces` [13] to generate graphs). This is why we formulate the weaker version of Conjecture 5.1:

Conjecture 5.2. *Line graphs of trees are imbalance graphic.*

Also, we should note that there are line graphs of imbalance graphic graphs that are imbalance non-graphic (see Figure 11). However, there are not many such connected graphs (see Table 1).

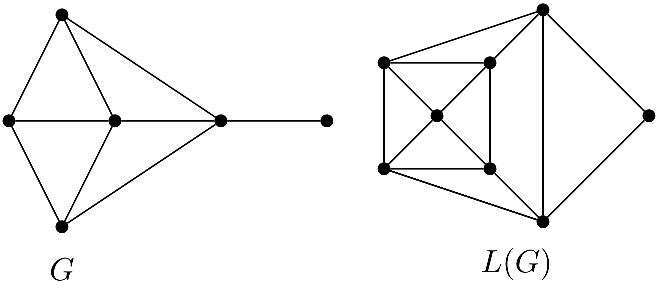


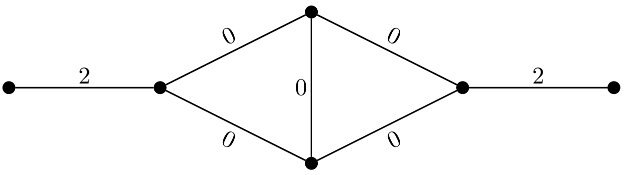
Fig. 11. An imbalance graphic graph G with imbalance non-graphic line graph $L(G)$

Table 1

Number of connected graphs with ≤ 10 vertices whose line graphs are imbalance non-graphic

n	Connected n-vertex graphs with imbalance non-graphic line graphs
1 – 5	0
6	1
7	1
8	9
9	5
10	64

Another interesting question arises in view of Corollary 3.3 and Theorem 3.4. Namely, whether *bicyclic graphs* (these are the n-vertex graphs with $n+1$ edges) are imbalance graphic. As it turns out, there is an imbalance non-graphic bicyclic graph depicted in Figure 12.



$M_G = \{2[2], 0[5]\}$

Fig. 12. Bicyclic graph G which is imbalance non-graphic

However, the graph mentioned above is the only connected bicyclic imbalance non-graphic graph among all such graphs up to 21 vertices. This is why we formulate the following conjecture:

Conjecture 5.3. *The graph shown in Figure 12 is the only connected imbalance non-graphic bicyclic graph.*

Now we turn our attention to the second imbalance conjecture formulated by Kozerenko and Skochko in [11]. It was believed that there is a universal constant $c > 0$ such that any graph G with $I(G) \geq c|E(G)|$ is necessarily imbalance graphic. Moreover, it was explicitly conjectured that $c = 2$. In the next proposition we show that this conjecture is false.

Proposition 5.4. *For every real number $c > 0$ there is an imbalance non-graphic graph G having $I(G) \geq c \cdot |E(G)|$.*

Proof. Consider the graph G given in Figure 13.

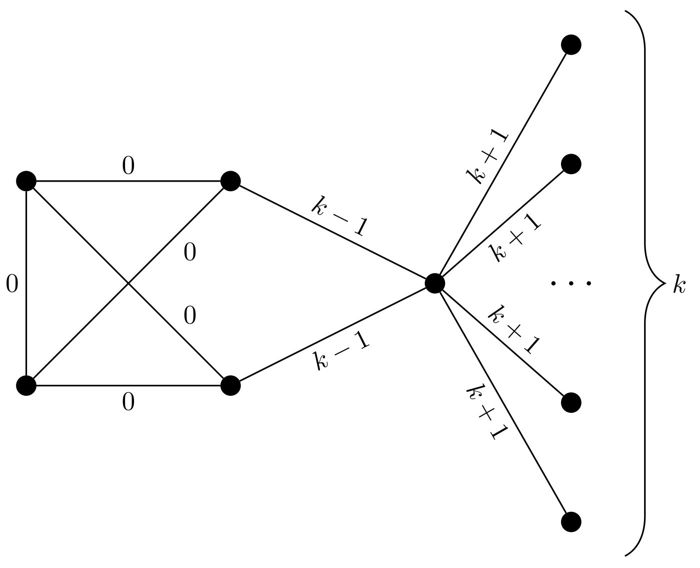


Fig. 13. Counterexample to the second imbalance conjecture from [11]

As we can see, $M_G \bmod 0 = \{k-1[2], k+1[k]\}$. Let us show that this multiset is not graphic. Suppose it is and let H be a realization of $M_G \bmod 0$. It is obvious that all the k vertices with the degree $k+1$ must be universal, because $|V(H)| = k+2$. However, since H is not a complete graph, it must be true that $\delta(H) \geq k$, which is a contradiction.

Now let us show that we can find such k that $I(G) \geq c \cdot |E(G)|$. On the one hand, $I(G) = 2(k-1) + k(k+1) = k^2 + 3k - 2$. On the other hand, $c \cdot |E(G)| = c(k+7)$. Hence, any k satisfying the inequality $k^2 + 3k - 2 \geq c(k+7)$ (equivalently, $k \geq \lceil \frac{c-3+\sqrt{c^2+22c+17}}{2} \rceil$) will do. □

Finally, we would like to recall the first and the most interesting hypothesis concerning imbalance graphic graphs that was also proposed in [11]. This conjecture was verified for all graphs up to 12 vertices.

Conjecture 5.5. *We note that in order to support this conjecture, in* [11] *Suppose that for all edges* $e \in E(G)$ *we have* $\mathrm{imb}_G(e) > 0$*. Then* G *is imbalance graphic.*

We note that in order to support this conjecture, in [10] it was showed that any *multigraph* (here multiple edges between a given pair of vertices are allowed) G with $\mathrm{imb}_G(e) > 0$ for all $e \in E(G)$ is *imbalance multigraphic* (i.e. its imbalance sequence M_G equals the multiset of vertex degrees of some multigraph).

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Sergiy Kozerenko (corresponding author)
kozerenkosergiy@ukr.net
https://orcid.org/0000-0002-5716-3084

National University of Kyiv-Mohyla Academy
Faculty of Computer Sciences
Department of Mathematics
Skovorody Str. 2, 04070 Kyiv, Ukraine

Andrii Serdiuk
serdiuk.andrii@ukr.net

National University of Kyiv-Mohyla Academy
Faculty of Computer Sciences
Department of Mathematics
Skovorody Str. 2, 04070 Kyiv, Ukraine

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