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GLOBAL ATTRACTIVITY OF A HIGHER ORDER NONLINEAR DIFFERENCE EQUATION WITH UNIMODAL TERMS

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Abstract. In the present paper, we study the asymptotic behavior of the following higher order nonlinear difference equation with unimodal terms

$$x(n+1) = ax(n) + bx(n)g(x(n)) + cx(n-k)g(x(n-k)), \quad n = 0, 1, \ldots,$$

where a, b and c are constants with $0 < a < 1$, $0 \leq b < 1$, $0 \leq c < 1$ and $a + b + c = 1$, $g \in C[[0,\infty),[0,\infty)]$ is decreasing, and k is a positive integer. We obtain some new sufficient conditions for the global attractivity of positive solutions of the equation. Applications to some population models are also given.

Keywords: higher order difference equation, positive equilibrium, unimodal term, global attractivity, population model.

Mathematics Subject Classification: 39A10, 92D25.

1. INTRODUCTION

In a recent paper [1], the asymptotic behavior of the following higher order nonlinear difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \quad n = 0, 1, \ldots, \tag{1.1}$$

is studied, where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0,\infty),[0,\infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. A sufficient condition for the global attractivity of positive solutions of Eq. (1.1) is obtained. Applications to some difference equation models are also given in [1].

Motivated by the work in [1], in the present paper, we are still interested in the study of asymptotic behavior of positive solutions of Eq. (1.1), but concentrate on the case that f is unimodal, that is, $f(x) = xg(x)$, where $g \in C[[0,\infty),[0,\infty)]$ is decreasing. Hence, in this case Eq. (1.1) can be written as

$$x(n+1) = ax(n) + bx(n)g(x(n)) + cx(n-k)g(x(n-k)), \quad n = 0,1,\dots \tag{1.2}$$

When $b = 0$, Eq. (1.2) reduces to

$$x(n+1) = ax(n) + cx(n-k)g(x(n-k)), \quad n = 0,1,\dots \tag{1.3}$$

Asymptotic behavior of positive solutions of Eq. (1.3) and some related forms has been studied by many authors, see for example, [3–7, 9–11, 15–22] and the references cited therein.

Clearly, if we let

$$x(-k), x(-k+1), \dots, x(0) \tag{1.4}$$

be $k+1$ given nonnegative numbers with $x(0) > 0$, then Eq. (1.2) has unique positive solution with the initial values in (1.4). In the present paper, by employing and extending an approach used in [14] for the global attractivity of the delay differential equation

$$x'(t) = -\alpha x(t) + x(t-\tau)g(x(t-\tau))$$

where α and τ are positive constants and g is as assumed above, we are able to establish a new sufficient condition for the global attractivity of the difference equation (1.2). Applications to some difference equation models are also given.

In the following discussion, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n} A(i) = 1$ and $\sum_{i=m}^{n} A(i) = 0$ whenever $\{A(n)\}$ is a real sequence and $m > n$.

2. MAIN RESULTS

In the following discussion, we assume that $g(\infty) < 1 < g(0)$. Then there is a positive number $\bar{x}$ such that $g(\bar{x}) = 1$ and so $\bar{x}$ is the unique positive equilibrium of Eq. (1.2).

The following theorem is our main result.

Theorem 2.1. *Assume that $x(a + bg(x))$ is increasing and g is differentiable such that either*

$$g'(x) > \frac{a^{k+1}}{\bar{x}(a^{k+1}-1)}, \quad x > 0 \tag{2.1}$$

or

$$(xg(x))' > \frac{2a^{k+1}-1}{a^{k+1}-1}, \quad x > 0. \tag{2.2}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.2) converges to $\bar{x}$ as $n \to \infty$.

Proof. By noting the decreasing property of g, we see that g satisfies

$$(x-\bar{x})(g(x)-1) < 0, \quad x > 0, x \neq \bar{x}.$$

Then it follows that $f(x) = xg(x)$ satisfies the negative feedback condition

$$(x - \bar{x})(f(x) - x) < 0, \quad x > 0, x \neq \bar{x}.$$

Hence, by [1, Lemma 2.1], every positive solution of Eq. (1.1) with $f(x) = xg(x)$, that is, Eq. (1.2) is bounded and persistent. Then by an argument similar to that in [1] we may show that every nonoscillatory solution (about $\bar{x}$) of Eq. (1.2) tends to $\bar{x}$ as $n \to \infty$.

Next, assume that $\{x(n)\}$ is a solution which oscillates about $\bar{x}$. In the following, we show that $x(n) \to \bar{x}$ also as $n \to \infty$. Since $\{x(n)\}$ is persistent and bounded, there are positive constants L and l such that

$$\lim_{n\to\infty} \sup x(n) = L \quad \text{and} \quad \lim_{n\to\infty} \inf x(n) = l. \tag{2.3}$$

We claim that L and l satisfy the inequalities

$$L \leq \bar{x}\left[1 + (a^{-(k+1)} - 1)(g(l) - 1)\right] \tag{2.4}$$

and

$$l \geq \bar{x}\left[1 + (a^{-(k+1)} - 1)(g(L) - 1)\right]. \tag{2.5}$$

Now, we show that (2.4) holds. The proof of (2.5) is similar and will be omitted. Let $x(i) < x(j)$ be two consecutive members of the solution $\{x(n)\}$ such that

$$x(i) \leq \bar{x}, \quad x(j+1) \leq \bar{x} \quad \text{and} \quad x(n) > \bar{x} \quad \text{for } i+1 \leq n \leq j$$

and let

$$x(n_i) = \max\{x(i+1), x(i+2), \ldots, x(j)\}.$$

Then $\{x(n_i)\}$ satisfies

$$x(n_i) \geq \bar{x},\ x(n_i) \geq x(n_i - 1), \quad i = 1, 2, \ldots$$

and $\{x(n_i)\}$ has a subsequence which converges to L. Hence, we see that there is a subsequence $\{x(n_r)\}$ of $\{x(n)\}$ such that

$$x(n_r) \geq \bar{x}, x(n_r) \geq x(n_r - 1), \quad r = 1, 2, \ldots, \quad \text{and} \quad \lim_{r\to\infty} x(n_r) = L. \tag{2.6}$$

Clearly, there are two possible cases for the behavior of the sequence $\{x(n_r - 1 - k)\}$:

(i) there is a subsequence $\{x(n_{r_s} - 1 - k)\}$ of $\{x(n_r - 1 - k)\}$ such that

$$x(n_{r_s} - 1 - k) \geq \bar{x}, \quad s = 1, 2, \ldots,$$

(ii) there is a positive integer N such that

$$x(n_r - 1 - k) < \bar{x}, \quad r \geq N.$$

If (i) holds, from (1.2) we know that

$$\begin{aligned} & x(n_{r_s}) - ax(n_{r_s}-1) - bx(n_{r_s}-1)g(x(n_{r_s}-1)) \\ & = cx(n_{r_s}-1-k)g(x(n_{r_s}-1-k)). \end{aligned} \tag{2.7}$$

Then by noting (2.6) holds, $g(x(n_{r_s})) \le 1$ and $ax + bxg(x)$ is increasing, we see that

$$\begin{aligned} & x(n_{r_s}) - ax(n_{r_s}-1) - bx(n_{r_s}-1)g(x(n_{r_s}-1)) \\ & = (1-a-b)x(n_{r_s}) + ax(n_{r_s}) + bx(n_{r_s}) \\ & \quad - ax(n_{r_s}-1) - bx(n_{r_s}-1)g(x(n_{r_s}-1)) \\ & \ge (1-a-b)x(n_{r_s}) + ax(n_{r_s}) + bx(n_{r_s})g(x(n_{r_s})) \\ & \quad - ax(n_{r_s}-1) - bx(n_{r_s}-1)g(x(n_{r_s}-1)) \\ & \ge (1-a-b)x(n_{r_s}) \end{aligned}$$

and so it follows from (2.7) that

$$(1-a-b)x(n_{r_s}) \le cx(n_{r_s}-1-k)g(x(n_{r_s}-1-k)). \tag{2.8}$$

Since $1-a-b=c$, (i) holds, g is decreasing and $g(\bar{x})=1$, we see that

$$cx(n_{r_s}) \le cx(n_{r_s}-1-k)g(\bar{x}) = cx(n_{r_s}-1-k). \tag{2.9}$$

Hence $x(n_{r_s}) \le x(n_{r_s}-1-k)$ and so it follows from (2.9) that

$$\lim_{s\to\infty} x(n_{r_s}-1-k) = L.$$

Then by taking limits on (2.8), we find that

$$(1-a-b)L \le cLg(L)$$

which implies that $g(L) \ge 1$ and so $L \le \bar{x}$. However, we know that $L \ge \bar{x}$. Hence $L = \bar{x}$.

Next assume that (ii) holds. By (2.3), given an $\epsilon > 0 (< l)$, there is a positive integer N_ϵ such that

$$l - \epsilon < x(n) < L + \epsilon, \quad n \ge N_\epsilon. \tag{2.10}$$

We claim that

$$(x(n) - \bar{x})g(x(n)) \le L + \epsilon - \bar{x}, \quad n \ge N_\epsilon. \tag{2.11}$$

In fact, if $x(n) \ge \bar{x}$, then by noting (2.10) and $g(x(n)) \le g(\bar{x}) = 1$, we see that (2.11) holds; while if $x(n) < \bar{x}$, then (2.11) holds also since the left side is negative. Hence, (2.11) holds for any case. From Eq. (1.2) we have

$$\begin{aligned} \frac{x(n+1)}{a^{n+1}} - \frac{x(n)}{a^n} &= \frac{b}{a^{n+1}}x(n)g(x(n)) + \frac{c}{a^{n+1}}x(n-k)g(x(n-k)) \\ &= \frac{b}{a^{n+1}}(x(n)) - \bar{x})g(x(n)) \\ &\quad + \frac{c}{a^{n+1}}(x(n-k) - \bar{x})g(x(n-k)) \\ &\quad + \frac{b}{a^{n+1}}\bar{x}g(x(n)) + \frac{c}{a^{n+1}}\bar{x}g(x(n-k)). \end{aligned} \tag{2.12}$$

Summing (2.12) from $n_r - k - 1$ to $n_r - 1$,

$$\begin{aligned}&\frac{x(n_r)}{a^{n_r}} - \frac{x(n_r - k - 1)}{a^{n_r - k - 1}}\\ &= \sum_{n=n_r-k-1}^{n_r-1} \frac{b}{a^{n+1}}(x(n)) - \bar{x})g(x(n))\\ &\quad + \sum_{n=n_r-k-1}^{n_r-1} \frac{c}{a^{n+1}}(x(n-k) - \bar{x})g(x(n-k))\\ &\quad + b\bar{x} \sum_{n=n_r-k-1}^{n_r-1} \frac{1}{a^{n+1}} g(x(n)) + c\bar{x} \sum_{n=n_r-k-1}^{n_r-1} \frac{1}{a^{n+1}} g(x(n-k))\end{aligned}$$

which yields

$$\begin{aligned}&x(n_r) - a^{k+1}x(n_r - k - 1)\\ &= b \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1}(x(n) - \bar{x})g(x(n))\\ &\quad + c \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1}(x(n-k) - \bar{x})g(x(n-k))\\ &\quad + b\bar{x} \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1} g(x(n)) + c\bar{x} \sum_{n=n_r-n-1}^{n_r-1} a^{n_r-k-1} g(x(n-k)).\end{aligned} \tag{2.13}$$

Then by noting (ii) and (2.11), it follows from (2.13) that when n_r is sufficiently large,

$$\begin{aligned}x(n_r) &< a^{k+1}\bar{x} + b \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1}(L + \epsilon - \bar{x})\\ &\quad + c \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1}(L + \epsilon - \bar{x})\\ &\quad + (b+c)g(l-\epsilon)\bar{x} \sum_{n=n_r-k-1}^{n_r-1} a^{n_r-n-1}\\ &= a^{k+1}\bar{x} + (b+c)(L + \epsilon - \bar{x})\frac{1 - a^{k+1}}{1-a}\\ &\quad + (b+c)g(l-\epsilon)\bar{x}\frac{1-a^{k+1}}{1-a}.\end{aligned} \tag{2.14}$$

Since $b + c = 1 - a$, we see that (2.14) yields

$$x(n_r) < a^{k+1}\bar{x} + (L + \epsilon - \bar{x})(1 - a^{k+1}) + g(l-\epsilon)\bar{x}(1 - a^{k+1}). \tag{2.15}$$

By taking limits on (2.15) and noting that ϵ is arbitrary, we obtain

$$L \le a^{k+1}\bar{x} + (L-\bar{x})(1-a^{k+1}) + g(l)\bar{x}(1-a^{k+1}).$$

Then it follows that

$$L \le \bar{x}\left[1 + (a^{-(k+1)}-1)(g(l)-1)\right]$$

which is (2.4). By a dual argument we may show that (2.5) holds also.

In the following, we show that $L = l = \bar{x}$. First, assume that (2.1) holds. Let

$$G(x) = \begin{cases} g(x), & x \ge 0, \\ g(0), & x < 0, \end{cases}$$

$$u(x) = \bar{x}\left[1 + (a^{-(k+1)}-1)(G(x)-1)\right] \quad \text{and} \quad U(x) = x - u(u(x)), \quad x \ge 0.$$

Here we extend the definition of $g(x)$ in case of $u(x) < 0$. Observe that $U(\bar{x}) = \bar{x} - u(u(\bar{x})) = 0$ and for $x > 0$,

$$\begin{aligned} U'(x) &= 1 - u'(u(x))u'(x) \\ &= 1 - ((a^{-(k+1)}-1)\bar{x})^2 G'(x)G'(u(x)) \\ &= \begin{cases} 1 - ((a^{-(k+1)}-1)\bar{x})^2 g'(x)g'(u(x)) & \text{if } u(x) > 0, \\ 1 & \text{if } u(x) < 0, \end{cases} \end{aligned}$$

which, in view of (2.1) implies that $U'(x) > 0$ for $x > 0$ and $u(x) \ne 0$. Clearly, (2.4) and (2.5) yield $l \ge u(u(l))$, that is, $U(l) = l - u(u(l)) \ge 0$. Hence, $l \ge \bar{x}$. However, we know that $l \le \bar{x}$. Hence, we must have $l = \bar{x}$. Then it follows that

$$L \le \bar{x}\left[1 + (a^{-(k+1)}-1)(g(l)-1)\right] = \bar{x}\left[1 + (a^{-(k+1)}-1)(g(\bar{x})-1)\right] = \bar{x}$$

which implies that $L = \bar{x}$.

Next assume that (2.2) holds. Let

$$v(x) = xu(x) = (2 - a^{-(k+1)})\bar{x}x + (a^{-(k+1)}-1)\bar{x}xg(x), \quad x \ge 0.$$

Since

$$v'(x) = (2 - a^{-(k+1)})\bar{x} + (a^{-(k+1)}-1)\bar{x}(xg(x))',$$

we see that under the condition (2.2), $v'(x) > 0$. However, (2.4) and (2.5) yield $v(L) \le Ll \le v(l)$ which implies that $L = l$.

Hence, in either case of (2.1) or (2.2), we have $L = l = \bar{x}$. Then it follows that $x(n) \to \bar{x}$ as $n \to \infty$. The proof of the theorem is complete. □

Remark 2.2. Consider the following difference equation in a more general form

$$x(n+1) = \alpha x(n) + \beta x(n)h(x(n)) + \gamma x(n-k)h(x(n-k)) \tag{2.16}$$

where $0 < \alpha < 1$, $\beta \ge 0$ and $\gamma \ge 0$ with $\beta + \gamma > 0$ are constants, $h \in C[[0,\infty),[0,\infty)]$ is a decreasing function, and k is a positive integer.

Observe that Eq. (2.16) can be written as

$$\begin{aligned} x(n+1) = \alpha x(n) + \frac{\beta(1-\alpha)}{\beta+\gamma} x(n) \left[\frac{\beta+\gamma}{1-\alpha} h(x(n)) \right] \\ + \frac{\gamma(1-\alpha)}{\beta+\gamma} x(n-k) \left[\frac{\beta+\gamma}{1-\alpha} h(x(n-k)) \right]. \end{aligned} \tag{2.17}$$

Clearly, Eq. (2.17) is in the form of (1.2) with

$$a = \alpha, \quad b = \frac{\beta(1-\alpha)}{\beta+\gamma}, \quad c = \frac{\gamma(1-\alpha)}{\beta+\gamma} \quad \text{and} \quad g(x) = \frac{\beta+\gamma}{1-\alpha} h(x).$$

Hence, the following result is a direct consequence of Theorem 2.1.

Corollary 2.3. *Assume that there is a positive number* $\bar{x}$ *such that* $h(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}$*. Suppose also that* $x(\alpha + \beta h(x))$ *is increasing and* h *is differentiable such that either*

$$h'(x) > \frac{(1-\alpha)\alpha^{k+1}}{\bar{x}(\beta+\gamma)(\alpha^{k+1}-1)}, \quad x > 0 \tag{2.18}$$

or

$$(xh(x))' > \frac{(1-\alpha)(2\alpha^{k+1}-1)}{(\beta+\gamma)(\alpha^{k+1}-1)}, \quad x > 0. \tag{2.19}$$

Then every positive solution $\{x(n)\}$ *of Eq.* (2.16) *converges to* $\bar{x}$ *as* $n \to \infty$*.*

3. APPLICATIONS

In this section, we apply the results obtained in the last section to some difference equations arising in biological applications.

Consider the following difference system

$$\begin{cases} x(n+1) = (1-\epsilon) f(x(n)) + \epsilon y(n), \\ y(n+1) = (1-\epsilon) y(n) + \epsilon f(x(n)), \qquad n = 0, 1, \ldots, \\ x(0) \geq 0, \ y(0) \geq 0, \ x(0) + y(0) > 0, \end{cases} \tag{3.1}$$

where $0 < \epsilon < 1$ is a positive constant and $f \in C[[0,\infty),[0,\infty)]$. Sys. (3.1) is a population model proposed by Newman *et al.* [12] which assumes symmetric dispersal between active population $x(n)$ and refuge population $y(n)$. The chaotic behavior of positive solutions of Sys. (3.1) is studied in [12] by numerical simulations. While in [2], various properties of solutions of (3.1) are studied and several results on the asymptotic behavior of solutions of (3.1) are obtained. In addition, a sufficient condition on the global stability of positive solutions of (3.1) is obtained in [1] recently.

Motivated by theoretical interest and plausible applications, we now consider the following more general difference system

$$\begin{cases} x(n+1) = \mu f(x(n)) + \delta y(n), \\ y(n+1) = \nu y(n) + \sigma f(x(n)), \qquad n = 0, 1, \ldots, \\ x(0) \geq 0, \ y(0) \geq 0, \ x(0) + y(0) > 0, \end{cases} \tag{3.2}$$

where μ, ν, δ, σ are nonnegative constants.

When $\mu = \nu = 1 - \epsilon$ and $\delta = \sigma = \epsilon$, Sys. (3.2) reduces to Sys. (3.1). By a simple calculation, Sys. (3.2) can be converted into the second order difference equation

$$x(n+1) = \nu x(n) + \mu f(x(n)) + (\delta\sigma - \mu\nu) f(x(n-1)), \quad n = 0, 1, \dots \tag{3.3}$$

In particular, when $f(x) = xh(x)$, Sys. (3.2) and Eq. (3.3) reduce to

$$\begin{cases} x(n+1) = \mu x(n) h(x(n)) + \delta y(n), \\ y(n+1) = \nu y(n) + \sigma x(n) h(x(n)), \qquad n = 0, 1, \dots, \\ x(0) \geq 0, \ y(0) \geq 0, \ x(0) + y(0) > 0, \end{cases} \tag{3.4}$$

and

$$x(n+1) = \nu x(n) + \mu x(n) h(x(n)) + (\delta\sigma - \mu\nu) x(n) h(x(n-1)), \quad n = 0, 1, \dots, \tag{3.5}$$

respectively. When h is decreasing and

$$0 < \nu < 1, \quad \delta\sigma - \mu\nu \geq 0, \tag{3.6}$$

Eq. (3.5) is in the form of (2.16). Furthermore, when

$$\mu + \delta\sigma - \mu\nu > 0 \quad \text{and} \quad h(\infty) < \frac{1-\nu}{\mu + \delta\sigma - \mu\nu} < h(0) \tag{3.7}$$

there is a positive number $\bar{x}$ such that $h(\bar{x}) = \frac{1-\nu}{\mu+\delta\sigma-\mu\nu}$. Then it is easy to check that $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right)$ is the only positive equilibrium of Sys. (3.4).

By Corollary 2.3, we may have the following result.

Theorem 3.1. *Assume that* (3.6) *and* (3.7) *hold,* h *is decreasing and* $x(\nu + \mu h(x))$ *is increasing. Suppose also that either*

$$h'(x) > \frac{-\nu^2}{\bar{x}(\mu + \delta\sigma - \mu\nu)(\nu + 1)}, \quad x > 0 \tag{3.8}$$

or

$$(xh(x))' > \frac{1 - 2\nu^2}{(\mu + \delta\sigma - \mu\nu)(\nu + 1)}, \quad x > 0. \tag{3.9}$$

Then every positive solution $(x(n), y(n))$ *of Sys.* (3.4) *tends to its positive equilibrium* $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right)$.

Proof. We know that Sys. (3.4) can be converted into (3.5). Eq. (3.5) is in the form of (2.16) with $\alpha = \nu, \beta = \mu$, $\gamma = \delta\sigma - \mu\nu$ and $k = 1$, and has the positive equilibrium $\bar{x}$. By the assumtions, $x(\alpha + \beta h(x)) = x(\nu + \mu h(x))$ is increasing. Observing that

$$\frac{(1-\alpha)\alpha^{k+1}}{\bar{x}(\beta + \gamma)(\alpha^{k+1} - 1)} = \frac{(1-\nu)\nu^2}{\bar{x}(\mu + \delta\sigma - \mu\nu)(\nu^2 - 1)} = \frac{-\nu^2}{\bar{x}(\mu + \delta\sigma - \mu\nu)(\nu + 1)}$$

and

$$\frac{(1-\alpha)(2\alpha^{k+1}-1)}{(\beta+\gamma)(\alpha^{k+1}-1)} = \frac{(1-\nu)(2\nu^2-1)}{(\mu+\delta\sigma-\mu\nu)(\nu^2-1)} = \frac{1-2\nu^2}{(\mu+\delta\sigma-\mu\nu)(\nu+1)}$$

we see that when (3.8) or (3.9) holds, then (2.18) or (2.19) is satisfied. Hence by Corollary 2.3, every positive solution $\{x(n)\}$ of Eq. (3.5) converges to $\bar{x}$ as $n \to \infty$. Then from (3.4) we see that

$$\delta y(n) = x(n+1) - \mu x(n)h(x(n)) \to \bar{x} - \mu\bar{x}h(\bar{x}) \quad \text{as} \quad n \to \infty.$$

Noting

$$\bar{x} - \mu\bar{x}h(\bar{x}) = \left(1 - \mu\frac{1-\nu}{\mu+\delta\sigma-\mu\nu}\right)\bar{x} = \frac{\delta\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}$$

we see that

$$\delta y(n) \to \frac{\delta\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x} \quad \text{as} \quad n \to \infty$$

which yields

$$y(n) \to \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x} \quad \text{as} \quad n \to \infty.$$

Hence, it follows that every positive solution $(x(n), y(n))$ of Sys. (3.4) converges to $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right)$. The proof is complete. □

When $h(x) = e^{-qx}$ where q is a positive constant, Sys. (3.4) can be converted into the second order difference equation

$$\begin{aligned} x(n+1) = \nu x(n) + \mu x(n)e^{-qx(n)} \\ + (\delta\sigma - \mu\nu)x(n-1)e^{-qx(n-1)}, \quad n = 0, 1, \dots \end{aligned} \tag{3.10}$$

With (3.6), Eq. (3.10) is a special case of the equation

$$x(n+1) = \alpha x(n) + \beta x(n)e^{-qx(n)} + \gamma x(n-k)e^{-qx(n-k)}, \quad n = 0, 1, \dots \tag{3.11}$$

where $0 < \alpha < 1, \beta \geq 0, \gamma \geq 0, q > 0$ and $k \in \{1, 2, \dots\}$. Clearly, Eq. (3.11) is in the form of (2.16). When $\beta = 0$, this equation reduces to

$$x(n+1) = \alpha x(n) + \gamma x(n-k)e^{-qx(n-k)}, \quad n = 0, 1, \dots \tag{3.12}$$

Eq. (3.12) is a discrete analogue of a model which has been used in describing the dynamics of Nicholson's blowflies [13], see also [8]. The asymptotic behavior of positive solutions of this equation and some related forms has been studied by numerous authors, see for example, [5, 9, 11, 17–22] and the references cited therein.

When $h(x) = e^{-qx}$ and $\alpha+\beta+\gamma > 1$, $\bar{x} = \frac{1}{q}\ln\frac{\beta+\gamma}{1-\alpha}$ is a positive constant satisfying $h(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}$ and so is a positive equilibrium of Eq. (3.11). Noting that

$$(\alpha x + \beta x h(x))' = (\alpha x + \beta x e^{-qx})' = \alpha + \beta(1-qx)e^{-qx}$$

and

$$(\alpha x+\beta xh(x))''=\beta q(qx-2)e^{-qx}$$

we see that $(\alpha x+\beta xh(x))'$ has minimum when $x=2/q$ and so

$$(\alpha x+\beta xh(x))'\geq(\alpha x+\beta xh(x))'|_{x=2/q}=\alpha-\beta e^{-2}.$$

Hence, when $\alpha-\beta e^{-2}\geq 0$, that is,

$$\beta\leq\alpha e^{2}, \tag{3.13}$$

$x(\alpha+\beta h(x))$ is increasing. Now, observe that

$$h'(x)=-qe^{-qx} \quad \text{and} \quad h''(x)=q^{2}e^{-qx}.$$

We see that $h'(x)\geq h'(0)=-q$. Hence if

$$-q>\frac{(1-\alpha)\alpha^{k+1}}{\bar{x}(\beta+\gamma)(\alpha^{k+1}-1)},$$

that is,

$$\frac{(1-\alpha)\alpha^{k+1}}{(\beta+\gamma)(1-\alpha^{k+1})\ln\frac{\beta+\gamma}{1-\alpha}}>1, \tag{3.14}$$

then (2.18) is satisfied. In addition, noting that

$$(xh(x))'=(1-qx)e^{-qx} \quad \text{and} \quad (xh(x))''=q(qx-2)e^{-qx}$$

we find that $(xh(x))'$ takes minimum when $x=2/q$ and so

$$(xh(x))'\geq(xh(x))'|_{x=2/q}=-e^{-2}.$$

Hence, if

$$-e^{-2}>\frac{(1-\alpha)(2-\alpha^{-(k+1)})}{(\beta+\gamma)(1-\alpha^{-(k+1)})},$$

that is,

$$\frac{(1-\alpha)(2\alpha^{k+1}-1)}{(\beta+\gamma)(1-\alpha^{k+1})}e^{2}>1, \tag{3.15}$$

then (2.19) is satisfied. Hence, from the above discussion, we have the following conclusion by Corollary 2.3: if (3.13) holds, and also either (3.14) holds or (3.15) holds, then every positive solution $\{x(n)\}$ of Eq. (3.11) tends to its positive equilibrium $\bar{x}$ as $n\to\infty$.

When $h(x)=e^{-qx}$, Sys. (3.4) becomes

$$\begin{cases} x(n+1)=\mu x(n)e^{-qx(n)}+\delta y(n), \\ y(n+1)=\nu y(n)+\sigma x(n)e^{-qx(n)}, \qquad n=0,1,\dots, \\ x(0)\geq 0,\ y(0)\geq 0,\ x(0)+y(0)>0, \end{cases} \tag{3.16}$$

and it can be converted into Eq. (3.10) which is in the form of (3.11) with $\alpha = \nu$, $\beta = \mu, \gamma = \delta\sigma - \mu\nu$ and $k = 1$.

From the above discussion, we know that $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right)$, where $\bar{x} = \frac{1}{q}\ln\frac{\mu+\delta\sigma-\mu\nu}{1-\nu}$, is the unique positive equilibrium of Sys. (3.16). In addition, (3.13), (3.14) and (3.15) become

$$\mu \leq \nu e^2, \tag{3.17}$$

$$\frac{\nu^2}{(\mu+\delta\sigma-\mu\nu)(\nu+1)\ln\frac{\mu+\delta\sigma-\mu\nu}{1-\nu}} > 1 \tag{3.18}$$

and

$$\frac{(1-2\nu^2)e^2}{(\mu+\delta\sigma-\mu\nu)(\nu+1)} > 1, \tag{3.19}$$

respectively. Hence, by Theorem 3.1 we have the following conclusion: when (3.17) holds, and either (3.18) or (3.19) holds, then every positive solution $(x(n), y(n))$ of Sys. (3.16) tends to its positive equilibrium $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right)$.

Example 3.2. Consider the difference system

$$\begin{cases} x(n+1) = \frac{1}{4}x(n)e^{-qx(n)} + \frac{1}{2}y(n), \\ y(n+1) = \frac{3}{4}y(n) + \frac{1}{2}x(n)e^{-qx(n)}, \qquad n = 0, 1, \ldots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.20}$$

which is in the form of (3.16) with $\mu = 1/4$, $\nu = 3/4$, $\delta = \sigma = 1/2$ and q is any positive constant. Hence, $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right) = (\bar{x}, (8/5)\bar{x})$, where $\bar{x} = \frac{1}{q}\ln\frac{\mu+\delta\sigma-\mu\nu}{1-\nu} = (1/q)\ln(5/4)$ is the unique positive equilibrium of Sys. (3.20). Clearly, (3.17) is satisfied. In addition, observing

$$\frac{\nu^2}{(\mu+\delta\sigma-\mu\nu)(\nu+1)\ln\frac{\mu+\delta\sigma-\mu\nu}{1-\nu}} = \frac{(3/4)^2}{(5/16)(7/4)\ln(5/4)} > 1$$

we see that (3.18) is satisfied. Hence, from the above conclusion, we see that every positive solution of Sys. (3.20) tends to its positive equilibrium $(\bar{x}, (8/5)\bar{x})$.

Example 3.3. Consider the difference system

$$\begin{cases} x(n+1) = \frac{3}{4}x(n)e^{-qx(n)} + \frac{3}{4}y(n), \\ y(n+1) = \frac{1}{4}y(n) + \frac{1}{3}x(n)e^{-qx(n)}, \qquad n = 0, 1, \ldots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.21}$$

which is in the form of (3.16) with $\mu = 3/4, \nu = 1/4, \delta = 3/4, \sigma = 1/3$ and q is any positive constant. Hence, $\left(\bar{x}, \frac{\sigma}{\mu+\delta\sigma-\mu\nu}\bar{x}\right) = (\bar{x}, (16/39)\bar{x})$, where

$\bar{x} = \frac{1}{q} \ln \frac{\mu+\delta\sigma-\mu\nu}{1-\nu} = \frac{1}{q} \ln(13/12)$ is the unique positive equilibrium of Sys. (3.20). Clearly, (3.17) is satisfied. In addition, observing

$$\frac{(1-2\nu^2)e^2}{(\mu+\delta\sigma-\mu\nu)(\nu+1)} = \frac{(1-2(1/4)^2)e^2}{(13/16)(5/4)} > 1$$

we see that (3.19) is satisfied. Hence, from the above conclusion, we see that every positive solution of Sys. (3.21) tends to its positive equilibrium $(\bar{x}, (16/39)\bar{x})$.

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