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SQUARE-ROOT BOUNDARIES FOR BESSEL PROCESSES AND THE HITTING TIMES OF RADIAL ORNSTEIN–UHLENBECK PROCESSES

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Abstract. This article deals with the first hitting times of a Bessel process to a square-root boundary. We obtain the explicit form of the distribution function of the hitting time by means of zeros of the confluent hypergeometric function with respect to the first parameter. In deducing the distribution function, the time that a radial Ornstein–Uhlenbeck process reaches a certain point is very useful and plays an important role. We also give its distribution function in the case that the starting point is closer to the origin than the arrival site.

Keywords: Bessel process, confluent hypergeometric function, first hitting time, radial Ornstein–Uhlenbeck process, square-root boundary.

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1. INTRODUCTION

In the theory of mathematical finance, Bessel processes are significant in analyzing CIR model, which gives the evolution of instant interest rates, and random times (e.g, default time and optimal stopping time) are quite important objects. In particular, the first hitting times of asset prices are useful for the theory of American options and exotic derivative securities. For applications of Bessel processes and first hitting times to mathematical finance, see [10, 30] for example.

In probability theory, the first hitting time of one-dimensional diffusion process is interesting and important in and of itself. The hitting time of Bessel process to a given point has been especially investigated. The explicit form of its distribution function is given in [16]. Asymptotic behavior of the tail probability is given in [16, 17] and the higher terms are discussed in [4, 14, 19, 20]. In addition, results on the density function are in [3, 5, 15, 23].

Our interest shifts towards the hitting time to a moving point rather than a fixed point. In this article, we investigate the hitting time of the Bessel process to a point which goes away from the origin with time of order the square root of the time. For $\nu \in \mathbb{R}$ and $a \geqq 0$ let $\{R_a^\nu(t)\}_{t \geqq 0}$ be the Bessel process with index ν starting from a. For $p, q > 0$ we write $\tau_a^\nu(p,q)$ for the infimum of $s > 0$ being subject to that $R_a^\nu(s) = \sqrt{p^2 + q^2 s}$. If $\nu \geqq 0$ and $a, b > 0$, the explicit form of the expectation of $\{1 + \tau_a(b,b)\}^{-\rho}$ for $\rho \geqq 0$ is given in [29] by a ratio of the confluent hypergeometric functions. Moreover it is mentioned in [9] that the Lamperti representation theorem (cf. [24]) allows the extension of several results in [6, 7, 29]. In addition, we remark that the time of the first arrival on a sphere of the d-dimensional Ornstein–Uhlenbeck process with parameter $1/2$ has high relevance to the first hitting time to a square-root boundary of the Brownian motion moving on $\mathbb{R}^d$. One-dimensional case is discussed precisely in [27].

We should mention that the contribution of $\tau_a^\nu(p,q)$ in the theory of mathematical finance. Pairs trading is an investment strategy to obtain profit from spread, which is the difference of prices between two assets. The spread is expected to converge to a certain level and thus mathematical formulation of optimization problem on the pairs trading requires information on the first hitting time of a stochastic process which has the mean-reversion. Ornstein–Uhlenbeck process is frequently used since it is considered that this process is one of good candidates for describing the pair value (e.g. [8]). With the help of the Cameron-Martin theorem and the time change formula, we can find that the first hitting time to a sphere of Ornstein–Uhlenbeck process moving on $\mathbb{R}^d$ is represented by $\tau_a^{d/2-1}(p,q)$ and the stochastic integral of a suitable function over 0 to $\tau_a^{d/2-1}(p,q)$.

The purpose in this article is to provide the explicit form of the distribution function of $\tau_a^\nu(p,q)$ in the case that $\nu > -1$. We deduce a formula for the distribution function of the first hitting time of the radial Ornstein–Uhlenbeck process, which is often called the Ornstein–Uhlenbeck-Bessel process, and prove that this formula leads to the explicit form of $P(\tau_a^\nu(p,q) \leqq t)$, which is represented by means of the zeros of the confluent hypergeometric function with respect to the first parameter.

This article is organized as follows. In Section 2, an explicit form of the distribution function of $\tau_a^\nu(p,q)$ is provided for $a, p \geqq 0$ and $q > 0$ with $a \neq p$. In Section 3, we show that the distribution function of $\tau_a(p,q)$ is represented by means of the hitting time of a suitable radial Ornstein–Uhlenbeck process. Section 4 deals with the case that a, p and q are all positive and investigate the first hitting time of the radial Ornstein–Uhlenbeck process for the proof of our results on $\tau_a^\nu(p,q)$. Section 5 and Section 6 are devoted to the proofs for $a = 0$ and $p = 0$, respectively. In addition, we will use $C_1, C_2, \ldots, C_{22}$ for suitable constants which are independent of variables throughout this paper.

2. SQUARE-ROOT BOUNDARY FOR BESSEL PROCESS

Bessel process with index $\nu \in \mathbb{R}$ means the one-dimensional diffusion process with the following generator:

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}.$$

In the case that $2\nu+2$ is a positive integer, this process can be represented as the radial part of $(2\nu+2)$-dimensional Brownian motion and thus $2\nu+2$ is often called the dimension of the Bessel process. The boundary point ∞ is natural for any $\nu \in \mathbb{R}$. The other boundary point is 0. It is an entrance but not exit boundary for $\nu \geqq 0$ and is an exit but not entrance boundary for $\nu \leqq -1$. When $-1 < \nu < 0$, the boundary 0 is regular. Although a regular boundary is classified into several cases, we assume that 0 is instantaneously reflecting in this paper. For more details, see [1, 22] for example. Throughout this paper we treat only the case that $\nu > -1$. In this case, the transition density function $p_0^\nu(t,x,y)$ with respect to the Lebesgue measure is given by

$$p_0^\nu(t,x,y) = \frac{y^{\nu+1}}{x^\nu t} \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right)$$

(e.g. [26, p. 446], [28, p. 579]), where I_ν denotes the modified Bessel function of the first kind of order ν.

Let $a \geqq 0$ and we write $\{R_a^\nu(t)\}_{t \geqq 0}$ for the Bessel process with index ν starting from a. For $p, q \geqq 0$ let

$$\tau_a^\nu(p,q) = \inf\{s > 0\,;\, R_a^\nu(s) = \sqrt{p^2 + q^2 s}\},$$

which is called the hitting time of the Bessel process $\{R_a^\nu(t)\}_{t \geqq 0}$ to the square-root boundary, and remark that $\tau_a^\nu(p,0)$ means the first hitting time of $\{R_a^\nu(t)\}_{t \geqq 0}$ to the given point p.

The purpose in this section is to give an explicit form of the distribution function of $\tau_a^\nu(p,q)$. In order to describe our results, we need to discuss the zeros of confluent hypergeometric functions with respect to the first parameter. We use the usual notation F and U for confluent hypergeometric functions of the first kind and the second kind, respectively. The functions F and U are often called Kummer function and Tricomi function, respectively.

Lemma 2.1. *Let $\nu > -1$, $\gamma > 0$ and $x > 0$.*

(1) *The function $\lambda \mapsto F(-\lambda, \nu+1, \gamma x^2)$ has countably many zeros and all zeros are positive and of multiplicity 1. Moreover we write $\{\lambda_{n,x}^{\nu,\gamma}\}_{n=1}^\infty$ for the increasing sequence of the zeros and have that*

$$\lim_{n\to\infty} \frac{\lambda_{n,x}^{\nu,\gamma}}{n^2} = \frac{\pi^2}{4\gamma x}. \tag{2.1}$$

(2) *The function $\lambda \mapsto U(-\lambda, \nu+1, \gamma x^2)$ has countably many zeros and all zeros are positive and of multiplicity 1. Moreover we write $\{\kappa_{n,x}^{\nu,\gamma}\}_{n=1}^\infty$ for the increasing sequence of the zeros and have that $\kappa_{n,x}^{\nu,\gamma} > n-1$ for each $n \geqq 1$.*

Theorem 4.4 in [13] implies that Lemma 2.1 (2) has been already established. Although it seems that we can carry out the similar computation for the proof of Lemma 2.1 (1), we need quite complicated calculations to obtain the necessary information on zeros of the function $\lambda \mapsto F(-\lambda, \nu+1, \gamma x^2)$. In this paper, we do not adopt the argument used in [13] and will apply the method given in [23]. The proof of the first claim is deferred to Section 4.

For a function f of several variables the notation f' will be used to denote the partial derivative with respect to the first variable and we finish the preparation for giving an explicit form of the distribution function of $\tau_a^\nu(p,q)$. The following theorem is one of our results and its proof is deferred to Section 4.

Theorem 2.2. *Let $\nu > -1$ and $q > 0$. For any $t > 0$ we have that, if $0 < a < p$,*

$$P(\tau_a^\nu(p,q) \leqq t) = 1 - \sum_{n=1}^\infty \frac{F(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2a^2/2p^2)}{\lambda_{n,q}^{\nu,1/2} F'(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2}t\right)^{-\lambda_{n,q}^{\nu,1/2}} \tag{2.2}$$

and that, if $0 < p < a$,

$$P(\tau_a^\nu(p,q) \leqq t) = 1 - \sum_{n=1}^\infty \frac{U(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2a^2/2p^2)}{\kappa_{n,q}^{\nu,1/2} U'(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2}t\right)^{-\kappa_{n,q}^{\nu,1/2}}. \tag{2.3}$$

For the proof of Theorem 2.2 it is useful to represent the distribution function of $\tau_a^\nu(p,q)$ by means of the first hitting time of a suitable radial Ornstein–Uhlenbeck process. Details will be described in Section 3.

The remainder of this section is devoted to giving the result on the limiting case, that is the case that either $a = 0$ or $p = 0$. We first give the result on the case that $a = 0$.

Theorem 2.3. *Let $\nu > -1$ and $p, q > 0$. We have that*

$$P(\tau_0^\nu(p,q) \leqq t) = 1 - \sum_{n=1}^\infty \frac{1}{\lambda_{n,q}^{\nu,1/2} F'(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2}t\right)^{-\lambda_{n,q}^{\nu,1/2}} \tag{2.4}$$

for any $t > 0$.

Since

$$\lim_{z\to 0} F(\alpha, \beta, z) = 1 \tag{2.5}$$

(cf. [25, p. 288]), it seems that (2.4) will be obtained by taking the limit of (2.2) as $a \downarrow 0$ and we find that this argument can be justified. Theorem 2.3 will be established in Section 5.

The following theorem is the result on the case that $p = 0$.

Theorem 2.4. *Let $\nu > -1$ and $a, q > 0$. We have that*

$$P(\tau_a^\nu(0,q) \leqq t) = 1 - \sum_{n=1}^\infty \frac{1}{\kappa_{n,q}^{\nu,1/2} U'(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(\frac{a^2}{2t}\right)^{\kappa_{n,q}^{\nu,1/2}} \tag{2.6}$$

for any $t > 0$.

It is expected that (2.6) can be obtained by taking limit of (2.3) as $p \downarrow 0$ with the help of the fact that

$$U(\alpha, \beta, x) = x^{-\alpha}(1 + O[x^{-1}]) \tag{2.7}$$

as $x \to \infty$ (cf. [25, p. 289]). Unfortunately we do not find an appropriate calculation for justifying this argument. However other argument leads us to Theorem 2.4 and the proof will be given in Section 6.

3. BASIC PROPERTIES OF $\tau_a^\nu(p,q)$

When we investigate $\tau_a^\nu(p,q)$, the first hitting time of a radial Ornstein–Uhlenbeck process plays an important role. We start to give information on this process. For $\nu \in \mathbb{R}$ and $\gamma > 0$ the radial Ornstein–Uhlenbeck process with index ν and parameter γ is the one-dimensional diffusion process with generator

$$L_{\nu,\gamma} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} - \gamma x\right)\frac{d}{dx}. \tag{3.1}$$

In the case when $\gamma = 0$, we can regard this process as the Bessel process. Similarly to the Bessel process, in the case when $2\nu + 2$ is a positive integer, this process is represented as the radial part of the $(2\nu+2)$-dimensional Ornstein–Uhlenbeck process. The classification of boundary points is the same as that of the Bessel process and we omit the details. In the case when $\nu > -1$, it is known that the transition density $p_\gamma^\nu(t,x,y)$ with respect to the Lebesgue measure is represented by

$$p_\gamma^\nu(t,x,y) = \frac{\gamma y^{\nu+1}}{x^\nu \sinh\gamma t}\exp\left\{\gamma(\nu+1)t - \frac{\gamma(e^{\gamma t}x^2 + e^{-\gamma t}y^2)}{2\sinh\gamma t}\right\} I_\nu\left(\frac{\gamma xy}{\sinh\gamma t}\right)$$

(cf. [1, pp. 139–140], [28, p. 581]). It is easy to show that $p_\gamma^\nu(t,x,y)$ is asymptotically equal to $p_0^\nu(t,x,y)$ as $\gamma \downarrow 0$.

Let $a \geqq 0$ and $p, q > 0$ with $a \neq p$. Since $\{\alpha^{-1}R_{\alpha c}^\nu(\alpha^2 t)\}_{t\geqq 0}$ is identical in law with $\{R_c^\nu(t)\}_{t\geqq 0}$ for each $c \geqq 0$ and $\alpha > 0$, we have that, if $p > 0$,

$$\begin{aligned} P(\tau_a^\nu(p,q) \leqq t) &= P\left(\inf\left\{s > 0\,;\, R_{qa/p}^\nu\left(\frac{q^2}{p^2}s\right) = q\sqrt{1 + \frac{q^2}{p^2}s}\right\} \leqq t\right) \\ &= P\left(\inf\{s > 0\,;\, R_{qa/p}^\nu(s) = q\sqrt{1+s}\} \leqq \frac{q^2}{p^2}t\right) \end{aligned}$$

for any $t > 0$, which coincides with

$$P\left(\inf\{s > 0\,;\, e^{-s/2}R_{qa/p}^\nu(e^s - 1) = q\} \leqq \log\left(1 + \frac{q^2}{p^2}t\right)\right). \tag{3.2}$$

For $\gamma > 0$ let $\{S_a^{\nu,\gamma}(t)\}_{t\geqq 0}$ be a radial Ornstein–Uhlenbeck process with index ν and parameter γ starting from a. We try to represent (3.2) by the distribution function of the first hitting time of $\{S_a^{\nu,\gamma}(t)\}_{t\geqq 0}$ for a suitable γ. Let

$$\sigma_{b,c}^{\nu,\gamma} = \inf\{s > 0\,;\, S_b^{\nu,\gamma}(s) = c\}$$

for $b, c \geqq 0$ with $b \neq c$.

Lemma 3.1. *If $p > 0$, we have that*

$$P(\tau_a^\nu(p,q) \leqq t) = P\left(\sigma_{qa/p,q}^{\nu,1/2} \leqq \log\left(1 + \frac{q^2}{p^2}t\right)\right) \tag{3.3}$$

for any $t > 0$.

Proof. It is known that the time change formula yields that $\{S_a^{\nu,\gamma}(t)\}_{t \geqq 0}$ is identical in law with $\{e^{-\gamma t} R_a^\nu((e^{2\gamma t} - 1)/2\gamma)\}_{t \geqq 0}$. Thus (3.3) can be easily deduced from (3.2). □

Formula (3.3) gives the probability that the Bessel process reaches a square-root boundary.

Proposition 3.2. *If $p > 0$ and $a \neq p$, we have that*

$$P(\tau_a^\nu(p,q) < \infty) = 1. \tag{3.4}$$

Proof. It follows from (3.3) that (3.4) is equivalent to $P(\sigma_{qa/p,q}^{\nu,1/2} < \infty) = 1$ and hence we concentrate on showing that

$$P(\sigma_{b,c}^{\nu,\gamma} < \infty) = 1 \tag{3.5}$$

for $\gamma > 0$, $b \geqq 0$ and $c > 0$ with $b \neq c$.

We first consider the case that $b \neq 0$. The Laplace transform of $\sigma_{b,c}^{\nu,\gamma}$ has been derived in [12, p. 325] and it follows that, for $0 < b < c$

$$E[e^{-\lambda \sigma_{b,c}^{\nu,\gamma}}] = \frac{F(\lambda/2\gamma, \nu+1, \gamma b^2)}{F(\lambda/2\gamma, \nu+1, \gamma c^2)} \tag{3.6}$$

and that, for $0 < c < b$

$$E[e^{-\lambda \sigma_{b,c}^{\nu,\gamma}}] = \frac{U(\lambda/2\gamma, \nu+1, \gamma b^2)}{U(\lambda/2\gamma, \nu+1, \gamma c^2)}. \tag{3.7}$$

Since $E[e^{-\lambda \sigma_{b,c}^{\nu,\gamma}}]$ converges to $P(\sigma_{b,c}^{\nu,\gamma} < \infty)$ as $\lambda \downarrow 0$, it is sufficient to give the limiting values of $F(\alpha, \beta, x)$ and $U(\alpha, \beta, x)$ as $\alpha \downarrow 0$ for $\beta > 0$ and $x > 0$.

In order to avoid complicated indices, we use $M(\kappa, \mu, z)$ and $W(\kappa, \mu, z)$ for Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$, respectively. It is known that both $M(\,\cdot\,, \mu, z)$ and $W(\,\cdot\,, \mu, z)$ are holomorphic on $\mathbb{C}$ (cf. [2], [13, pp. 68–89]). Hence the formulas

$$F(\alpha, \beta, x) = e^{x/2} x^{-\beta/2} M\left(\frac{\beta}{2} - \alpha, \frac{\beta - 1}{2}, x\right), \tag{3.8}$$

$$U(\alpha, \beta, x) = e^{x/2} x^{-\beta/2} W\left(\frac{\beta}{2} - \alpha, \frac{\beta - 1}{2}, x\right) \tag{3.9}$$

(cf. [25, p. 304]) give that

$$\lim_{\alpha \to 0} F(\alpha, \beta, x) = e^{x/2} x^{-\beta/2} M\left(\frac{\beta}{2}, \frac{\beta - 1}{2}, x\right),$$

$$\lim_{\alpha \to 0} U(\alpha, \beta, x) = e^{x/2} x^{-\beta/2} W\left(\frac{\beta}{2}, \frac{\beta - 1}{2}, x\right).$$

Since

$$M\left(\frac{\beta}{2}, \frac{\beta-1}{2}, x\right) = W\left(\frac{\beta}{2}, \frac{\beta-1}{2}, x\right) = e^{-x/2}x^{\beta/2}$$

(cf. [25, p. 305]), we conclude that

$$\lim_{\alpha\to 0} F(\alpha,\beta,x) = \lim_{\alpha\to 0} U(\alpha,\beta,x) = 1. \tag{3.10}$$

Hence the right hand sides of (3.6) and (3.7) converge to 1 as $\lambda \downarrow 0$, which yields (3.5) for $b \neq 0$.

We next consider the case that $b = 0$ and the calculation is easy since

$$E[e^{-\lambda\sigma_{0,c}^{\nu,\gamma}}] = \frac{1}{F(\lambda/2\gamma, \nu+1, \gamma c^2)} \tag{3.11}$$

(cf. [12, p. 324]). In virtue of (3.10), we can obtain (3.5) for $b = 0$ by taking the limit as $\lambda \downarrow 0$. □

4. THE DISTRIBUTION FUNCTION OF $\tau_a^\nu(p,q)$

Our goal of this section is to prove Theorem 2.2. Formula (3.3) implies that it is sufficient to give an explicit form of the distribution function of $\sigma_{b,c}^{\nu,\gamma}$ for $b, c > 0$ with $b \neq c$.

The case that $0 < c < b$ has been discussed. Since the Laplace transform of $\sigma_{b,c}^{\nu,\gamma}$ is given as formula (3.7), the Heaviside expansion theorem (cf. [21, p. 281]) yields that

$$P(\sigma_{b,c}^{\nu,\gamma} \leqq t) = 1 - \sum_{n=1}^{\infty} \frac{U(-\kappa_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{\kappa_{n,c}^{\nu,\gamma} U'(-\kappa_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} e^{-2\gamma\kappa_{n,c}^{\nu,\gamma} t}$$

for any $t > 0$ (cf. [13, Theorem 5.1]). Hence we can conclude (2.3) by replacement of b, c, γ and t with qa/p, q, $1/2$ and $\log(1 + q^2t/p^2)$, respectively.

We concentrate on proving (2.2). When we discuss the case that $0 < b < c$, it seems that the similar computation to the case that $0 < c < b$ can be carried out since the Laplace transform of $\sigma_{b,c}^{\nu,\gamma}$ is given by the similar form. However we need quite complicated calculations when we try to obtain the necessary information on zeros of the function $\lambda \mapsto F(-\lambda, \nu+1, \gamma b^2)$ similarly. In this paper, we do not adopt the argument used in [13] and will apply the method given in [23] to derive the density function of $\sigma_{a,b}^{\nu,\gamma}$.

Since the generator of the radial Ornstein–Uhlenbeck process is given by (3.1), a simple calculation shows that the speed measure m and the scale function s can be chosen in the following way:

$$m(dx) = 2x^{2\nu+1}e^{-\gamma x^2}\,dx, \quad s(x) = \int_1^x y^{-2\nu-1}e^{\gamma y^2}\,dy \tag{4.1}$$

(cf. [1, p. 138], [28, p. 518]).

For a real valued function u on $(0,\infty)$ we define

$$u^+(x) = \lim_{\varepsilon\downarrow 0} \frac{u(x+\varepsilon)-u(x)}{s(x+\varepsilon)-s(x)}$$

for $x>0$ if the limit exists. In the case that u is differentiable, we have by (4.1) that

$$u^+(x) = u'(x)x^{2\nu+1}e^{-\gamma x^2}. \tag{4.2}$$

Lemma 4.1. *For each $\lambda>0$ the solution of the equation*

$$L_{\nu,\gamma}u(x) + \lambda u(x) = 0, \quad x>0 \tag{4.3}$$

together with the boundary condition

$$\lim_{x\downarrow 0} u(x) = 1, \tag{4.4}$$

$$\lim_{x\downarrow 0} u^+(x) = 0 \tag{4.5}$$

is given by

$$u_\lambda(x) = F\Big(-\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big). \tag{4.6}$$

Proof. We put $v(x) = u(\sqrt{x/\gamma})$ for $x>0$. The standard calculation shows that (4.3) yields that v satisfies the following equation:

$$xv''(x) + (\nu+1-x)v'(x) + \frac{\lambda}{2\gamma}v(x) = 0. \tag{4.7}$$

It is known that the general solution of (4.7) is

$$u(x) = \zeta_1 F\Big(-\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big) + \zeta_2 U\Big(-\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big), \tag{4.8}$$

where ζ_1 and ζ_2 are arbitrary constants (e.g. [25, p. 270]).

We shall decide the constants ζ_1 and ζ_2 satisfying (4.4) and (4.5). When we apply asymptotic behavior of $U(\alpha,\beta,x)$ as $x\downarrow 0$ given in [25, p. 288], we need to consider that α is a negative integer or not since the leading term of $U(\alpha,\beta,x)$ contains $1/\Gamma(\alpha)$.

We start to investigate the case that $\lambda/2\gamma$ is not an integer. Since the asymptotic behavior of $U(\alpha,\beta,x)$ as $x\downarrow 0$ is different between the case that $\nu \geqq 0$ and the case that $-1<\nu<0$, we need to consider these two cases individually and first treat the case that $\nu\geqq 0$. It can be easily obtained that

$$\lim_{z\to 0} |U(\alpha,\beta,z)| = \infty \tag{4.9}$$

for $\beta \geqq 1$ unless α is an integer with $\alpha \leqq 0$ (cf. [25, p. 288]). This gives that ζ_2 must be 0 and we deduce from (2.5) that ζ_1 is equal to 1. Hence the function u satisfying (4.3) and (4.4) is

$$u(x) = F\Big(-\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big),$$

which is the analytic function on $\mathbb{R}$. With the help of (4.2) and (2.5), we can easily check that this function satisfies (4.5) since

$$\frac{d}{dz}F(\alpha,\beta,z) = \frac{\alpha}{\beta}F(\alpha+1,\beta+1,z) \tag{4.10}$$

holds (cf. [25, p. 264]).

We shall discuss the case that $-1<\nu<0$. Note that

$$\frac{d}{dz}U(\alpha,\beta,z) = -\alpha U(\alpha+1,\beta+1,z)$$

(cf. [25, p. 265]). Combining it with (4.10), we have that

$$\begin{aligned} u^+(x) = &-\frac{\lambda}{\nu+1}\zeta_1 x^{2\nu+2}e^{-\gamma x^2}F\Big(-\frac{\lambda}{2\gamma}+1,\nu+2,\gamma x^2\Big) \\ &+\lambda\zeta_2 x^{2\nu+2}e^{-\gamma x^2}U\Big(-\frac{\lambda}{2\gamma}+1,\nu+2,\gamma x^2\Big). \end{aligned} \tag{4.11}$$

It follow from (2.5) that the first term of the right hand side of (4.11) converges to 0. However (4.9) is not adequate for the decision of ζ_2. We need further asymptotic behavior and the following is useful:

$$U(\alpha,\beta,z) = \frac{\Gamma(\beta-1)}{\Gamma(\alpha)}z^{1-\beta} + O[1] \tag{4.12}$$

as $z\to 0$ for $1<\beta<2$ unless α is an integer with $\alpha \leqq 0$ (cf. [25, p. 288]). This gives that the second term of the right hand side of (4.11) converges to a constant multiple of ζ_2 as $x \downarrow 0$. In virtue of (4.5), we need to take $\zeta_2=0$. Moreover, combining (2.5) and (4.4), we easily have that $\zeta_1=1$, which yields (4.6) for $-1<\nu<0$.

It remains to consider the case that $\lambda/2\gamma = n$ for an integer $n \geqq 1$. In this case, the fundamental solutions of (4.7) are $L_n^{(\nu)}(x)$ and $l_n^{(\nu)}(x)$, where

$$\begin{aligned} L_n^{(\nu)}(x) &= \sum_{m=0}^{n}(-1)^m\binom{n+\nu}{n-m}\frac{x^m}{m!}, \\ l_n^{(\nu)}(x) &= x^{-(\nu+1)/2}e^{x/2}W\Big(-n-\frac{\nu+1}{2},\frac{\nu}{2},xe^{i\pi}\Big) \end{aligned}$$

(cf. [2, p. 34]). The function $L_n^{(\nu)}$ is called the generalized (or associated) Laguerre polynomial. We remark that

$$L_n^{(\nu)}(x) = \binom{n+\nu}{n}F(-n,\nu+1,x)$$

(cf. [25, p. 240]) and that

$$l_n^{(\nu)}(x) = e^{i\pi(\nu+1)/2}e^{x}U(n+\nu+1,\nu+1,xe^{i\pi}),$$

which is obtained by (3.9). Hence the general solution of (4.3) can be expressed by

$$u(x) = \zeta_1 F(-n, \nu+1, \gamma x^2) + \zeta_2 e^{\gamma x^2} U(n+\nu+1, \nu+1, \gamma x^2 e^{i\pi}).$$

This form is more useful than (4.8). When $\nu \geqq 0$, we can easily see by (4.9) that the second term diverges as $x \downarrow 0$ if $\zeta_2 \neq 0$. In virtue of (2.5), we obtain that the solution of (4.3) satisfying (4.23) is the following:

$$u(x) = F(-n, \nu+1, \gamma x^2). \tag{4.13}$$

We can show that this satisfies (4.5) by (2.5) and (4.10).

When $-1 < \nu < 0$, we begin to decide ζ_2 by using (4.5). Formula (4.2) yields that $u^+(x)$ is equal to

$$-\frac{2\gamma n}{\nu+1}\zeta_2 x^{2\nu+2} e^{-\gamma x^2} F(-n+1, \nu+2, \gamma x^2) \tag{4.14}$$
$$+ 2\gamma\zeta_2 x^{2\nu+2} U(n+\nu+1, \nu+1, -\gamma x^2) \tag{4.15}$$
$$+ 2\gamma(n+\nu+1)\zeta_2 x^{2\nu+2} U(n+\nu+2, \nu+2, \gamma x^2 e^{i\pi}). \tag{4.16}$$

We easily obtain that (2.5) gives that (4.14) vanishes as $x \downarrow 0$ and it follows from (4.12) that (4.16) is

$$2\gamma(n+\nu+1)\zeta_2 x^{2\nu+2}\left\{\frac{\Gamma(\nu+1)}{\Gamma(n+\nu+2)}(\gamma x^2 e^{i\pi})^{-\nu-1} + O[1]\right\}.$$

which converges to

$$2\gamma^{-\nu}(n+\nu+1)e^{-i\pi(\nu+1)}\frac{\Gamma(\nu+1)}{\Gamma(n+\nu+2)}\zeta_2 \tag{4.17}$$

as $x \downarrow 0$. For calcultion of (4.15) we need another precise asymptotic behavior of $U(\alpha, \beta, z)$. For $0 < \beta < 1$ we have that

$$U(\alpha, \beta, z) = \frac{\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)} + O[|z|^{1-\beta}]$$

as $z \to 0$ (cf. [25, p. 288]). This yields that (4.15) converges to 0 as $x \downarrow 0$ and hence $u^+(x)$ is asymptotically equal to (4.17). We obtain by (4.5) that $\zeta_2 = 0$ and can conclude (4.13) in virtue of (4.4) and (2.5). □

We are now ready to prove Lemma 2.1 (1). To avoid the complicated indices, we write $u(\lambda, x)$ instead of $u_\lambda(x)$. The following lemma is a straightforward consequence of the combination of Lemma 4.1 and Theorem 4.1 in [23].

Lemma 4.2. *For each $x > 0$ the function $\lambda \mapsto u(\lambda, x)$ has countably many zeros. Moreover all zeros are positive and of multiplicity* 1.

Let $\{\rho_{n,x}^{\nu,\gamma}\}_{n=1}^{\infty}$ be the increasing sequence of zeros of $u(\,\cdot\,,x)$. Lemma 4.2 gives that $u'(\rho_{n,x}^{\nu,\gamma},x) \neq 0$ for each $n \geqq 1$. For each $x>0$ we have by (4.1) that

$$\int_0^x y^{-2\nu-1/2} m(dy) = 2\int_0^x y^{1/2} e^{-\gamma y^2} dy < \infty,$$
$$\int_0^x y^{2\nu+1/2} s'(y) dy = \int_0^x y^{-1/2} e^{\gamma y^2} dy < \infty.$$

Hence the formula (4.2) in [23] yields that, for $x>0$

$$\lim_{n\to\infty} \frac{\rho_{n,x}^{\nu,\gamma}}{n^2} = \left[\frac{1}{\pi}\int_0^x \left\{\frac{m'(y)}{s'(y)}\right\}^{1/2} s'(y)dy\right]^{-2}, \tag{4.18}$$

where m' is the density of the measure m with respect to the Lebesgue measure. In virtue of (4.1) we have that

$$\frac{m'(x)}{s'(x)} = 2x^{4\nu+2}e^{-2\gamma x^2} \tag{4.19}$$

and thus (4.18) gives that

$$\lim_{n\to\infty} \frac{\rho_{n,x}^{\nu,\gamma}}{n^2} = \frac{\pi^2}{2x}. \tag{4.20}$$

For each $n \geqq 1$ let $\lambda_{n,x}^{\nu,\gamma} = \rho_{n,x}^{\nu,\gamma}/2\gamma$ and then (2.1) can be obtained obviously by (4.20). It follows from Lemma 4.2 that

$$F(-\lambda_{n,x}^{\nu,\gamma}, \nu+1, \gamma x^2) = u(\rho_{n,x}^{\nu,\gamma}, x) = 0, \tag{4.21}$$

which implies that each $\lambda_{n,x}^{\nu,\gamma}$ is a zero of the function $\lambda \mapsto F(-\lambda, \nu+1, \gamma x^2)$. In addition, the formula

$$u'(\lambda, x) = -\frac{1}{2\gamma}F'\left(-\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\right), \tag{4.22}$$

which is derived by (4.6), implies that $F'(-\lambda_{n,x}^{\nu,\gamma}, \nu+1, \gamma x^2) \neq 0$. The proof of the first claim of Lemma 2.1 is completed.

We next deal with the tail probability of $\sigma_{b,c}^{\nu,\gamma}$ in the case that $0<b<c$ and first try to derive a formula for the density function of $\sigma_{b,c}^{\nu,\gamma}$, which is denoted by $p_{b,c}^{\nu,\gamma}$.

Proposition 4.3. *Let $\nu>-1$, $\gamma>0$ and $0<b<c$. For any $t>0$ we have that*

$$p_{b,c}^{\nu,\gamma}(t) = 2\gamma \sum_{n=1}^{\infty} \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} e^{-2\gamma\lambda_{n,c}^{\nu,\gamma} t}. \tag{4.23}$$

Proof. According to Theorem 6.1 in [23], if we succeed in proving that

$$\frac{1}{u'(\rho_{n,x}^{\nu,\gamma}, x)} = O[e^{\varepsilon \rho_{n,x}^{\nu,\gamma}}] \tag{4.24}$$

as $n \to \infty$ for any $x > 0$ and $\varepsilon > 0$, we can obtain that, for $t > 0$

$$p_{b,c}^{\nu,\gamma}(t) = -\sum_{n=1}^{\infty} \frac{u(\rho_{n,c}^{\nu,\gamma}, b)}{u'(\rho_{n,c}^{\nu,\gamma}, c)} e^{-\rho_{n,c}^{\nu,\gamma} t}. \tag{4.25}$$

In virtue of (4.22), it can be concluded that (4.25) is equivalent to (4.23). Hence we may concentrate on showing (4.24).

With the help of Lemma 6.1 in [23], it is sufficient to prove that

$$\frac{m'(x)}{s'(x)} = O\left[|s(x)|^{-2-\delta}\right] \tag{4.26}$$

as $x \downarrow 0$ for a suitable constant $\delta > 0$. In the case that $\nu > 0$, it is easy to see by (4.1) that, for $0 < x < 1$

$$|s(x)| = \int_x^1 y^{-2\nu-1} e^{\gamma y^2} dy \leqq \frac{1}{2\nu} e^{\gamma} x^{-2\nu}$$

and hence it follows from (4.19) that, for sufficiently small $x > 0$

$$0 \leqq |s(x)|^{2+1/\nu} \frac{m'(x)}{s'(x)} \leqq C_1.$$

This leads to (4.26) for $\delta = 1/\nu$.

When $\nu = 0$, the proof of (4.26) is also easy. Indeed, since

$$|s(x)| = \int_x^1 y^{-1} e^{\gamma y^2} dy \leqq e^{\gamma} \log \frac{1}{x}$$

for $0 < x < 1$, we have by (4.19) that, for sufficiently small $x > 0$

$$0 \leqq |s(x)|^{2+\delta} \frac{m'(x)}{s'(x)} \leqq C_2 x^2 \left(\log \frac{1}{x}\right)^{2+\delta},$$

where $\delta > 0$ is arbitrarily given.

In the case that $-1 < \nu < 0$, Lemma 6.2 in [23] immediately gives that (4.24) holds since 0 is a regular boundary. □

Proposition 4.3 leads us to the formula for the tail probability of $\sigma_{b,c}^{\nu,\gamma}$. It follows from (3.5) and (4.23) that

$$P(\sigma_{b,c}^{\nu,\gamma} > t) = 2\gamma \int_t^{\infty} \sum_{n=1}^{\infty} \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} e^{-2\lambda_{n,c}^{\nu,\gamma} s} ds.$$

In order to change the order of the integral and the summation, we need to show the convergence of

$$\sum_{n=1}^{\infty} \left| \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{\lambda_{n,c}^{\gamma,\nu} F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} \right| e^{-2\lambda_{n,c}^{\nu,\gamma} t} \tag{4.27}$$

for each $t > 0$. The argument used to obtain (6.1) in [13] works well.

We start with giving an estimate of $F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)$. For simplicity let

$$\xi_n = \lambda_{n,c}^{\nu,\gamma} + \frac{\nu+1}{2}, \quad \nu_0 = \frac{\nu}{2}$$

and we note that (4.20) gives that ξ_n is asymptotically equal to a constant multiple of n^2 for large n. It follows from (3.8) that

$$F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2) = e^{\gamma b^2/2}(\gamma b^2)^{-(\nu+1)/2} M(\xi_n, \nu_0, \gamma b^2).$$

Asymptotic behavior of $M(\xi, \mu, x)$ for large ξ is given by

$$M(\xi, \mu, z) = \frac{\Gamma(1+2\mu) z^{1/4}}{\sqrt{\pi}\xi^{\mu+1/4}} \left\{ \sin\left(2\sqrt{\xi z} - \pi\mu + \frac{\pi}{4}\right) + O\left[|\xi|^{-1/2}\right] \right\} \tag{4.28}$$

if $|\arg(\xi z)| < 2\pi$ (cf. [25, p. 318]). Hence we deduce that

$$|F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)| \leqq C_3 \xi_n^{-\nu_0 - 1/4} \tag{4.29}$$

for any $n \geqq 1$.

We next aim to give a lower bound of $|F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)|$ and recall obtaining that $F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)$ does not varnish for any $n \geqq 1$. In virtue of (3.8), we have that

$$F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2) = -e^{\gamma c^2/2}(\gamma c^2)^{-(\nu+1)/2} M'(\xi_n, \nu_0, \gamma c^2). \tag{4.30}$$

The following behavior of M' is useful for the estimate of (4.30):

$$M'(\xi, \mu, x) = \frac{\Gamma(1+2\mu) x^{3/4}}{\sqrt{\pi}\xi^{\mu+3/4}} \left\{ \cos\left(2\sqrt{\xi x} - \pi\mu + \frac{\pi}{4}\right) + O\left[\xi^{-1/4}\right] \right\}$$

as $\xi \to \infty$ along the real line for $\mu \in \mathbb{R}$ and $x > 0$ (cf. [13, Lemma 3.2]). This yields that (4.30) is equal to a constant multiple of $\xi_n^{-\nu_0 - 3/4}(\cos\psi_n^1 + O[\xi_n^{-1/4}])$, where $\psi_n^1 = 2c\sqrt{\xi_n \gamma} - \pi\nu_0 + \pi/4$. We need to deduce the limiting behavior of $\cos\psi_n^1$ as $n \to \infty$ in order to obtain the estimate of the absolute value of (4.30). The combination of (3.8) and (4.21) gives that $M(\xi_n, \nu_0, \gamma c^2) = 0$ for each $n \geqq 1$ and thus (4.28) yields that $\sin\psi_n^1$ converges to 0 as $n \to \infty$, which is equivalent to the convergence of $|\cos\psi_n^1|$ to 1. This implies that there exists an integer $n_0 \geqq 1$ such that $|\cos\psi_n^1| \geqq 1/2$ for any $n \geqq n_0$ and then we have that, for sufficiently large n

$$|F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)| \geqq C_4 \xi_n^{-\nu_0 - 3/4}. \tag{4.31}$$

Since ξ_n is asymptotically equal to $\lambda_{n,c}^{\nu,\gamma}$, we obtain by (4.29) and (4.31) that

$$\left| \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} \right| \leqq C_5 (\lambda_{n,c}^{\nu,\gamma})^{1/2} \tag{4.32}$$

for any $n \geqq 1$. In virtue of (2.1), we can conclude that (4.27) converges for each $t > 0$. The dominated convergence theorem yields that

$$P(\sigma_{b,c}^{\nu,\gamma} > t) = \sum_{n=1}^{\infty} \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{\lambda_{n,c}^{\nu,\gamma} F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} e^{-2\gamma \lambda_{n,c}^{\nu,\gamma} t}$$

for any $t > 0$, which is equivalent to the following formula:

$$P(\sigma_{b,c}^{\nu,\gamma} \leqq t) = 1 - \sum_{n=1}^{\infty} \frac{F(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma b^2)}{\lambda_{n,c}^{\nu,\gamma} F'(-\lambda_{n,c}^{\nu,\gamma}, \nu+1, \gamma c^2)} e^{-2\gamma \lambda_{n,c}^{\nu,\gamma} t} \tag{4.33}$$

Therefore, combining it with (3.3), we can deduce Theorem 2.2.

Remark 4.4. Let $\phi_a^\nu(t; p, q)$ be the density function of $\tau_a^\nu(p, q)$. Formula (3.3) yields that $\phi_a^\nu(t; p, q)$ can be obtained by the density function $p_{b,c}^{\nu,\gamma}$ of $\sigma_{b,c}^{\nu,\gamma}$. We have already derived $p_{b,c}^{\nu,\gamma}$ in (4.23) for $0 < b < c$ and in [13, p. 86] for $0 < c < b$. For any $t > 0$ we have that

$$\phi_a^\nu(t; p, q) = \frac{q^2}{p^2 + q^2 t} \sum_{n=1}^{\infty} \frac{F(-\lambda_{n,q}^{\nu,1/2}, \nu+1, a^2 q^2/2p^2)}{F'(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2} t\right)^{-\lambda_{n,q}^{\nu,1/2}}$$

if $0 < a < p$ and that

$$\phi_a^\nu(t; p, q) = \frac{q^2}{p^2 + q^2 t} \sum_{n=1}^{\infty} \frac{U(-\kappa_{n,q}^{\nu,1/2}, \nu+1, a^2 q^2/2p^2)}{U'(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2} t\right)^{-\kappa_{n,q}^{\nu,1/2}}$$

if $0 < p < a$.

5. THE DISTRIBUTION FUNCTION OF $\tau_0^\nu(p, q)$

In this section we treat the case that $a = 0$ and our purpose is to give a proof of Theorem 2.3. It seems that (2.4) will be obtained by taking the limit of (2.2) as $a \downarrow 0$. Even if we succeed in justifying to change the order of the limit for a and the summation on n, we are not sure whether the distribution function of $\tau_a^\nu(p, q)$ converges to that of $\tau_0^\nu(p, q)$ or not. However we can get around this difficulty by using the Laplace transform.

Assume that $\nu > -1$ and $p, q > 0$. For $b, c \geqq 0$ let

$$\sigma_{b,c}^\nu = \inf\{s > 0\, ;\, R_b^\nu(s) = c\}.$$

Applying the Dynkin formula (cf. [22, p. 99]), we can easily derive by the standard calculation that $E[\sigma_{b,c}^\nu] = (c^2 - b^2)/(2\nu + 2)$ for $0 \leqq b < c$, which gives that

$$P(\sigma_{b,c}^\nu < \infty) = 1 \tag{5.1}$$

in this case. The strong Markov property yields that the Laplace transform of the function $t \mapsto P(\tau_0^\nu(p,q) \leqq t)$ is

$$\int_0^\infty dt\, e^{-\lambda t} \int_0^t P(\inf\{s>0\,;\, R_a^\nu(s) = \sqrt{p^2+q^2T+q^2s}\} \leqq t-T) d\mu_a^\nu(T) \tag{5.2}$$

for any $a \in (0,p)$, where μ_a^ν has been used to denote the probability law of $\sigma_{0,a}^\nu$. Change the order of the integrals and thus (5.2) is equal to

$$\int_0^\infty d\mu_a^\nu(T) \int_T^\infty e^{-\lambda t} P(\tau_a^\nu(\sqrt{p^2+q^2T}, q) \leqq t-T) dt. \tag{5.3}$$

We let

$$G(a,T,t) = P(\tau_a^\nu(\sqrt{p^2+q^2T}, q) > t)$$

and thus obtain by (3.4) that (5.3) is equal to

$$\int_0^\infty d\mu_a^\nu(T)\, e^{-\lambda T} \int_0^\infty e^{-\lambda t} \{1 - G(a,T,t)\}\, dt,$$

which coincides with

$$\frac{1}{\lambda}\int_0^\infty e^{-\lambda T} d\mu_a^\nu(T) - \int_0^\infty dt\, e^{-\lambda t} \int_0^\infty e^{-\lambda T} G(a,T,t) d\mu_a^\nu(T).$$

Hence we have that the Laplace transform of $P(\tau_0^\nu(p,q) \leqq t)$ is equal to

$$\frac{1}{\lambda} E\left[e^{-\lambda \sigma_{0,a}^\nu}\right] - \int_0^\infty e^{-\lambda t} E\left[e^{-\lambda \sigma_{0,a}^\nu} G(a, \sigma_{0,a}^\nu, t)\right] dt. \tag{5.4}$$

In order to calculating the limit of (5.4) with respect to a, we need to derive the limiting values of $\sigma_{0,a}^\nu$ and $G(a, \sigma_{0,a}^\nu, t)$ as $a \downarrow 0$ and start with deriving the limiting value of $\sigma_{0,a}^\nu$.

Lemma 5.1. *We have that*

$$\lim_{a \downarrow 0} \sigma_{0,a}^\nu = 0 \tag{5.5}$$

almost surely.

Proof. We can deduce (5.5) from the general theory of one-dimensional diffusion processes (cf. [22, p. 108]).

Since $\sigma_{0,a}^\nu \leqq \sigma_{0,a'}^\nu$ with probability 1 for $0 < a < a'$, there exists a random variable σ_0^ν such that $\sigma_{0,a}^\nu$ converges to σ_0^ν as $a \downarrow 0$ almost surely. The Blumenthal 0–1 law yields that the probability that $\sigma_0^\nu = 0$ can be chosen only either 0 or 1. In the case that

$\nu \geqq 0$, since the boundary 0 is entrance but not exit, it is known that $\sigma_0^\nu = 0$ almost surely. If $-1 < \nu < 0$, the boundary 0 is regular. In general, we can not decide the value of $P(\sigma_0^\nu = 0)$. However, since we assume that 0 is an instantaneously reflecting boundary, we conclude that $P(\sigma_0^\nu = 0) = 1$. □

The following lemma gives the limiting value of $G(a, \sigma_{0,a}^\nu, t)$ as $a \downarrow 0$.

Lemma 5.2. *For any $t > 0$ we have that*

$$\lim_{a\downarrow 0} G(a, \sigma_{0,a}^\nu, t) = G(t) \tag{5.6}$$

almost surely, where G is a function on $(0,\infty)$ defined by

$$G(t) = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n,q}^{\nu,1/2} F'(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2} t\right)^{-\lambda_{n,q}^{\nu,1/2}}.$$

Before proving Lemma 5.2, we shall complete the proof of Theorem 2.3. It follows from (5.5) that the first term of (5.4) converges to $1/\lambda$ as $a \downarrow 0$. Moreover, applying the dominated convergence theorem, we can easily show by (5.6) that the second term of (5.4) converges to the Laplace transform of the function G as $a \downarrow 0$, which yields that

$$\int_0^\infty e^{-\lambda t} P(\tau_0^\nu(p,q) \leqq t) dt = \int_0^\infty e^{-\lambda t} \{1 - G(t)\} dt.$$

Note that the function $t \mapsto P(\tau_0^\nu(p,q) \leqq t)$ is non-decreasing and hence we eventually obtain that

$$P(\tau_0^\nu(p,q) \leqq t) = 1 - G(t) \tag{5.7}$$

for any $t > 0$, which implies (2.4), by the following lemma.

Lemma 5.3. *The function G is continuously differentiable on $(0,\infty)$ and*

$$G'(t) = -\phi_0^\nu(t; p, q) \tag{5.8}$$

holds for any $t > 0$, where

$$\phi_0^\nu(t; p, q) = \frac{q^2}{p^2 + q^2 t} \sum_{n=1}^{\infty} \frac{1}{F'(-\lambda_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2} t\right)^{-\lambda_{n,q}^{\nu,1/2}}.$$

This lemma will be shown after the proof of Lemma 5.2.

Remark 5.4. We have by (5.7) and (5.8) that the function $t \mapsto \phi_0^\nu(t; p, q)$ is the density function of $\tau_0^\nu(p,q)$.

The remainder of this section is devoted to showing Lemmas 5.2 and 5.3. We start with the proof of Lemma 5.2. For simplicity we use the notation λ_n for $\lambda_{n,q}^{\nu,1/2}$. Let

$$G_n(a,t) = \frac{F(-\lambda_n, \nu+1, a^2 q^2/2(p^2 + q^2 \sigma_{0,a}^\nu))}{\lambda_n F'(-\lambda_n, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2 + q^2 \sigma_{0,a}^\nu} t\right)^{-\lambda_n}.$$

Combining (2.2) with (3.4), we have that $G(a, \sigma_{0,a}^{\nu}, t)$ is the sum of $G_n(a,t)$ on n over $[1,\infty)$, which gives that (5.6) is equivalent to

$$G(t) = \lim_{a\downarrow 0} \sum_{n=1}^{\infty} G_n(a,t). \tag{5.9}$$

Since (2.5) and (5.5) yield that

$$\lim_{a\downarrow 0} G_n(a,t) = \frac{1}{\lambda_n F'(-\lambda_n, \nu+1, q^2/2)} \left(1 + \frac{q^2}{p^2} t\right)^{-\lambda_n},$$

the formula (5.9) can be obtained if we succeed in showing that the order of the limit for a and the summation on n can be changed in the right hand side of (5.9). Hence we need to derive an upper bound of $|G_n(a,t)|$ uniformly for small a. The following estimate is quite useful.

Lemma 5.5. *Let $\lambda > 0$ and $|x| \leqq 1$. For any $\delta > 0$ we have that*

$$|F(-\lambda, \nu+1, x)| \leqq e\left\{\frac{(1+\delta)^{2\lambda}}{\Gamma(\lambda+1)} + \left(1+\frac{1}{\delta}\right)^{2\lambda}\right\}. \tag{5.10}$$

The proof of Lemma 5.5 is deferred to the last part of this section. Let a be given in the interval $(0, \min\{q/p, p\})$. Note that $0 < \sigma_{0,a}^{\nu} < \sigma_{0,p}^{\nu} < \infty$ almost surely. It follows from (5.1) that $q^2 t/(p^2 + q^2 \sigma_{0,p}^{\nu})$ is positive and hence we can take $\delta > 0$ satisfying that

$$0 < \frac{1}{\delta} < \sqrt{1 + \frac{q^2 t}{p^2 + q^2 \sigma_{0,p}^{\nu}}} - 1. \tag{5.11}$$

Applying (5.10) for $\lambda = \lambda_n$ and $x = a^2 q^2 / 2(p^2 + q^2 \sigma_{0,a}^{\nu})$, we obtain that

$$\left| F\left(-\lambda_n, \nu+1, \frac{a^2 q^2}{2(p^2 + q^2 \sigma_{0,a}^{\nu})}\right)\right| \leqq e\left\{\frac{(1+\delta)^{2\lambda_n}}{\Gamma(\lambda_n+1)} + \left(1+\frac{1}{\delta}\right)^{2\lambda_n}\right\}$$

for any $n \geqq 1$. Combining it with (4.31), we can deduce that

$$\lambda_n |G_n(a,t)| \leqq C_6 \lambda_n^{\nu/2+3/4} \left\{\frac{(1+\delta)^{2\lambda_n}}{\Gamma(\lambda_n+1)} + \delta_1^{\lambda_n}\right\} \tag{5.12}$$

for sufficiently large n, where

$$\delta_1 = \frac{(1+1/\delta)^2}{1 + q^2 t/(p^2 + q^2 \sigma_{0,p}^{\nu})}.$$

Note that (5.11) gives $0 < \delta_1 < 1$. Since

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} e^{z(\log z - 1)} \{1 + O[|z|^{-1}]\} \tag{5.13}$$

as $|z| \to \infty$ (cf. [25, p. 12]), we obtain that the right hand side of (5.12) converges to 0 as $n \to \infty$ and, in particular, is bounded with respect to n. Hence we have by (2.1) that $|G_n(a,t)| \leqq C_7 n^{-2}$ for any $n \geqq 1$. This leads to (5.9) and also (5.6). The proof of Lemma 5.2 is finished.

We next try to show Lemma 5.3. Let $t > 0$ be fixed. The mean value theorem yields that

$$\left|\left(1+\frac{q^2}{p^2}t\right)^{-\lambda_n} - \left(1+\frac{q^2}{p^2}T\right)^{-\lambda_n}\right| \leqq \frac{\lambda_n q^2}{p^2+q^2t/2}\left(1+\frac{q^2}{2p^2}t\right)^{-\lambda_n}|t-T|$$

for any $T > t/2$. Hence we have that by (4.31) that

$$\begin{aligned}&\left|\frac{1}{F'(-\lambda_n,\nu+1,q^2/2)}\left\{\left(1+\frac{q^2}{p^2}t\right)^{-\lambda_n} - \left(1+\frac{q^2}{p^2}T\right)^{-\lambda_n}\right\}\right| \\ &\leqq \frac{C_8 q^2}{p^2+q^2t/2}\lambda_n^{\nu/2+7/4}\left(1+\frac{q^2}{2p^2}t\right)^{-\lambda_n}|t-T|\end{aligned} \tag{5.14}$$

uniformly for $T > t/2$. Note that (2.1) gives the summation of the right hand side of (5.14) is a constant multiple of $|t-T|$ and thus $\phi_0^\nu(\,\cdot\,;p,q)$ is continuous at any $t > 0$. Moreover (5.14) yields that the absolute value of

$$\frac{1}{\lambda_n F'(-\lambda_n,\nu+1,q^2/2)}\frac{(1+q^2t/p^2)^{-\lambda_n} - (1+q^2T/p^2)^{-\lambda_n}}{t-T}$$

is dominated by a constant multiple of

$$\frac{q^2}{p^2+q^2t/2}\lambda_n^{\nu/2+3/4}\left(1+\frac{q^2}{2p^2}t\right)^{-\lambda_n} \tag{5.15}$$

uniformly for $T > t/2$. Since the summation of (5.15) on n over $[1,\infty)$ converges, we have that

$$\lim_{T\to t}\frac{G(t)-G(T)}{t-T} = \sum_{n=1}^\infty \frac{1}{\lambda_n F'(-\lambda_n,\nu+1,q^2/2)}\frac{d}{dt}\left(1+\frac{q^2}{p^2}t\right)^{-\lambda_n},$$

which coincides with $-\phi_0^\nu(t;p,q)$. Hence (5.8) holds and we finish the proof of Lemma 5.3.

It remains to show Lemma 5.5. The definition of the Kummer function F shows that

$$|F(-\lambda,\nu+1,x)| \leqq 1 + \sum_{n=1}^\infty \frac{|(-\lambda)_n|}{(\nu+1)_n}\frac{|x|^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $\alpha \in \mathbb{R}$ and $n \geqq 1$. Since

$$|(-\lambda)_n| = \left|\prod_{k=0}^{n-1}(-\lambda+k)\right| \leqq \prod_{k=0}^{n-1}(\lambda+k) = (\lambda)_n,$$

we have that $|F(-\lambda,\nu+1,x)| \leqq F(\lambda,\nu+1,1)$ for $|x| \leqq 1$. The integral representation of F gives that

$$F(\lambda,\nu+1,1) = \frac{\Gamma(\nu+1)}{\Gamma(\lambda)} \int_0^\infty \xi^{\lambda-\nu/2-1} e^{-\xi} I_\nu(2\sqrt{\xi}) d\xi$$

(cf. [25, p. 275]), where I_ν is the modified Bessel function of the first kind of order ν. Similarly to Theorem A.1 in [18], it can be proved that

$$I_\nu(y) \leqq \frac{y^\nu e^y}{2^\nu \Gamma(\nu+1)}$$

for $y > 0$. Hence, if $|x| \leqq 1$,

$$|F(-\lambda,\nu+1,x)| \leqq \frac{e}{\Gamma(\lambda)} \int_0^\infty \xi^{\lambda-1} e^{-(\sqrt{\xi}-1)^2} d\xi.$$

Let $\delta > 0$ be arbitrarily given. We divide the integral on ξ in the right hand side into the following two parts;

$$\Lambda_1 = \int_0^{(1+\delta)^2} \xi^{\lambda-1} e^{-(\sqrt{\xi}-1)^2} d\xi, \quad \Lambda_2 = \int_{(1+\delta)^2}^\infty \xi^{\lambda-1} e^{-(\sqrt{\xi}-1)^2} d\xi.$$

The estimate of Λ_1 is easy. Indeed, it follows that

$$\Lambda_1 \leqq \int_0^{(1+\delta)^2} \xi^{\lambda-1} d\xi = \frac{(1+\delta)^{2\lambda}}{\lambda}.$$

For the estimate of Λ_2 we change the variables from ξ to η given by $\eta = \sqrt{\xi} - 1$ and then have that

$$\Lambda_2 = 2\int_\delta^\infty (\eta+1)^{2\lambda-1} e^{-\eta^2} d\eta.$$

If $\lambda \geqq 1/2$, it follows from $\lambda > 1/2$ that Λ_2 is dominated by

$$2\int_\delta^\infty \left(\eta + \frac{\eta}{\delta}\right)^{2\lambda-1} e^{-\eta^2} d\eta \leqq 2\left(1+\frac{1}{\delta}\right)^{2\lambda-1} \int_0^\infty \eta^{2\lambda-1} e^{-\eta^2} d\eta$$
$$\leqq \left(1+\frac{1}{\delta}\right)^{2\lambda} \Gamma(\lambda).$$

When $0 < \lambda < 1/2$, the calculation is easy. Indeed, we have that

$$\Lambda_2 \leqq 2\int_0^\infty \eta^{2\lambda-1} e^{-\eta^2} d\eta = \Gamma(\eta).$$

Hence we obtain the claim of Lemma 5.5.

6. THE DISTRIBUTION FUNCTION OF $\tau_a^\nu(0,q)$

The purpose in this section is to show Theorem 2.4, which gives the explicit form of the distribution function of $\tau_a^\nu(0,q)$. It can be easily shown by (2.7) that each summand in the right hand side of (2.3) converges to the corresponding term of the right hand side of (2.6) as $p \downarrow 0$. Although we can justify to change the order of the limit for p and the summation on n in the right hand side of (2.3), we do not find any proofs of the convergence of $\tau_a^\nu(p,q)$ to $\tau_a^\nu(0,q)$ as $p \downarrow 0$ in a suitable sense. The basic idea for the proof of (2.6) is quite different from (2.4).

Let $a, q > 0$ be fixed and π_p^ν be the law of $\tau_a^\nu(p,q)$ for $p > 0$. The Laplace transform of the function $t \mapsto P(\tau_a^\nu(0,q) \leqq t)$ is equal to

$$\int_0^\infty dt\, e^{-\lambda t} \int_0^t P(\inf\{s > 0\,;\, R^\nu_{\alpha(p,q,T)}(s) = q\sqrt{T+s}\} \leqq t - T) d\pi_p^\nu(T) \tag{6.1}$$

for any $p \in (0,a)$, where $\alpha(p,q,T) = \sqrt{p^2 + q^2 T}$. Change the order of the integrals and thus (6.1) is equal to

$$\int_0^\infty d\pi_p^\nu(T) \int_T^\infty e^{-\lambda t} P(\tau^\nu_{\alpha(p,q,T)}(q\sqrt{T}, q) \leqq t - T) dt. \tag{6.2}$$

We let

$$H(p,T,t) = P(\tau^\nu_{\alpha(p,q,T)}(q\sqrt{T}, q) > t)$$

and thus obtain by (3.4) that (6.2) is equal to

$$\int_0^\infty d\pi_p^\nu(T)\, e^{-\lambda T} \int_0^\infty e^{-\lambda t} \left\{1 - H(p,T,t)\right\} dt,$$

which coincides with

$$\frac{1}{\lambda} \int_0^\infty e^{-\lambda T} d\pi_p^\nu(T) - \int_0^\infty dt\, e^{-\lambda t} \int_0^\infty e^{-\lambda T} H(p,T,t) d\pi_p^\nu(T).$$

Hence the Laplace transform of $P(\tau_a^\nu(0,q) \leqq t)$ is equal to

$$\frac{1}{\lambda} E[e^{-\lambda \tau_a^\nu(p,q)}] - \int_0^\infty e^{-\lambda t} E[e^{-\lambda \tau_a^\nu(p,q)} H(p, \tau_a^\nu(p,q), t)] dt, \tag{6.3}$$

Asymptotic behavior of $H(p, \tau_a^\nu(p,q), T)$ for p near a works essentially for the proof of (2.6). Namely it is important to investigate the limiting behavior as $p \uparrow a$ not as $p \downarrow 0$.

It is easy to give the limit of $\tau_a^\nu(p,q)$ as $p \uparrow a$. Note that $0 < \tau_a^\nu(p,q) \leqq \sigma^\nu_{a,p}$ almost surely for $0 < p < a$. Since $\sigma^\nu_{a,p}$ converges to 0 as $p \uparrow a$ (cf. [22, p. 106]), we have that

$$\lim_{p \uparrow a} \tau_a^\nu(p,q) = 0 \tag{6.4}$$

almost surely.

For $t > 0$ let

$$H(t) = \sum_{n=1}^{\infty} \frac{1}{\kappa_{n,q}^{\nu,1/2} U'(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(\frac{a^2}{2t} \right)^{\kappa_{n,q}^{\nu,1/2}}, \tag{6.5}$$

$$\phi_a^{\nu}(t;0,q) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{1}{U'(-\kappa_{n,q}^{\nu,1/2}, \nu+1, q^2/2)} \left(\frac{a^2}{2t} \right)^{\kappa_{n,q}^{\nu,1/2}}. \tag{6.6}$$

The following lemma holds.

Lemma 6.1. *For any $t > 0$ we have that*

$$\lim_{p \uparrow a} H(p, \tau_a^{\nu}(p,q), t) = H(t) \tag{6.7}$$

almost surely. Moreover the function H on $(0, \infty)$ is continuously differentiable and

$$H'(t) = -\phi_a^{\nu}(t;0,q) \tag{6.8}$$

for any $t > 0$.

Before proving Lemma 6.1, we shall show that (6.4) and Lemma 6.1 complete the proof of Theorem 2.4. The dominated convergence theorem yields that (6.3) converges to

$$\frac{1}{\lambda} - \int_0^{\infty} e^{-\lambda t} H(t) dt,$$

which implies that

$$\int_0^{\infty} e^{-\lambda t} P(\tau_a^{\nu}(0,q) \leqq t) dt = \int_0^{\infty} e^{-\lambda t} \left\{ 1 - H(t) \right\} dt.$$

Since H is continuously differentiable function, we have that $P(\tau_a^{\nu}(0,q) \leqq t)$ coincides with $1 - H(q,t)$ for any $t > 0$. Hence (2.6) holds.

Remark 6.2. We have by (6.8) that the function $\phi_a^{\nu}(\,\cdot\,;0,q)$ is the density of $\tau_a^{\nu}(0,q)$.

The remainder of this section is devoted to showing Lemma 6.1. We start with the proof of (6.7) and simply write κ_n and τ_p instead of $\kappa_{n,q}^{\nu,1/2}$ and $\tau_a^{\nu}(p,q)$, respectively. Combining (2.3) and (3.4), we can obtain that

$$H(p, \tau_p, t) = \sum_{n=1}^{\infty} H_n(p,t), \tag{6.9}$$

where

$$H_n(p,t) = \frac{U(-\kappa_n, \nu+1, q^2/2 + p^2/2\tau_p)}{\kappa_n U'(-\kappa_n, \nu+1, q^2/2)} \left(1 + \frac{t}{\tau_p} \right)^{-\kappa_n}.$$

Hence (6.9) gives that (6.7) is equivalent to

$$H(t) = \lim_{p\uparrow a} \sum_{n=1}^{\infty} H_n(p,t). \tag{6.10}$$

We now concentrate on proving (6.10). It follows from (2.7) and (6.4) that

$$\lim_{p\uparrow a} U\left(-\kappa_n, \nu+1, \frac{q^2}{2} + \frac{p^2}{2\tau_p}\right)\left(1+\frac{t}{\tau_p}\right)^{-\kappa_n} = \left(\frac{a^2}{2t}\right)^{\kappa_n}$$

almost surely, which implies that

$$\lim_{p\uparrow a} H_n(p,t) = \frac{1}{\kappa_n U'(-\kappa_n, \nu+1, q^2/2)}\left(\frac{a^2}{2t}\right)^{\kappa_n}$$

with probability 1. Since the right hand side coincides with the n-th term of the summation in the right hand side of (6.5), it is sufficient to check that the order of the limit for p and the summation on n can be changed in the right hand side of (6.10). In order to apply the dominated convergence theorem, we aim to give a uniform estimate of the absolute value of $H_n(p,t)$ for $a/2 < p < a$. Let

$$h_n(p) = \frac{U(-\kappa_n, \nu+1, q^2/2 + p^2/2\tau_p)}{\kappa_n U'(-\kappa_n, \nu+1, q^2/2)}$$

and then $H_n(p,t) = h_n(p)(1+t/\tau_p)^{-\kappa_n}$ holds. Thus we need to give a bound of the absolute value of $h_n(p)$ for the estimate of $H_n(p,t)$. Recall that ν_0 has been used for $\nu/2$ in Section 4 and, in addition, let $q_0 = q^2/2$ for simplicity. It follows from (3.9) that

$$|h_n(p)| = e^{p^2/4\tau_p}\left(1+\frac{p^2}{q^2\tau_p}\right)^{-(\nu+1)/2}\left|\frac{W(\eta_n, \nu_0, q_0 + p^2/2\tau_p)}{\kappa_n W'(\eta_n, \nu_0, q_0)}\right|, \tag{6.11}$$

where

$$\eta_n = \kappa_n + \frac{\nu+1}{2}.$$

We first give a bound of the first derivative of the Whittaker function of the second kind with respect to the first parameter and can apply the argument which has been used to derive (4.31).

Lemma 6.3. *We have that*

$$|W'(\eta_n, \nu_0, q_0)| \geqq C_9 e^{-\eta_n} \eta_n^{\eta_n - 1/4} \tag{6.12}$$

for large n.

Proof. Since each κ_n is a zero of the function $\kappa \mapsto U(-\kappa, \nu+1, q_0)$, it follows from (3.9) that

$$W(\eta_n, \nu_0, q_0) = 0. \tag{6.13}$$

Recall that, for $\mu \in \mathbb{R}$ and $x > 0$

$$W'(\eta,\mu,x) = W(\eta,\mu,x)\log\eta - \sqrt{2}\pi x^{1/4}e^{-\eta}\eta^{\eta-1/4}\left\{\sin\left(2\sqrt{\eta x}-\pi\eta+\frac{\pi}{4}\right)+O[\eta^{-1/2}]\right\}$$

as $\eta \to \infty$ in $\eta \in \mathbb{R}$ (cf. [13, Lemma 3.3]). Hence (6.13) yields that

$$|W'(\eta_n,\nu_0,q_0)| = \sqrt{2}\pi q_0^{1/4}e^{-\eta_n}\eta_n^{\eta_n-1/4}\left|\sin\psi_n^2 + O[\eta_n^{-1/2}]\right|, \tag{6.14}$$

where $\psi_n^2 = 2\sqrt{\eta_n q_0} - \eta_n\pi + \pi/4$. We need to give the limiting value of $|\sin\psi_n^2|$ for large n.

Since

$$W(\eta,\mu,x) = \sqrt{2}x^{1/4}e^{-\eta}\eta^{\eta-1/4}\left\{\cos\left(2\sqrt{\eta x}-\pi\eta+\frac{\pi}{4}\right)+O[\eta^{-1/2}]\right\} \tag{6.15}$$

as $\eta \to \infty$ along the real line for $\mu \in \mathbb{R}$ and $x > 0$ (cf. [13, Lemma 3.1]), we obtain by (6.13) that $\cos\psi_n^2$ converges to 0 as $n \to \infty$, which instantly gives that $|\sin\psi_n^2|$ converges to 1. Hence we can conclude that the right hand side of (6.14) is asymptotically equal to $\sqrt{2}\pi q_0^{1/4}e^{-\eta_n}\eta_n^{\eta_n-1/4}$ for large n, which implies (6.12). □

Since $\eta_n = \kappa_n(1+o[1])$ as $n \to \infty$, we have by (6.11) and (6.12) that $|h_n(p,q)|$ is dominated by a constant multiple of

$$e^{p^2/4\tau_p}\left(1+\frac{p^2}{q^2\tau_p}\right)^{-(\nu+1)/2} e^{\eta_n}\eta_n^{-\eta_n-3/4}\left|W\left(\eta_n,\nu_0,q_0+\frac{p^2}{2\tau_p}\right)\right|. \tag{6.16}$$

In order to obtain an upper bound of (6.16) with respect to p, we need to give a bound of $|W(\eta,\mu,x)|$ with respect to η and x. However, since the error term of the right hand side of (6.15) depends on x, a necessary estimate of $W(\eta,\mu,x)$ cannot be derived by (6.15). The following lemma gives an appropriate estimate for obtaining the suitable upper bound of (6.16).

Lemma 6.4. *Let $\eta > 0$, $\mu \geqq 0$ and $x > q_0$. For any $\delta > 0$ we have that*

$$|W(\eta,\mu,x)| \leqq C_{10}e^{-x/2}x^\eta(1+\delta)^\eta + C_{11}e^{-x/2}e^{-\eta}\left(\eta+\frac{1}{2}\right)^\eta\left(1+\frac{1}{\delta}\right)^\eta. \tag{6.17}$$

Proof. It is known that

$$W(\eta,\mu,x) = \frac{2x^{1/2}e^{x/2}}{\pi i}\int_{c-\infty i}^{c+\infty i} e^{y^2}y^{2\eta}K_{2\mu}(2y\sqrt{x})dy \tag{6.18}$$

for an arbitrary $c > 0$ (cf. [2, p. 73]), where $K_{2\mu}$ is the modified Bessel function of the second kind of order 2μ. We here put $c = \sqrt{x}$ and have that the right hand side of (6.18) is equal to

$$\frac{2x^{1/2}e^{x/2}}{\pi}\int_{-\infty}^{\infty} e^{(\sqrt{x}+yi)^2}(\sqrt{x}+yi)^{2\eta}K_{2\mu}(2(\sqrt{x}+yi)\sqrt{x})dy.$$

Recall that

$$|K_{2\mu}(z)| \leqq C_{12}|z|^{-1/2}e^{-\operatorname{Re} z} \tag{6.19}$$

if $|z| \geqq 2q_0$ and $|\arg z| < \pi$ (cf. [25, p. 139]). Since $|2(\sqrt{x}+yi)\sqrt{x}| \geqq 2x$ for $y \in \mathbb{R}$, we can apply (6.19) and then have that

$$|K_{2\mu}(2(\sqrt{x}+yi)\sqrt{x})| \leqq C_{12}x^{-1/2}e^{-2x},$$

which yields that

$$|W(\eta,\mu,x)| \leqq C_{13}e^{-x/2}\int_0^\infty e^{-y^2}(y^2+x)^\eta dy. \tag{6.20}$$

Let $\delta > 0$ be arbitrarily given and we will give a bound of the integral in (6.20) by dividing it into the following two parts:

$$\varXi_1 = \int_0^{\sqrt{\delta x}} e^{-y^2}(y^2+x)^\eta dy, \quad \varXi_2 = \int_{\sqrt{\delta x}}^\infty e^{-y^2}(y^2+x)^\eta dy.$$

The estimate of $\varXi_1$ is easy. Indeed, it follows that

$$\varXi_1 \leqq \int_0^{\sqrt{\delta x}} e^{-y^2}(\delta x+x)^\eta dy \leqq \frac{\sqrt{\pi}}{2}(1+\delta)^\eta x^\eta.$$

On the other hand, $\varXi_2$ is dominated by

$$\int_{\sqrt{\delta x}}^\infty e^{-y^2}\left(y^2+\frac{y^2}{\delta}\right)^\eta dy \leqq \left(1+\frac{1}{\delta}\right)^\eta \int_0^\infty e^{-y^2}y^{2\eta}dt = \frac{1}{2}\left(1+\frac{1}{\delta}\right)^\eta \varGamma\left(\eta+\frac{1}{2}\right).$$

Since (5.13) gives that $\varGamma(x)$ is not larger than $C_{14}x^{x-1/2}e^{-x}$ for any $x > 1/2$, a bound of $\varXi_2$ is a constant multiple of

$$e^{-\eta}\left(1+\frac{1}{\delta}\right)^\eta \left(\eta+\frac{1}{2}\right)^\eta.$$

Hence we can conclude (6.17). □

Let δ be fixed in $(\tau_{a/2}/t, \infty)$ and p be arbitrarily given in $(a/2, a)$. Lemma 6.4 yields that $|W(\eta_n, \nu_0, q_0 + p^2/2\tau_p)|$ is bounded by

$$C_{15}e^{-q_0/2-p^2/4\tau_p}(1+\delta)^{\eta_n}q_0^{\eta_n}\left(1+\frac{p^2}{q^2\tau_p}\right)^{\eta_n}$$
$$+ C_{16}e^{-q_0/2-p^2/4\tau_p}e^{-\eta_n}\left(\eta_n+\frac{1}{2}\right)^{\eta_n}\left(1+\frac{1}{\delta}\right)^{\eta_n}.$$

Therefore $|h_n(p,t)|$ is not larger than the sum of the following;

$$C_{15}\eta_n^{-\eta_n-3/4}\{e(1+\delta)q_0\}^{\eta_n}\left(\frac{1+p^2/q^2\tau_p}{1+t/\tau_p}\right)^{\kappa_n}, \tag{6.21}$$

$$C_{17}\eta_n^{-3/4}\left(1+\frac{1}{2\eta_n}\right)^{\eta_n}\left(\frac{1+t/\tau_p}{1+p^2/q^2\tau_p}\right)^{\nu_0+1/2}\left(\frac{1+1/\delta}{1+t/\tau_p}\right)^{\eta_n}. \tag{6.22}$$

Let

$$\theta_1=\frac{a^2/4q^2}{\tau_{a/2}+t},\quad \theta_2=\frac{\tau_{a/2}+a^2/q^2}{t},\quad \theta_3=\frac{q^2e(1+\delta)}{2}.$$

Since $0<\tau_p<\tau_{a/2}<\infty$, we have that

$$\theta_1\leqq\frac{\tau_p+a^2/4q^2}{\tau_p+t}\leqq\frac{1+p^2/q^2\tau_p}{1+t/\tau_p}\leqq\frac{\tau_p+a^2/q^2}{\tau_p+t}\leqq\theta_2.$$

This gives that (6.21) is dominated by $C_{18}\eta_n^{-\eta_n-3/4}\theta_2^{\kappa_n}\theta_3^{\eta_n}$ and that (6.22) is less than or equal to a constant multiple of

$$\eta_n^{-3/4}\left(1+\frac{1}{2\eta_n}\right)^{\eta_n}\theta_1^{-\nu_0-1/2}\delta_2^{\eta_n}, \tag{6.23}$$

where

$$\delta_2=\frac{1+1/\delta}{1+t/\tau_{a/2}}.$$

It is obvious that $1<(1+1/2\eta_n)^{\eta_n}\leqq e^{1/2}$ for $n\geqq 1$ since $e^x\geqq 1+x$ for $x\in\mathbb{R}$. It follows that (6.23) is bounded by $C_{19}\eta_n^{-3/4}\delta_2^{\eta_n}$. Hence we obtain that

$$|H_n(p,t)|\leqq C_{18}\eta_n^{-\eta_n-3/4}\theta_2^{\kappa_n}\theta_3^{\eta_n}+C_{19}\eta_n^{-3/4}\delta_2^{\eta_n}. \tag{6.24}$$

Since $\delta>\tau_{a/2}/t$, we have that $0<\delta_2<1$. We remark that η_n is asymptotically equal to κ_n for large n and then obtain that the right hand side of (6.24) times κ_n^2 converges to 0 as $n\to\infty$. This gives that $|H_n(p,t)|\leqq C_{20}\kappa_n^{-2}$ for any $n\geqq 1$. Recall that $\kappa_n>n-1$, which has been indicated in Section 3, and then we have that the sum of $|H_n(p,t)|$ on n over $[1,\infty)$ converges. Hence we can apply the dominated convergence theorem to (6.9) and obtain (6.4). This completes the proof of (6.7).

We lastly show the second claim of Lemma 6.1. Let $t>0$ be given. Note that (3.9) and (6.12) yield that

$$|U'(-\kappa_n,\nu+1,q^2/2)|\geqq C_{21}e^{-\eta_n}\eta_n^{\eta_n-1/4}.$$

Since $|t^{-\kappa_n}-T^{-\kappa_n}|\leqq\kappa_n(t/2)^{-\kappa_n-1}|t-T|$ for $T>t/2$, we have that

$$\begin{aligned}&\left|\frac{1}{U'(-\kappa_n,\nu+1,q^2/2)}\left\{\left(\frac{a^2}{2t}\right)^{\kappa_n}-\left(\frac{a^2}{2T}\right)^{\kappa_n}\right\}\right|\\&\leqq C_{22}\left(\frac{a^2}{t}\right)^{\kappa_n}e^{\eta_n}\eta_n^{-\eta_n+5/4}|t-T|.\end{aligned} \tag{6.25}$$

Since $\eta_n > n + (\nu - 1)/2$ for $n \geqq 1$, the sum of the right hand side of (6.25) on n is dominated by a constant multiple of $|t - T|$, which implies that $\phi_a^\nu(\,\cdot\,; 0, q)$ is a continuous function on $(0, \infty)$. Moreover, applying (6.25), we have that a bound of the absolute value of

$$\frac{1}{\kappa_n U'(-\kappa_n, \nu + 1, q^2/2)} \frac{(a^2/2t)^{\kappa_n} - (a^2/2T)^{\kappa_n}}{t - T}$$

is a constant multiple of $(a^2 e/t)^{\eta_n} \eta_n^{-\eta_n + 1/4}$, of which the summation on n converges. Hence it follows that

$$\lim_{T \to t} \frac{H(t) - H(T)}{t - T} = \sum_{n=1}^{\infty} \frac{1}{\kappa_n U'(-\kappa_n, \nu + 1, q^2/2)} \left(\frac{a^2}{2}\right)^{\kappa_n} \frac{d}{dt} t^{-\kappa_n} = -\phi_a^\nu(t; 0, q),$$

which implies (6.8). The proof of Lemma 6.1 is completed.

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