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SELF-COALITION GRAPHS

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Abstract. A coalition in a graph $G = (V, E)$ consists of two disjoint sets V_1 and V_2 of vertices, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set of G. A coalition partition in a graph G of order $n = |V|$ is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i either is a dominating set consisting of a single vertex of degree $n - 1$, or is not a dominating set but forms a coalition with another set V_j which is not a dominating set. Associated with every coalition partition π of a graph G is a graph called the coalition graph of G with respect to π, denoted $CG(G, \pi)$, the vertices of which correspond one-to-one with the sets $V_1, V_2, \ldots, V_k$ of π and two vertices are adjacent in $CG(G, \pi)$ if and only if their corresponding sets in π form a coalition. The singleton partition π_1 of the vertex set of G is a partition of order $|V|$, that is, each vertex of G is in a singleton set of the partition. A graph G is called a self-coalition graph if G is isomorphic to its coalition graph $CG(G, \pi_1)$, where π_1 is the singleton partition of G. In this paper, we characterize self-coalition graphs.

Keywords: coalitions in graphs, coalition partitions, coalition graphs, domination.

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1. INTRODUCTION

Motivated by the real-world concept of a coalition, that is, two entities joining together for joint action, coalitions in graphs were introduced by the authors in [1] and studied in [2–4]. In this paper, we characterize self-coalition graphs, which were first defined in [4]. Before presenting the definition of self-coalition graphs, we need some other definitions and terminology.

Let $G = (V, E)$ be a graph and $\overline{G}$ be the complement of G. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Each vertex $u \in N(v)$ is called a *neighbor* of v, and $|N(v)|$ is the *degree* of v, denoted $\deg(v)$. In a graph G of order $n = |V|$, a vertex of degree $n - 1$ is called a *dominating vertex*, while a vertex of degree 0 is an *isolated vertex* or just an *isolate*. The minimum degree of a vertex in G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. For a set S of vertices, we denote the *subgraph induced by* S by $G[S]$. A non-empty subset $X \subseteq V$ is called a *singleton set* if $|X| = 1$ or a *non-singleton set* if $|X| \geq 2$. For an integer k, we use the standard notation $i \in [k]$ to mean that i is an integer and $1 \leq i \leq k$.

A *vertex cover* in G is a set of vertices that covers all the edges of G. A set S of vertices is called *independent* if no two vertices in S are adjacent in G. The *vertex independence number* $\alpha(G)$ is the maximum cardinality of an independent set of vertices in G, and the *vertex cover number* $\beta(G)$ is the minimum cardinality of a vertex cover of G. A set $S \subseteq V$ is a *dominating set* of a graph G if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of any dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a *γ-set* of G. If X and Y are sets of vertices, we say that $[X, Y]$ is *full* if all possible edges between the vertices of X and the vertices of Y exist.

The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the maximum distance between two vertices in G. We denote the family of paths, cycles, and complete graphs of order n by P_n, C_n, and K_n, respectively, and the complete bipartite graph having r vertices in one partite set and s vertices in the other by $K_{r,s}$. The *union* $G \cup H$ of two disjoint graphs G and H is the disconnected graph composed of a copy of G and a copy of H. The *join* $G + H$ of two vertex-disjoint graphs G and H is the graph obtained from the union of G and H by adding every possible edge between the vertices of $V(G)$ and the vertices of $V(H)$.

The concept of coalitions in graphs was introduced by the authors in 2020 [1] as follows.

Definition 1.1. A *coalition* in a graph G consists of two disjoint sets V_1 and V_2 of vertices, where neither V_1 nor V_2 is a dominating set but the union $V_1 \cup V_2$ is a dominating set of G. We say that the sets V_1 and V_2 *form a coalition* and are *coalition partners*.

Definition 1.2. A *coalition partition*, henceforth called a *c-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a singleton dominating set, or is not a dominating set but forms a coalition with another set V_j in π.

It was proven in [1] that every graph has a *c*-partition. Note that if G has no dominating vertex, then no set V_i in a *c*-partition is a dominating set, and hence must form a coalition with another set V_j in the partition. Naturally associated with each coalition partition is a coalition graph defined in [4] as follows.

Definition 1.3. Let G be a graph with a *c*-partition $\pi = \{V_1, V_2, \ldots, V_k\}$. The *coalition graph* $CG(G, \pi)$ of G is the graph with vertex set $V_1, V_2, \ldots, V_k$, corresponding one-to-one with the sets of π, and two vertices V_i and V_j are adjacent in $CG(G, \pi)$ if and only if the sets V_i and V_j are coalition partners in π, that is, neither V_i nor V_j is a dominating set of G, but $V_i \cup V_j$ is a dominating set of G.

The path $P_4 = (v_1, v_2, v_3, v_4)$ with the c-partition $\pi = \{\{v_1, v_2\}, \{v_3, v_4\}\}$ yields the coalition graph $CG(P_4, \pi) \simeq K_2$, for example. In [3], the authors show that there are only finitely many coalition graphs of paths and finitely many coalition graphs of cycles, and they determine all such coalition graphs. On the other hand, the authors also show in [3] that there are infinitely many coalition graphs of trees and they characterize these coalition graphs.

The c-partitions of interest in this paper are singleton partitions defined as follows.

Definition 1.4. The *singleton partition* of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ is the partition $\pi_1 = \{V_1, V_2, \ldots, V_n\}$, where for $i \in [n]$, $V_i = \{v_i\}$. We shall use the notation π_1 throughout to denote the singleton partition of any graph G.

Abusing notation slightly, if G is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and singleton c-partition $\pi_1 = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$, we refer to a vertex $\{v_i\}$ of $CG(G, \pi_1)$ simply as v_i. That is, for simplicity we use the same labels for a vertex of G and its corresponding vertex in $CG(G, \pi_1)$. We also say that $CG(G, \pi_1)$ is a *singleton coalition graph*.

Note that the singleton partition π_1 is not always a c-partition, as can be seen with a path P_n for $n \geq 6$, since no set of π_1 containing an endvertex of P_n forms a coalition with another set of π_1. For examples of graphs whose singleton partition π_1 is a c-partition, consider the cycle C_4 for which $CG(C_4, \pi_1) \simeq K_4$, and the cycle C_5 for which $CG(C_5, \pi_1) \simeq C_5$. Note that C_5 is isomorphic to its singleton coalition graph. Clearly, the only way a graph G can be isomorphic to one of its coalition graphs is if both graphs have the same order, implying that the partition yielding the coalition graph is the singleton partition as formally stated below.

Definition 1.5. A graph G is called a *self-coalition graph* if for the singleton c-partition π_1 of G, $CG(G, \pi_1) \simeq G$.

Examples of self-coalition graphs include the trivial graph K_1, the cycle C_5, and the complete bipartite graph $K_{r,s}$ for $3 \leq r \leq s$. More generally, we observe the following.

Observation 1.6. *For the multipartite graph $G = K_{n_1, n_2, \ldots, n_k}$ with $n_i \geq 3$ for all $i \in [k]$, $CG(G, \pi_1) \simeq G$.*

2. MAIN RESULTS

Our main result characterizes the self-coalition graphs. We let $\mathcal{F}$ be the family of graphs that is defined recursively from the following rules:

1. The complete bipartite graphs $K_{r,s}$, where $3 \leq r \leq s$, are in $\mathcal{F}$.
2. The graphs, denoted G_{2k+1}, that can be obtained from an odd complete graph K_{2k+1}, for $k \geq 2$, by removing the edges of a spanning cycle C_{2k+1} are in $\mathcal{F}$.
3. If F is in $\mathcal{F}$, then so is the join $F + \overline{K}_p$, for any $p \geq 3$.
4. If F is in $\mathcal{F}$, then so is the join $F + G_{2k+1}$, for any $k \geq 2$.
5. No graph is in $\mathcal{F}$ unless it is obtained from a finite number of applications of Rules 1, 2, 3, and 4.

Note that $C_5 \in \mathcal{F}$ since $C_5 \simeq G_5$, that is, the cycle C_5 can be obtained by removing the edges of a spanning cycle from the complete graph K_5. Also, the family of complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$, where $n_i \geq 3$ for all $i \in [k]$, is a subfamily of $\mathcal{F}$ since these graphs can be constructed from the graph K_{n_1,n_2} by repeated applications of Rule 3.

By Observation 1.6, the complete multipartite graphs having partites sets of cardinality three or more are self-coalition graphs. We observe that the graphs G_{2k+1} are also, so the graphs described in Rules 1 and 2 are self-coalition graphs. We state this straightforward observation.

Observation 2.1. *If $G \simeq K_{r,s}$, for $3 \leq r \leq s$, or $G \simeq G_{2k+1}$, for $k \geq 2$, then $CG(G, \pi_1) \simeq G$.*

We are now ready to present our main result, which characterizes the self-coalition graphs. We shall prove the following theorem in Section 4.

Theorem 2.2. *A graph G is a self-coalition graph if and only if G is the trivial graph K_1 or $G \in \mathcal{F}$.*

3. PRELIMINARY RESULTS

In this section, we develop some useful properties of self-coalition graphs. The first property we prove applies to any graph G and any c-partition of G. Recall that the vertex cover number and the vertex independence number of a graph G are denoted by $\beta(G)$ and $\alpha(G)$, respectively.

Proposition 3.1. *For a graph G with minimum degree $\delta(G)$ and c-partition π,*

$$\beta(CG(G, \pi)) \leq \delta(G) + 1.$$

Proof. Let G with a graph with c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$. Let u be a vertex of minimum degree in G. Furthermore, let π' be the subset of π, such that $V_i \in \pi'$ if and only if V_i contains a vertex of $N[u]$. Hence, $|\pi'| \leq |N[u]| = \delta(G) + 1$. Now consider an arbitrary edge V_iV_j in $CG(G, \pi)$. Since the set $V_i \cup V_j$ is a dominating set of G, at least one of V_i and V_j contains a vertex from $N[u]$ in G. Thus, at least one of V_i and V_j is in π'. Hence, π' is a vertex cover of $CG(G, \pi)$. □ □

Considering the singleton c-partition of a graph G, we have the following corollaries.

Corollary 3.2. *Let $G = (V, E)$ be a graph with singleton c-partition π_1. For any vertex $u \in V$ and set $U = N_G[u]$, the set U is a vertex cover of $CG(G, \pi_1)$, while $V(G) - U$ is an independent set in $CG(G, \pi_1)$.*

Corollary 3.3. *For a graph G with order n, minimum degree $\delta(G)$, and singleton c-partition π_1,*

$$\alpha(CG(G, \pi_1)) \geq n - \delta(G) - 1.$$

We next prove another useful property of graphs and their singleton coalition graphs.

Proposition 3.4. *Let G be a graph with singleton c-partition π_1. If S is an independent set in G and $|S| \geq 3$, then the same set S of vertices is an independent set in $CG(G, \pi_1)$.*

Proof. Let S be an independent set in G, with $|S| \geq 3$. No pair of vertices in S dominates G because the pair does not dominate the other vertices in S. Hence, their corresponding singleton sets in π_1 do not form a coalition. It follows that S is independent in $CG(G, \pi_1)$. □

We conclude this section by turning our attention to self-coalition graphs. A graph G is called γ-*excellent* if every vertex of G is in some γ-set of G.

Proposition 3.5. *If G is a self-coalition graph of order $n \geq 2$ and size m, then each of the following holds:*

(a) $\gamma(G) = 2$,
(b) *G is a γ-excellent graph,*
(c) *G has exactly m minimum dominating sets,*
(d) *G is connected,*
(e) $\mathrm{diam}(G) = 2$,
(f) $2 \leq \delta(G) \leq \Delta(G) \leq n - 2$.

Proof. Assume that G is a self-coalition graph of order $n \geq 2$ and size m. By definition, $G \simeq CG(G, \pi_1)$.

Since π_1 is a singleton c-partition of G, every element of π_1 either contains a dominating vertex or is a coalition partner with another singleton set of π_1. Hence, $\gamma(G) \leq 2$. Assume that $\gamma(G) = 1$. Then G has a dominating vertex v, and so, v is an isolate in $CG(G, \pi_1)$. Since $G \simeq CG(G, \pi_1)$, G has an isolated vertex. But it is not possible for G to have both a dominating vertex and an isolated vertex if $n \geq 2$. Thus, G has no dominating vertex, that is, $\gamma(G) \geq 2$ and so $\gamma(G) = 2$. This proves (a). Further, G has no dominating vertex, so G has maximum degree $\Delta(G) \leq n - 2$, proving the upper bound of (f). Moreover, every singleton set of π_1 is in a coalition with another singleton set of π_1, implying that every vertex of G is in a γ-set of G and (b) holds.

Note that two vertices are in a γ-set of G if and only if the vertices corresponding to their singleton sets in $CG(G, \pi_1)$ are adjacent. Thus, the number of edges in $CG(G, \pi_1)$ is precisely the number of γ-sets of G. Since $G \simeq CG(G, \pi_1)$, G has exactly m γ-sets, proving (c).

To prove (d), suppose that G is disconnected. Since $\gamma(G) = 2$ and every vertex of G is in a γ-set of G, it follows that $G \simeq K_r \cup K_s$ for $1 \leq r \leq s$. But $CG(K_r \cup K_s, \pi_1) \simeq K_{r,s}$, contradicting that $G \simeq CG(G, \pi_1)$. Hence, G is connected.

To prove (e), suppose to the contrary that $\mathrm{diam}(G) \neq 2$. Since G has no dominating vertex, G is not complete and so $\mathrm{diam}(G) \neq 1$, that is, $\mathrm{diam}(G) \geq 3$. Let $u, v \in V$ where $d(u, v) = \mathrm{diam}(G)$. Suppose first that $\mathrm{diam}(G) \geq 4$. Let u_2 be a vertex at distance 2 from u on a (u, v)-path. Now $\{u_2, x\}$ dominates G for some $x \in V$. But x must be in $N[u]$ to dominate u and x must be in $N[v]$ to dominate v, a contradiction since $N[u] \cap N[v] = \emptyset$. Thus, $\mathrm{diam}(G) \leq 3$. It suffices to prove that $\mathrm{diam}(G) \neq 3$. Let A, B, and C denote the sets of vertices at distance 1, 2, and 3, respectively, from u in G. Note that $v \in C$ and none of A, B, and C is empty. By (a) and (b), $\gamma(G) = 2$

and every vertex of G is in some γ-set. Thus, each vertex of $B \cup C$ must be in a dominating set of G with a vertex from $N[u] = \{u\} \cup A$ to dominate u. But since no vertex of $\{u\} \cup A$ is adjacent to a vertex in C in G, it follows that $G[C]$ is complete and $[B, C]$ is full in G.

Now consider $CG(G, \pi_1)$. Note that no pair of vertices in $B \cup C$ dominates u in G, so the vertices of $B \cup C$ form an independent set in $CG(G, \pi_1)$. Similarly, no two vertices of $A \cup \{u\}$ dominate G, so $A \cup \{u\}$ is an independent set in $CG(G, \pi_1)$. Thus, $CG(G, \pi_1)$ is a bipartite graph with non-empty partite sets $A \cup \{u\}$ and $B \cup C$.

Since $G \simeq CG(G, \pi_1)$, it follows that G is bipartite. Thus, G has no odd cycles. But since $G[C]$ is complete and $[B, C]$ is full in G, this implies that $|C| = 1$, otherwise G has a triangle. Thus, $C = \{v\}$. Moreover, each of A and B is an independent set in G. Now in $CG(G, \pi_1)$, uv is an edge since $\{u, v\}$ dominates G.

If $|A| \geq 2$ and $|B| \geq 2$, then u is in only one γ-set, namely $\{u, v\}$ of G and similarly, v is in only the one γ-set $\{u, v\}$ of G. Thus, the edge uv is a component of in $CG(G, \pi_1)$, implying that $CG(G, \pi_1)$ is disconnected since $n \geq 4$. But G is connected and $G \simeq CG(G, \pi_1)$, a contradiction.

Hence, at least one of A and B has exactly one vertex. If both A and B have cardinality 1, then $n = 4$ and G is the path P_4. But P_4 is not a self-coalition graph. Hence, without loss of generality, we may assume that $|A| = 1$ and $|B| \geq 2$. Let $A = \{a\}$. By definition, the vertices of B are distance 2 from u in G, so a is adjacent to every vertex of B. Now consider $CG(G, \pi_1)$. If $b \in B$, then in $CG(G, \pi_1)$, the only neighbor of b is a, implying that $CG(G, \pi_1)$ has at least two vertices of degree 1 since $|B| \geq 2$, while G has exactly one vertex, namely u, of degree 1. Again, we have a contradiction since $G \simeq CG(G, \pi_1)$. Hence, $\mathrm{diam}(G) = 2$, proving (e).

For the lower bound of (f), assume to the contrary that $\delta(G) < 2$. Since G has order $n \geq 2$ and is connected, G has no isolated vertices, so $\delta(G) \geq 1$. Suppose that G has a vertex v of degree 1 and let x be the neighbor of v in G. Since G has diameter 2, every vertex that is not adjacent to v in G must be adjacent to x. But then x is a dominating vertex in G, contradicting that $\gamma(G) = 2$. Thus, $\delta(G) \geq 2$. □

4. PROOF OF THEOREM 2.2

In this section, we prove Theorem 2.2. We begin with two lemmas. For the join $G \simeq F + H$, we use the notation $V(F)$ and $V(H)$ to refer to these sets of vertices in both G and $CG(G, \pi_1)$.

Lemma 4.1. *Let $G \simeq F + \overline{K}_p$, for some graph F and $p \geq 3$. Then G is a self-coalition graph if and only if F is a nontrivial self-coalition graph or F is the empty graph $\overline{K}_q$, for $q \geq 3$.*

Proof. Let $G \simeq F + \overline{K}_p$, for some graph F and $p \geq 3$. If F is the empty graph $\overline{K}_q$ for $q \geq 3$, then G is the complete bipartite graph $K_{p,q}$, where $p, q \geq 3$, and G is a self-coalition graph by Observation 2.1.

Next assume that F is a nontrivial self-coalition graph. By Proposition 3.5(a), F has no dominating vertex, so $G \simeq F + \overline{K}_p$ for $p \geq 3$ has no dominating vertex. Note that $\gamma(G) = 2$ and $\{f, h\}$ is a γ-set of G, for every vertex $f \in V(F)$ and every vertex $h \in V(\overline{K}_p)$. Thus, $[V(F), V(\overline{K}_p)]$ is full in $CG(G, \pi_1)$. Moreover, by Proposition 3.4, any independent set of G having cardinality at least 3 is also an independent set in $CG(G, \pi_1)$. Thus, $V(\overline{K}_p)$ is an independent set in $CG(G, \pi_1)$. It follows that

$$CG(G, \pi_1) \simeq CG(F, \pi_1) + \overline{K}_p \simeq F + \overline{K}_p \simeq G,$$

and so G is a self-coalition graph.

For the converse, assume that $G \simeq F + \overline{K}_p$, for $p \geq 3$, is a self-coalition graph. We must show that F is either a self-coalition graph or $F \simeq \overline{K}_q$, for some $q \geq 3$. By Proposition 3.5(a), $\gamma(G) = 2$, and so $\{f, h\}$ is a γ-set of G, for every $f \in V(F)$ and every vertex $h \in V(\overline{K}_p)$. Hence, $[V(F), V(\overline{K}_p)]$ is full in $CG(G, \pi_1)$. Moreover, by Proposition 3.4, any independent set of G having cardinality at least 3 is also an independent set in $CG(G, \pi_1)$, so $V(\overline{K}_p)$ is also an independent set in $CG(G, \pi_1)$.

Since G has no dominating vertex, it follows that F has no dominating vertex and F has order at least 2. If F has order 2, then $V(F)$ is a γ-set of G, implying that the vertices of $V(F)$ are adjacent in $CG(G, \pi_1)$. But then each vertex of $V(F)$ is a dominating vertex in $CG(G, \pi_1)$, a contradiction since $CG(G, \pi_1) \simeq G$ and G has no dominating vertex. Thus, we may assume that F has order at least 3.

If F is the empty graph $\overline{K}_q$, then the result holds. Henceforth, we may assume that F has at least one edge. Since $[V(F), V(\overline{K}_p)]$ is full in $CG(G, \pi_1)$ and $V(\overline{K}_p)$ is an independent set in $CG(G, \pi_1)$, we deduce that the subgraph induced by $V(F)$ in $CG(G, \pi_1)$ has at least one edge. Thus, there exist two vertices $x, y \in V(F)$ such that $\{x, y\}$ is a dominating set of G, and hence of F.

Next we show that F has a singleton c-partition. Assume, to the contrary, that F does not have a singleton c-partition. Then there exists a vertex $u \in V(F)$ that is not in any dominating set of cardinality 2 of F. Thus, $u \notin \{x, y\}$. Moreover, u has no neighbors in $V(F)$ in $CG(G, \pi_1)$. Since $G \simeq CG(G, \pi_1)$, there is an isomorphism ϕ between G and $CG(G, \pi_1)$, $\phi : G \to CG(G, \pi_1)$, such that u' maps to u under ϕ. In other words, there is a vertex u' in G that has no neighbors in $V(F)$. It follows that $u' \in \{x, y\}$, say $u' = x$, and u' is not adjacent to y in F. Note that $u' \neq u$ since $u \notin \{x, y\}$. Then y is adjacent to every vertex in $V(F) - \{u', y\}$, including u, in G. Moreover, there are at least two vertices in $V(F) - \{u', y\}$, otherwise F has a singleton c-partition.

Under ϕ, there is a vertex y^* in $CG(G, \pi_1)$ such that $\phi(y) = y^*$, that is, y^* is adjacent to every vertex of $V(F)$ except u in $CG(G, \pi_1)$. But since $|V(F)| \geq 4$, y^* is adjacent to a vertex $w \in V(F) - \{y^*, u, u'\}$ in $CG(G, \pi_1)$, implying $\{y^*, w\}$ dominates G, a contradiction since neither y^* nor w is adjacent to u' in G.

Hence, we may assume that F has a singleton c-partition. Since G is a self-coalition graph, $CG(G, \pi_1) \simeq G \simeq F + \overline{K}_p$. Note that any coalition of $CG(F, \pi_1)$ is a coalition of $CG(G, \pi_1)$. Futher, since $[V(F), V(\overline{K}_p)]$ is full in $CG(G, \pi_1)$, and $V(\overline{K}_p)$ is an independent set in $CG(G, \pi_1)$, we have

$$CG(G, \pi_1) \simeq CG(F, \pi_1) + \overline{K}_p \simeq F + \overline{K}_p \simeq G.$$

It follows that the subgraph of $CG(G, \pi_1)$ denoted by $CG(F, \pi_1)$ is indeed a coalition graph of F and that $CG(F, \pi_1) \simeq F$. Thus, F is a nontrivial self-coalition graph. □

Lemma 4.2. *Let $G \simeq F + G_{2k+1}$, for some graph F and $k \geq 2$. Then G is a self-coalition graph if and only if F is the empty graph $\overline{K}_q$ for $q \geq 3$ or F is a nontrivial self-coalition graph.*

Proof. First assume that F is a nontrivial self-coalition graph, and consider the join $G \simeq F + G_{2k+1}$, for $k \geq 2$. Then $CG(F, \pi_1) \simeq F$, and by Observation 2.1, $CG(G_{2k+1}, \pi_1) \simeq G_{2k+1}$. Note that for every $u \in V(F)$ and $v \in V(G_{2k+1})$, $\{u, v\}$ is a dominating set of G, implying that $[V(F), V(G_{2k+1})]$ is full in $CG(G, \pi_1)$. Moreover, any coalition of $CG(F, \pi_1)$ is a coalition of $CG(G, \pi_1)$. Thus,

$$CG(G, \pi_1) \simeq CG(F, \pi_1) + CG(G_{2k+1}, \pi_1) \simeq F + G_{2k+1} \simeq G.$$

Hence, G is a self-coalition graph.

For the converse, let $G \simeq F + G_{2k+1}$, for $k \geq 2$, and assume that G is a self-coalition graph. We must show that F is either the empty graph $\overline{K}_q$, for some $q \geq 3$, or F is a nontrivial self-coalition graph. By Proposition 3.5(a), G has no dominating vertex, which implies that F has no dominating vertex. Using the same arguments used in the proof of Lemma 4.1, we can show that F has order at least three and that $[V(F), V(G_{2k+1})]$ is full in $CG(G, \pi_1)$. If F is the empty graph $\overline{K}_q$ for $q \geq 3$, then the result holds. Hence, we may assume that F has at least one edge. Again, the same argument as in the proof of Lemma 4.1 shows that F has a singleton c-partition. And by our previous comments,

$$CG(G, \pi_1) \simeq CG(F, \pi_1) + CG(G_{2k+1}, \pi_1) \simeq CG(F, \pi_1) + G_{2k+1} \simeq F + G_{2k+1} \simeq G,$$

and so $CG(F, \pi_1) \simeq F$. Thus, F is a nontrivial self-coalition graph, completing the proof. □

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Obviously, the trivial graph is a self-coalition graph. If $G \in \mathcal{F}$, then Observation 2.1 together with Lemmas 4.1 and 4.2 show that G is a self-coalition graph.

In order to prove the converse, we show, by induction on the order n of G, that if G is an arbitrary nontrivial self-coalition graph, then $G \in \mathcal{F}$. One can easily show by inspection that the only nontrivial self-coalition graphs of order $n \leq 6$ are the cycle $C_5 = G_5$ and the complete bipartite graph $K_{3,3}$. Thus, all self-coalition graphs having order at most 6 are in $\mathcal{F}$.

Assume that all nontrivial self-coalition graphs of order less than n are in $\mathcal{F}$. Let G be a self-coalition graph of order $n \geq 2$, with vertices $V = \{u_1, u_2, \ldots, u_n\}$. Label the corresponding vertices of $CG(G, \pi_1)$ the same. It is important to note, however, that this labeling does not necessarily preserve adjacencies.

By Proposition 3.5, we have that $\gamma(G) = 2$, every vertex of G is in a γ-set of G, and

$$2 \leq \delta(G) = \delta(CG(G, \pi_1)) \leq \Delta(G) = \Delta(CG(G, \pi_1)) \leq n - 2.$$

Let u be a vertex of minimum degree $\delta(G)$ in G, and let $U = N_G(u) = \{u_1, u_2, \ldots, u_\delta\}$. Let $X = V - N_G[u] = \{x_1, x_2, \ldots, x_t\}$. By Corollary 3.2, $U \cup \{u\}$ is a vertex cover of $CG(G, \pi_1)$ and X is an independent set in $CG(G, \pi_1)$, although X may not be an independent set in G. Consider the set of vertices X in $CG(G, \pi_1)$, where $t = |X| = n - (\delta(G) + 1)$.

If $|X| = 1$, then since x_1 is not adjacent to u in G and x_1 must have at least $\delta(G)$ neighbors in G, it follows that $N_G(x_1) = U$. Then $\{u\}$ forms a coalition with every other set of π_1, that is, u is a dominating vertex in $CG(G, \pi_1)$. Since $G \simeq CG(G, \pi_1)$, G also has a dominating vertex, contradicting Proposition 3.5(a). Thus, we may assume that $t = |X| \geq 2$.

Assume that $t \geq 3$. Note that since $G \simeq CG(G, \pi_1)$, we may think of G as being the singleton coalition graph of $CG(G, \pi_1)$. Since X is an independent set in $CG(G, \pi_1)$ and $|X| = t \geq 3$, Proposition 3.4 implies that X is also an independent set in G. Note that no vertex in X is in a γ-set with u in G since such a set does not dominate X. Thus, no vertex in X is adjacent to u in $CG(G, \pi_1)$, that is, $X \cup \{u\}$ is independent in $CG(G, \pi_1)$. Since $G \simeq CG(G, \pi_1)$ and $\delta(G) = |U|$, it follows that every vertex in $X \cup \{u\}$ is adjacent to every vertex of U, that is, $[U, X \cup \{u\}]$ is full in $CG(G, \pi_1)$. Let F be the subgraph induced by U in $CG(G, \pi_1)$. Therefore, $G \simeq CG(G, \pi_1) \simeq F + \overline{K}_{t+1}$, where $t + 1 \geq 4$. By Lemma 4.1, F is the empty graph $\overline{K}_q$ for $q \geq 3$ or F is a nontrivial self-coalition graph. If $F \simeq \overline{K}_q$, then $G \simeq K_{q,t+1}$, for $q \geq 3$ and $t + 1 \geq 4$, and so $G \in \mathcal{F}$. If F is a nontrivial self-coalition graph having order less than n, then, by our inductive hypothesis, $F \in \mathcal{F}$. Therefore, by Rule 3, $G \simeq F + \overline{K}_{t+1} \in \mathcal{F}$, and the result holds for $t \geq 3$.

Henceforth, we may assume that $t = 2$. Thus, G has order $n = \delta(G) + 3$. To determine the graph G, we consider its complement $\overline{G}$. Observe that

$$\Delta(\overline{G}) = n - \delta(G) - 1 = \delta(G) + 3 - \delta(G) - 1 = 2.$$

Since G has no dominating vertex, $\overline{G}$ has no isolated vertex, that is, $1 \leq \delta(\overline{G}) \leq \Delta(\overline{G}) = 2$. It follows that every component of $\overline{G}$ is either a nontrivial path or a cycle. Note that if Y and Z are the vertex sets of two components of $\overline{G}$, then $[Y, Z]$ is full in G.

We show that $\overline{G}$ has a component that is an odd cycle C_{2k+1}, for some integer $k \geq 2$, by proving two claims.

Claim 1. Every component of $\overline{G}$ is a cycle.

Assume, to the contrary, that $\overline{G}$ has a path component. Among all path components, let $P = (v_1, v_2, \ldots, v_m)$, $m \geq 2$, be a shortest path in $\overline{G}$. If $m = 2$, then $\{v_1, v_2\}$ is a γ-set of of G, where each of v_1 and v_2 dominates every vertex in $G - \{v_1, v_2\}$. Therefore, the singleton set $\{v_1\}$ is in a coalition with every other singleton set in π_1, implying that v_1 is a dominating vertex in $CG(G, \pi_1)$. Since $CG(G, \pi_1) \simeq G$, G has a dominating vertex, contradicting Proposition 3.5(a).

Hence, we may assume that $m \geq 3$. Observe that for every vertex v_i on P in $\overline{G}$, $\{v_i\}$ forms a coalition in G with every other singleton set in π_1 except for the vertices at distance 2 from v_i on P. In other words, v_i is adjacent to every other vertex of $CG(G, \pi_1)$ except for the vertices at distance 2 from it on P in $\overline{G}$. It follows that $P' = (v_1, v_3, \ldots, v_w)$, where $w = \lfloor (2m+1)/2 \rfloor$, is a path component of $\overline{CG(G, \pi_1)}$.

But since $G \simeq CG(G, \pi_1)$, and so $\overline{G} \simeq \overline{CG(G, \pi_1)}$, P' is a path component of $\overline{G}$ with a shorter length than P, contradicting our choice of P. Hence, no component of the graph $\overline{G}$ is a path, proving the claim.

As a result of Claim 1, we know that every component of $\overline{G}$ is a cycle.

Claim 2. $\overline{G}$ has a component that is an odd cycle.

Assume, to the contrary, that every component of $\overline{G}$ is an even cycle. Let $C = (x_0, x_1, \ldots, x_{2k+1}, x_0)$, for $k \geq 1$, be a component of smallest order among all such even length cycle components of $\overline{G}$. If $C = C_4$, then each of the paths (x_0, x_2) and (x_1, x_3) is a component of $\overline{CG(G, \pi_1)}$. But since $\overline{G} \simeq \overline{CG(G, \pi_1)}$, it follows that these paths are components in $\overline{G}$, contradicting Claim 1.

Hence, we may assume that C has order at least 6. Using an argument similar to the proof of Claim 1, the vertices in C in $\overline{G}$ result in two cycle components in $\overline{CG(G, \pi_1)}$, namely $(x_0, x_2, \ldots, x_{2k})$ and $(x_1, x_3, x_5, \ldots, x_{2k+1})$. If these components are even cycles, then $\overline{CG(G, \pi_1)}$ has a smaller even cycle than $\overline{G}$, contradicting that $\overline{G} \simeq \overline{CG(G, \pi_1)}$. Therefore, both of these cycles have odd length and $\overline{G}$ has a component that is an odd cycle.

It follows from Claim 2 that $\overline{G}$ has a component that is an odd cycle C_{2k+1} for some $k \geq 1$. If $\overline{G}$ has exactly one component, then G is the graph G_{2k+1}. Since $\delta(G) \geq 2$, it follows that $k \geq 2$ and $G \in \mathcal{F}$. Thus, we may assume that $\overline{G}$ has at least two components and so $G \simeq F + G_{2k+1}$ for some graph F.

If $k = 1$, then $G \simeq F + \overline{K}_3$. Lemma 4.1 implies that F is a nontrivial self-coalition graph or F is the empty graph $\overline{K}_q$, for $q \geq 3$. If F is the empty graph $\overline{K}_q$, for $q \geq 3$, then $G \simeq K_{q,3}$ and $G \in \mathcal{F}$. If F is nontrivial self-coalition graph having order less than n, then by our inductive hypothesis, $F \in \mathcal{F}$. Therefore, by Rule 3, $G \simeq F + \overline{K}_3 \in \mathcal{F}$.

Hence, $k \geq 2$. By Lemma 4.2, F is the empty graph $\overline{K}_q$ for $q \geq 3$ or F is a nontrivial self-coalition graph. If $F \simeq \overline{K}_q$, then $G \simeq \overline{K}_q + G_{2k+1} \simeq G_{2k+1} + \overline{K}_q$, for $q \geq 3$. Since $G_{2k+1} \in \mathcal{F}$ for $k \geq 2$, we have $G \in \mathcal{F}$ by Rule 3. If F is nontrivial self-coalition graph having order less than n, then by our inductive hypothesis, $F \in \mathcal{F}$. Therefore, by Rule 4, $G \simeq F + G_{2k+1} \in \mathcal{F}$, completing the proof. □

We conclude with the following corollary.

Corollary 4.3. *A bipartite graph G is a self-coalition graph if and only if $G \simeq K_{r,s}$ for $3 \leq r \leq s$.*

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