Opuscula Math. 43, no. 3 (2023), 275–333
https://doi.org/10.7494/OpMath.2023.43.3.275 Opuscula Mathematica

OPERATORS INDUCED BY CERTAIN HYPERCOMPLEX SYSTEMS

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Communicated by P.A. Cojuhari

Abstract. In this paper, we consider a family $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ of rings of hypercomplex numbers, indexed by the real numbers, which contain both the quaternions and the split-quaternions. We consider natural Hilbert-space representations $\left\{\left(\mathbb{C}^2, \pi_t\right)\right\}_{t\in\mathbb{R}}$ of the hypercomplex system $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, and study the realizations $\pi_t(h)$ of hypercomplex numbers $h \in \mathbb{H}_t$, as (2×2)-matrices acting on $\mathbb{C}^2$, for an arbitrarily fixed scale $t \in \mathbb{R}$. Algebraic, operator-theoretic, spectral-analytic, and free-probabilistic properties of them are considered.

Keywords: scaled hypercomplex ring, scaled hypercomplex monoids, representations, scaled-spectral forms, scaled-spectralization.

Mathematics Subject Classification: 20G20, 46S10, 47S10.

1. INTRODUCTION

In this paper, we study representations of the hypercomplex numbers (a, b) of complex numbers a and b, constructing a ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2,\ +,\ \cdot_t\right),$$

scaled by a real number $t \in \mathbb{R}$, where $(+)$ is the usual vector addition on the 2-dimensional vector space $\mathbb{C}^2$, and $(\cdot_t)$ is the t-scaled vector-multiplication on $\mathbb{C}^2$, defined by

$$(a_1, b_1) \cdot_t (a_2, b_2) = \left(a_1a_2 + tb_1\overline{b_2},\ a_1b_2 + b_1\overline{a_2}\right),$$

where $\overline{z}$ are the conjugates of z in $\mathbb{C}$.

Motivated by the canonical Hilbert-space representation $\left(\mathbb{C}^2, \pi\right)$ of the quaternions $\mathbb{H}$, introduced in [2, 3] and [20], we consider the canonical representation,

$$\Pi_t = \left(\mathbb{C}^2,\ \pi_t\right),$$

of the ring $\mathbb{H}_t$, and understand each element $h = (a, b)$ of $\mathbb{H}_t$ as its realization,

$$\pi_t(h) \overset{\text{denote}}{=} [h]_t \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

where $M_2(\mathbb{C}) = B(\mathbb{C}^2)$ is the matricial algebra (or, the operator algebra acting on $\mathbb{C}^2$) of all (2×2)-matrices over $\mathbb{C}$ (respectively, all bounded linear transformations, or simply operators on $\mathbb{C}^2$), for each $t \in \mathbb{R}$. Under our setting, one can check that the ring $\mathbb{H}_{-1}$ is nothing but the noncommutative field $\mathbb{H}$ of all quaternions (e.g., [2, 3] and [20]) and the ring $\mathbb{H}_1$ is the ring of all bicomplex numbers (e.g., [1]).

The spectral-analytic, operator-theoretic (or, matrix-theoretic), and free-probabilistic properties of $\mathbb{H}_t$ are considered and characterized under the canonical representation Π_t. In particular, certain decompositional properties on $\mathbb{H}_t$ are studied algebraically, and spectral-theoretically. And then, it is considered how those properties affect the spectral-analytic, operator-theoretic, and free-probabilistic properties of hypercomplex numbers of $\mathbb{H}_t$, for $t \in \mathbb{R}$.

1.1. MOTIVATION

The quaternions $\mathbb{H}$ is an interesting object not only in pure mathematics (e.g., [5, 10–12, 15, 16, 18, 20, 23], but also in applied mathematics (e.g., [4, 7, 13, 14, 17] and [21]). Independently, spectral analysis on $\mathbb{H}$ is considered in [2] and [3], under representation, "over $\mathbb{C}$", different from the usual quaternion-eigenvalue problems of quaternion-matrices studied in [13, 15] and [17].

Motivated by the generalized setting of the quaternions so-called the split-quaternions of [1], and by the main results of [2] and [3], we study a new type of hypercomplex numbers induced by the pairs of $\mathbb{C}^2$. Especially, we construct a system of the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, and study how the hypercomplex numbers act as (2×2)-matrices over $\mathbb{C}$ for given scales $t \in \mathbb{R}$, under our canonical Hilbert-space representations $\left\{\Pi_t = \left(\mathbb{C}^2, \pi_t\right)\right\}_{t \in \mathbb{R}}$. We are interested in algebraic, operator-theoretic, spectral-theoretic, free-probabilistic properties of $\mathbb{H}_t$ under Π_t, for $t \in \mathbb{R}$. Are they similar to those of the quaternions $\mathbb{H}_{-1} = \mathbb{H}$, shown in [2] and [3]? The answers are determined differently case-by-case, up to scales (see below).

1.2. OVERVIEW

In Section 2, we define our main objects, the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, and their canonical Hilbert-space representations $\{\Pi_t\}_{t \in \mathbb{R}}$. We understand each hypercomplex number of $\mathbb{H}_t$ as an operator, a (2×2)-matrix over $\mathbb{C}$. We concentrate on studying the invertibility on $\mathbb{H}_t$, for an arbitrarily fixed scale t. It is shown that if $t < 0$, then $\mathbb{H}_t$ forms a noncommutative field like the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, however, if $t \geq 0$, then it becomes a ring with unity, which is not a noncommutative field.

In Section 3, the spectral theory on (the realizations of) $\mathbb{H}_t$ is studied over $\mathbb{C}$. After finding the spectra of hypercomplex numbers, we define so-called the t-spectral forms whose main diagonal entries are from the spectra, and off-diagonal entries are 0's. As we have seen in [2] and [3], such spectral forms are similar to the realizations of quaternions of $\mathbb{H}_{-1}$. However, if a scale $t \in \mathbb{R} \setminus \{-1\}$ is arbitrary, then such a similarity does not hold in general. We focus on studying such a similarity in detail.

In Section 4, we briefly discuss about how the usual adjoint on $M_2(\mathbb{C})$ acts on the sub-structure $\mathcal{H}_2^t$ of $M_2(\mathbb{C})$, consisting of all realizations of $\mathbb{H}_t$, for a scale $t \in \mathbb{R}$. Different from the quaternionic case of [2] and [3], in general, the adjoints (conjugate-transposes) of many matrices of $\mathcal{H}_2^t$ are not contained in $\mathcal{H}_2^t$, especially, if $t \neq -1$. It shows that a bigger, operator-algebraically-better $*$-algebraic structure generated by $\mathcal{H}_2^t$ is needed in $M_2(\mathbb{C})$, to consider operator-theoretic, and free-probabilistic properties on $\mathcal{H}_2^t$.

In the final Section 5, on the C^*-algebraic structure of Section 4, we study operator-theoretic, and free-probabilistic properties up to the usual trace, and the normalized trace.

2. THE SCALED HYPERCOMPLEX SYSTEMS $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$

In this section, we define a ring $\mathbb{H}_t$ of hypercomplex numbers, and establish the corresponding canonical Hilbert-space representations Π_t, for an arbitrary fixed scale $t \in \mathbb{R}$. Throughout this section, we let

$$\mathbb{C}^2 = \{(a, b) : a, b \in \mathbb{C}\}$$

be the Cartesian product of two copies of the complex field $\mathbb{C}$. One may understand $\mathbb{C}^2$ as the usual 2-dimensional Hilbert space equipped with its canonical orthonormal basis, $\{(1,0), (0,1)\}$.

2.1. A t-SCALED HYPERCOMPLEX RING $\mathbb{H}_t$

In this section, we fix an arbitrary real number t in the real field $\mathbb{R}$. On the vector space $\mathbb{C}^2$ (over $\mathbb{C}$), define the t-scaled vector-multiplication $(\cdot_t)$ by

$$(a_1, b_1) \cdot_t (a_2, b_2) \overset{\text{def}}{=} \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2}\right), \tag{2.1}$$

for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2$, where $\overline{z}$ are the conjugates of z in $\mathbb{C}$. It is not difficult to check that such an operation $(\cdot_t)$ is closed on $\mathbb{C}^2$. Moreover, it satisfies that

$$\begin{aligned}
&((a_1, b_1) \cdot_t (a_2, b_2)) \cdot_t (a_3, b_3) \\
&= \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2}\right) \cdot_t (a_3, b_3) \\
&= \left(a_1 a_2 a_3 + t\left(b_1 \overline{b_2} a_3 + a_1 b_2 \overline{b_3} + b_1 \overline{a_2}\,\overline{b_3}\right), a_1 a_2 b_3 + a_1 b_2 \overline{a_3} + b_1 \overline{a_2 a_3} + t b_1 \overline{b_2} b_3\right),
\end{aligned}$$

and

$$\begin{aligned}
&(a_1, b_1) \cdot_t ((a_2, b_2) \cdot_t (a_3, b_3)) \\
&= (a_1, b_1) \cdot_t \left(a_2 a_3 + t b_2 \overline{b_3},\ a_2 b_3 + b_2 \overline{a_3}\right) \\
&= \left(a_1\left(a_2 a_3 + t b_2 \overline{b_3}\right) + t b_1\left(\overline{a_2}\,\overline{b_3} + \overline{b_2} a_3\right), a_1\left(a_2 b_3 + b_2 \overline{a_3}\right) + b_1\left(\overline{a_2 a_3} + t \overline{b_2} b_3\right)\right),
\end{aligned}$$

implying the equality

$$((a_1, b_1) \cdot_t (a_2, b_2)) \cdot_t (a_3, b_3) = (a_1, b_1) \cdot_t ((a_2, b_2) \cdot_t (a_2, b_3)), \tag{2.2}$$

in $\mathbb{C}^2$, for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2, 3$.

Furthermore, if $\vartheta = (1, 0) \in \mathbb{C}^2$, then

$$\vartheta \cdot_t (a, b) = (a, b) = (a, b) \cdot_t \vartheta \tag{2.3}$$

by (2.1), for all $(a, b) \in \mathbb{C}^2$.

By (2.2) and (2.3), if

$$\mathbb{C}^{2\times} = \mathbb{C}^2 \setminus \{(0, 0)\},$$

then the pair $\left(\mathbb{C}^{2\times}, \cdot_t\right)$ forms a monoid (i.e., semigroup with its identity $(1, 0)$).

Lemma 2.1. *Let $\mathbb{C}^{2\times} = \mathbb{C}^2 \setminus \{(0, 0)\}$, and $(\cdot_t)$ be the closed operation* (2.1) *on $\mathbb{C}^2$. Then the algebraic structure $\left(\mathbb{C}^{2\times}, \cdot_t\right)$ forms a monoid with its identity* $(1, 0)$.

Proof. The proof is done by (2.2) and (2.3). □

Therefore, one can obtain the following ring structure.

Proposition 2.2. *The algebraic triple $\left(\mathbb{C}^2, +, \cdot_t\right)$ forms a unital ring with its unity* (*or the multiplication-identity*) $(1, 0)$, *where* $(+)$ *is the usual vector addition on* $\mathbb{C}^2$, *and* $(\cdot_t)$ *is the vector multiplication* (2.1).

Proof. Clearly, the algebraic pair $\left(\mathbb{C}^2, +\right)$ is an Abelian group under the usual addition $(+)$ with its $(+)$-identity $(0, 0)$. While, by Lemma 2.1, the pair $\left(\mathbb{C}^{2\times}, \cdot_t\right)$ forms a monoid (and hence, a semigroup). Observe now that

$$\begin{aligned}
&(a_1, b_1) \cdot_t ((a_2, b_2) + (a_3, b_3)) \\
&= (a_1, b_1) \cdot_t (a_2 + a_3, b_2 + b_3) \\
&= \left(a_1 (a_2 + a_3) + t b_1 \left(\overline{b_2} + \overline{b_3}\right),\ a_1 (b_2 + b_3) + b_1 (\overline{a_2} + \overline{a_3})\right) \\
&= \left(a_1 a_2 + a_1 a_3 + t b_1 \overline{b_2} + t b_1 \overline{b_3},\ a_1 b_2 + a_1 b_3 + b_1 \overline{a_2} + b_1 \overline{a_3}\right) \\
&= \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2}\right) + \left(a_1 a_3 + t b_1 \overline{b_3},\ a_1 b_3 + b_1 \overline{a_3}\right) \\
&= (a_1, b_1) \cdot_t (a_2, b_2) + (a_1, b_1) \cdot_t (a_3, b_3),
\end{aligned}$$

and, similarly,

$$((a_1, b_1) + (a_2, b_2)) \cdot_t (a_3, b_3) = (a_1, b_1) \cdot_t (a_3, b_3) + (a_2, b_2) \cdot_t (a_3, b_3), \tag{2.4}$$

in $\mathbb{C}^2$. So, the operations $(+)$ and $(\cdot_t)$ are left-and-right distributive by (2.4).

Therefore, the algebraic triple $\left(\mathbb{C}^2, +, \cdot_t\right)$ forms a unital ring with its unity $(1, 0)$. □

The above proposition characterizes the algebraic structure of $\left(\mathbb{C}^2, +, \cdot_t\right)$ as a well-defined unital ring for a fixed $t \in \mathbb{R}$. Remark here that, since a scale t is arbitrary in $\mathbb{R}$, in fact, we obtain the unital rings $\{\mathbb{H}_t\} =_{t \in \mathbb{R}}$.

Definition 2.3. For a fixed $t \in \mathbb{R}$, the ring $\left(\mathbb{C}^2, +, \cdot_t\right)$ is called the hypercomplex ring with its scale t (in short, the t-scaled hypercomplex ring). By $\mathbb{H}_t$, we denote the t-scaled hypercomplex ring.

2.2. THE CANONICAL REPRESENTATION $\Pi_t = \left(\mathbb{C}^2,\ \pi_t\right)$ OF $\mathbb{H}_t$

In this section, we fix $t \in \mathbb{R}$, and the corresponding t-scaled hypercomplex ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2,\ +,\ \cdot_t\right),$$

where $(\cdot_t)$ is the vector-multiplication (2.1). We consider a natural finite-dimensional--Hilbert-space representation Π_t of $\mathbb{H}_t$, and understand each hypercomplex number $h \in \mathbb{H}_t$ as an operator acting on a Hilbert space determined by Π_t. In particular, as in the quaternionic case of [2, 3] and [20], a 2-dimensional-Hilbert-space representation of the hypercomplex ring $\mathbb{H}_t$ is established naturally.

Define now a morphism,

$$\pi_t : \mathbb{H}_t \to B\left(\mathbb{C}^2\right) = M_2\left(\mathbb{C}\right),$$

by

$$\pi_t\left((a,b)\right) = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \forall (a,b) \in \mathbb{H}_t, \tag{2.5}$$

where $B\left(H\right)$ is the operator algebra consisting of all bounded (or, continuous linear) operators on a Hilbert space H, and $M_k\left(\mathbb{C}\right)$ is the matricial algebra of all $(k \times k)$-matrices over $\mathbb{C}$, isomorphic to $B\left(\mathbb{C}^k\right)$, for all $k \in \mathbb{N}$ (e.g., [9] and [8]).

By definition, the function π_t of (2.5) is an injective map from $\mathbb{H}_t$ into $M_2\left(\mathbb{C}\right)$. Indeed, if

$$(a_1, b_1) \neq (a_2, b_2) \ \text{ in } \mathbb{H}_t,$$

then

$$\pi_t\left((a_1,b_1)\right) = \begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} \neq \begin{pmatrix} a_2 & tb_2 \\ \overline{b_2} & \overline{a_2} \end{pmatrix} = \pi_t\left((a_2,b_2)\right), \tag{2.6}$$

in $M_2\left(\mathbb{C}\right)$. Furthermore, it satisfies that

$$\begin{aligned} \pi_t\left((a_1,b_1) + (a_2,b_2)\right) &= \begin{pmatrix} a_1 + a_2 & t\left(b_1 + b_2\right) \\ \overline{b_1 + b_2} & \overline{a_1 + a_2} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{b_2} \end{pmatrix} + \begin{pmatrix} a_2 & tb_2 \\ \overline{b_2} & \overline{a_2} \end{pmatrix} \\ &= \pi_t\left((a_1,b_1)\right) + \pi_t\left((a_2,b_2)\right). \end{aligned} \tag{2.7}$$

Also, one has

$$\pi_t\left((a_1,b_1) \cdot_t (a_2,b_2)\right) = \pi_t\left(\left(a_1a_2 + tb_1\overline{b_2},\ a_1b_2 + b_1\overline{a_2}\right)\right)$$

by (2.1)

$$\begin{aligned} &= \begin{pmatrix} a_1a_2 + tb_1\overline{b_2} & t\left(a_1b_2 + b_1\overline{a_2}\right) \\ \overline{a_1b_2 + b_1\overline{a_2}} & \overline{a_1a_2 + tb_1\overline{b_2}} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} \begin{pmatrix} a_2 & tb_2 \\ \overline{b_2} & \overline{a_2} \end{pmatrix} = \pi_t\left((a_1,b_1)\right) \pi_t\left((a_2,b_2)\right), \end{aligned} \tag{2.8}$$

where the multiplication $(\cdot)$ in the far-right-hand side of (2.8) is the usual matricial multiplication on $M_2(\mathbb{C})$.

Since our t-scaled hypercomplex ring $\mathbb{H}_t = \left(\mathbb{C}^2, +, \cdot_t\right)$ is identified with the 2-dimensional space $\mathbb{C}^2$ (set-theoretically), one may / can understand this ring $\mathbb{H}_t$ as a topological ring equipped with the usual topology for $\mathbb{C}^2$, for any $t \in \mathbb{R}$. From below, we regard the ring $\mathbb{H}_t$ as a topological unital ring under the usual topology for $\mathbb{C}^2$.

Lemma 2.4. *The pair $\left(\mathbb{C}^2, \pi_t\right)$ is an injective Hilbert-space representation of the t-scaled hypercomplex ring $\mathbb{H}_t$, where π_t is an action* (2.5)*.*

Proof. The morphism $\pi_t : \mathbb{H}_t \to M_2(\mathbb{C})$ of (2.5) is a well-defined injective function by (2.6). Moreover, this map π_t satisfies the relations (2.7) and (2.8), and hence, it is a(n algebraic) ring-action of $\mathbb{H}_t$, acting on the 2-dimensional vector space $\mathbb{C}^2$. So, the pair $\left(\mathbb{C}^2, \pi_t\right)$ forms an algebraic representation of $\mathbb{H}_t$. By regarding $\mathbb{H}_t$ and $M_2(\mathbb{C})$ as topological spaces equipped with their usual topologies, then it is not difficult to check that the ring-action π_t is continuous from $\mathbb{H}_t$ (which is homeomorphic to $\mathbb{C}^2$ as a topological space) into $M_2(\mathbb{C})$ (which is $*$-isomorphic to the C^*-algebra $B\left(\mathbb{C}^2\right)$). Thus, the algebraic representation $\left(\mathbb{C}^2, \pi_t\right)$ forms a Hilbert-space representation of $\mathbb{H}_t$ acting on $\mathbb{C}^2$ via π_t. $\square$

The above lemma shows that the t-scaled hypercomplex ring $\mathbb{H}_t$ is realized in the matricial algebra $M_2(\mathbb{C})$ as

$$\pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a,b) \in \mathbb{H}_t \right\},$$

as an embedded topological ring in $M_2(\mathbb{C})$.

Definition 2.5. The realization $\pi_t(\mathbb{H}_t)$ of the t-scaled hypercomplex ring $\mathbb{H}_t$ is called the t-scaled (hypercomplex-)realization of $\mathbb{H}_t$ (in $M_2(\mathbb{C})$), for a scale $t \in \mathbb{R}$. And we denote $\pi_t(\mathbb{H}_t)$ by $\mathcal{H}_2^t$, i.e.,

$$\mathcal{H}_2^t \overset{\text{denote}}{=} \pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} : (a,b) \in \mathbb{H}_t \right\}.$$

Also, by $[\xi]_t$, we denote $\pi_t(\xi) \in \mathcal{H}_2^t$, for all $\xi \in \mathbb{H}_t$.

By the above lemma and definition, we obtain the following result.

Theorem 2.6. *For $t \in \mathbb{R}$, the corresponding t-scaled hypercomplex ring $\mathbb{H}_t$ is topological-ring-isomorphic to the t-scaled realization $\mathcal{H}_2^t$ in $M_2(\mathbb{C})$, i.e.,*

$$\mathbb{H}_t \overset{T.R}{=} \mathcal{H}_2^t \quad \text{in} \quad M_2(\mathbb{C}), \tag{2.9}$$

where "$\overset{T.R}{=}$" means "being topological-ring-isomorphic to".

Proof. The relation (2.9) is proven by Lemma 2.4 and the injectivity (2.6) of π_t. $\square$

By the above theorem, one can realize that $\mathbb{H}_t$ and $\mathcal{H}_2^t$ as an identical topological ring, for a fixed $t \in \mathbb{R}$. Recall that the relation (2.9) is independently shown in [2] and [3], only for the quaternionic case where $t = -1$.

2.3. SCALED HYPERCOMPLEX MONOIDS

Throughout this section, we fix a scale $t \in \mathbb{R}$, and the corresponding t-scaled hypercomplex ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2, +, \cdot_t\right),$$

which is isomorphic to the t-scaled realization,

$$\mathcal{H}_2^t = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\},$$

in $M_2(\mathbb{C})$. Let

$$\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \mathbb{H}_t \setminus \{(0,0)\},$$

set-theoretically, where $(0,0) \in \mathbb{H}_t$ is the $(+)$-identity of the Abelian group $\left(\mathbb{C}^2, +\right)$. Thus, by Proposition 2.2, this set forms a well-defined semigroup,

$$\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \left(\mathbb{H}_t^{\times}, \cdot_t\right),$$

equipped with its $(\cdot_t)$-identity $(1,0)$, and hence, the pair $\mathbb{H}_t^{\times}$ is the maximal monoid embedded in $\mathbb{H}_2^t$ up to the operation $(\cdot_t)$.

Definition 2.7. The maximal monoid $\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{\times}, \cdot_t\right)$, embedded in the t-scaled hypercomplex ring $\mathbb{H}_t$, is called the t-scaled hypercomplex monoid.

By (2.9), the following corollary is trivial.

Corollary 2.8. *The t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is monoid-isomorphic to the monoid $\mathcal{H}_2^{t\times} \overset{\text{denote}}{=} \left(\mathcal{H}_2^{t\times}, \cdot\right)$, equipped with its identity,*

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \cdot 0 \\ 0 & 1 \end{pmatrix} = [(1,0)]_t,$$

the (2×2)-identity matrix of $M_2(\mathbb{C})$, where $(\cdot)$ is the usual matricial multiplication inherited from that on $M_2(\mathbb{C})$, i.e.,

$$\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{\times}, \cdot_t\right) \overset{\text{Monoid}}{=} \left(\mathcal{H}_2^{t\times}, \cdot\right) = \mathcal{H}_2^{t\times}, \tag{2.10}$$

where "$\overset{\text{Monoid}}{=}$" means "being monoid-isomorphic".

Proof. The isomorphic relation (2.10) is proven by the proof of Proposition 2.2, and that of Theorem 2.6. $\square$

2.4. INVERTIBILITY ON $\mathbb{H}_t$

In this section, by identifying our t-scaled hypercomplex ring $\mathbb{H}_t$ as its isomorphic realization $\mathcal{H}_2^t$, we consider invertibility of elements of $\mathbb{H}_t$, for an arbitrarily fixed $t \in \mathbb{R}$.

Observe first that, for any $(a, b) \in \mathbb{H}_t$ realized to be $[(a, b)]_t \in \mathcal{H}_2^t$, one can get that

$$\det\left([(a,b)]_t\right) = \det\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 - t|b|^2,$$

i.e.,

$$\det\left([(a,b)]_t\right) = |a|^2 - t|b|^2, \tag{2.11}$$

where $\det : M_2(\mathbb{C}) \to \mathbb{C}$ is the determinant, and $|\cdot|$ is the modulus on $\mathbb{C}$.

Theorem 2.9. *Let $(a, b) \in \mathbb{H}_t$, realized to be $[(a, b)]_t \in \mathcal{H}_2^t$. Then the following assertions hold.*

(i) $\det\left([(a,b)]_t\right) = |a|^2 - t|b|^2$.

(ii) *If either $|a|^2 > t|b|^2$, or $|a|^2 < t|b|^2$, then $[(a, b)]_t$ is invertible "in $M_2(\mathbb{C})$", with its inverse matrix,*

$$[(a,b)]_t^{-1} = \frac{1}{|a|^2 - t|b|^2}\begin{pmatrix} \overline{a} & t(-b) \\ \overline{(-b)} & a \end{pmatrix}.$$

(iii) *If $|a|^2 - t|b|^2 \neq 0$, then $(a, b) \in \mathbb{H}_t$ is invertible in the sense that there exists a unique $(c, d) \in \mathbb{H}_t$, such that*

$$(a, b) \cdot_t (c, d) = (1, 0) = (c, d) \cdot_t (a, b).$$

In particular, one has that

$$(c, d) = \left(\frac{\overline{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2}\right) \in \mathbb{C}^2$$

(iv) *Assume that (a, b) is invertible in $\mathbb{H}_t$ in the sense of* (iii). *Then the inverse is also contained "in $\mathbb{H}_t$".*

Proof. The statement (i) is shown by (2.11).

Note-and-recall that a matrix $A \in M_n(\mathbb{C})$ is invertible in $M_n(\mathbb{C})$, if and only if $\det(A) \neq 0$, for all $n \in \mathbb{N}$. Therefore,

$$\det\left([(a,b)]_t\right) \neq 0 \Longleftrightarrow [(a,b)]_t \text{ is invertible in } M_2(\mathbb{C}).$$

So, by (i),

$$|a|^2 - t|b|^2 \neq 0 \Longleftrightarrow [(a,b)]_t \text{ is invertible in } M_2(\mathbb{C}).$$

Moreover, $|a|^2 - t|b|^2 \neq 0$, if and only if

$$[(a,b)]_t^{-1} = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}^{-1} = \frac{1}{|a|^2 - t|b|^2}\begin{pmatrix} \overline{a} & -tb \\ -\overline{b} & a \end{pmatrix},$$

in $M_2(\mathbb{C})$. Therefore, the statement (ii) holds true in $M_2(\mathbb{C})$.

By (ii), one has $\det\left([(a,b)]_t\right) \neq 0$, if and only if

$$[(a,b)]_t^{-1} = \begin{pmatrix} \frac{\overline{a}}{|a|^2-t|b|^2} & t\left(\frac{-b}{|a|^2-t|b|^2}\right) \\ \overline{\left(\frac{-b}{|a|^2-t|b|^2}\right)} & \frac{a}{|a|^2-t|b|^2} \end{pmatrix} \in M_2\left(\mathbb{C}\right),$$

and it is actually contained "in $\mathcal{H}_2^t$", satisfying

$$\pi_t^{-1}\begin{pmatrix} \frac{\overline{a}}{|a|^2-t|b|^2} & t\left(\frac{-b}{|a|^2-t|b|^2}\right) \\ \overline{\left(\frac{-b}{|a|^2-t|b|^2}\right)} & \frac{a}{|a|^2-t|b|^2} \end{pmatrix} = \left(\frac{\overline{a}}{|a|^2-t|b|^2}, \frac{-b}{|a|^2-t|b|^2}\right),$$

in $\mathbb{H}_t$, by the injectivity of π_t. It shows that $[(a,b)]_t^{-1}$ exists in $M_2\left(\mathbb{C}\right)$, if and only if it is contained "in $\mathcal{H}_2^t$", i.e., if $[(a,b)]_t$ is invertible, then its inverse is also contained in $\mathcal{H}_2^t$, too, and vice versa. So, the statements (2.8) and (2.9) hold. □

The above theorem not only characterizes the invertibility of the monoidal elements of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$, but also confirms that the inverses (if exist) are contained in the monoid $\mathbb{H}_t^{\times}$, i.e.,

$$(a,b)^{-1} \text{ exists } \Longleftrightarrow (a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2-t|b|^2}, \frac{-b}{|a|^2-t|b|^2}\right),$$

"in $\mathbb{H}_t^{\times}$", equivalently,

$$\left[(a,b)^{-1}\right]_t = [(a,b)]_t^{-1} \text{ in } \mathcal{H}_2^{\times}.$$

Corollary 2.10. *Let $(a,b) \in \mathbb{H}_t^{\times}$. Then it is invertible, if and only if*

$$\left[(a,b)^{-1}\right]_t = \left[\left(\frac{\overline{a}}{|a|^2-t|b|^2}, \frac{-b}{|a|^2-t|b|^2}\right)\right]_t = [(a,b)]_t^{-1}, \tag{2.12}$$

in $\mathcal{H}_2^{\times}$, where $[(a,b)]_t^{-1}$ means the matricial inverse in $M_2\left(\mathbb{C}\right)$.

Proof. The proof of (2.12) is immediately done by Theorem 2.9(ii)–(iv). □

The above corollary can be re-stated by that: if $\xi \in \mathbb{H}_t^{\times}$ is invertible, then

$$\pi_t\left(\xi^{-1}\right) = \left(\pi_t(\xi)\right)^{-1} \text{ in } \mathcal{H}_2^{t\times}.$$

Now consider the cases where

$$|a|^2 - t|b|^2 = 0 \Longleftrightarrow |a|^2 = t|b|^2, \tag{2.13}$$

in $\mathbb{R}$. As we have seen above, the condition (2.13) holds for $(a,b) \in \mathbb{H}_t$, if and only if (a,b) is not invertible in $\mathbb{H}_t$ (and hence, its realization $[(a,b)]_t$ is not invertible in $M_2\left(\mathbb{C}\right)$, and hence, in $\mathcal{H}_2^t$). Clearly, we are not interested in the $(+)$-identity $(0,0)$

of $\mathbb{H}_t$ automatically satisfying the condition (2.13). So, without loss of generality, we focus on elements (a, b) of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ (or, its realizations $[(a, b)]_t$ of $\mathcal{H}_2^{t\times}$), satisfying the condition (2.13).

Recall that an algebraic triple, $(X, +, \cdot)$, is a noncommutative field, if (i) $(X, +)$ is an Abelian group, (ii) $(X^{\times}, \cdot)$ forms a non-Abelian group, and (iii) the operations $(+)$ and $(\cdot)$ are left-and-right distributive. For instance, the quaternions $\mathbb{H} = \mathbb{H}_{-1}$ is a noncommutative field (e.g., [2] and [3]).

Theorem 2.11. *Suppose the fixed scale $t \in \mathbb{R}$ is negative, i.e., $t < 0$ in $\mathbb{R}$. Then "all" elements (a, b) of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ are invertible in $\mathbb{H}_t$, with their inverses,*

$$\left(\frac{\overline{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right) \in \mathbb{H}_t^{\times},$$

i.e.,

$$t < 0 \text{ in } \mathbb{R} \Longrightarrow \mathbb{H}_t \textit{ is a noncommutative field.} \tag{2.14}$$

Proof. Suppose the scale $t \in \mathbb{R}$ is negative. Then, for any $(a, b) \in \mathbb{H}_t^{\times}$,

$$|a|^2 \neq t|b|^2 \Longleftrightarrow |a|^2 - t|b|^2 > 0,$$

since $(a, b) \neq (0, 0)$, i.e., if $t < 0$, then every element $(a, b) \in \mathbb{H}_t^{\times}$ does "not" satisfy the condition (2.13). It implies that if $t < 0$, then every element $(a, b) \in \mathbb{H}_t^{\times}$ is invertible in $\mathbb{H}_t^{\times}$, by Theorem 2.9(iii)-(iv); and the inverse is determined to be (2.12) in $\mathbb{H}_t^{\times}$. Thus, the pair $\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{\times}, \cdot_t\right)$ forms a group which is not Abelian by (2.1) and (2.8).

Therefore, if $t < 0$ in $\mathbb{R}$, then the t-scaled hypercomplex ring $\mathbb{H}_t$ becomes a noncommutative field, proving the statement (2.14). □

The above theorem characterizes that the algebraic structure of scaled hypercomplex rings $\{\mathbb{H}_t\}_{t<0}$ as noncommutative fields.

Theorem 2.12. *Suppose $t = 0$ in $\mathbb{R}$. Then an element (a, b) of the 0-scaled hypercomplex monoid $\mathbb{H}_0^{\times}$ is invertible in $\mathbb{H}_0$, with their inverses,*

$$\left(\frac{\overline{a}}{|a|^2}, \frac{-b}{|a|^2} \right) \in \mathbb{H}_0^{\times},$$

if and only if $a \neq 0$ in $\mathbb{C}$, if and only if only the elements of the subset,

$$\left\{ (a, b) \in \mathbb{H}_0^{\times} : a \neq 0 \right\} \ \textit{of } \mathbb{H}_0^{\times} \tag{2.15}$$

are invertible in $\mathbb{H}_0^{\times}$, if and only if $(0, b) \in \mathbb{H}_0^{\times}$ are not invertible in $\mathbb{H}_0^{\times}$, for all $b \in \mathbb{C}$.

Proof. Assume that we have the zero scale, i.e., $t = 0$ in $\mathbb{R}$. Then, by (2.13),

$$|a|^2 = 0 \cdot |b|^2 \Longleftrightarrow |a|^2 = 0 \Longleftrightarrow a = 0 \text{ in } \mathbb{C},$$

if and only if $(0, b) \in \mathbb{H}_0^{\times}$ are not invertible in $\mathbb{H}_0^{\times}$, for all $b \in \mathbb{C}$, if and only if all elements (a, b), contained in the subset (2.15), are invertible in $\mathbb{H}_0^{\times}$.

Observe that (a,b) is contained in the subset (2.15) of $\mathbb{H}_0^{\times}$, if and only if

$$[(a,b)]_0 \left[\left(\frac{\overline{a}}{|a|^2}, \frac{-b}{|a|^2}\right)\right]_0 = \begin{pmatrix} a & 0 \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} \frac{\overline{a}}{|a|^2} & 0 \\ \overline{\frac{-b}{|a|^2}} & \frac{a}{|a|^2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\overline{a}}{|a|^2} & 0 \\ \overline{\frac{-b}{|a|^2}} & \frac{a}{|a|^2} \end{pmatrix} \begin{pmatrix} a & 0 \\ \overline{b} & \overline{a} \end{pmatrix}$$
$$= \left[\left(\frac{\overline{a}}{|a|^2}, \frac{-b}{|a|^2}\right)\right]_0 [(a,b)]_0,$$

in $\mathbb{H}_0^{\times}$. Therefore, if exists, $(a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2}, \frac{-b}{|a|^2}\right)$ in $\mathbb{H}_0^{\times}$. □

The above theorem shows that if we have the zero-scale in $\mathbb{R}$, then our 0-scaled hypercomplex ring $\mathbb{H}_0$ cannot be a noncommutative field. It directly illustrates that the algebra on the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, and the algebra on the scaled-hypercomplex rings $\{\mathbb{H}_t\}_{t\in\mathbb{R}\setminus\{-1\}}$ can be different in general, especially, when $t \geq 0$.

Theorem 2.13. *Suppose the scale $t \in \mathbb{R}$ is positive, i.e., $t > 0$ in $\mathbb{R}$. Then an element $(a,b) \in \mathbb{H}_t^{\times}$ is invertible in $\mathbb{H}_t^{\times}$ with its inverse,*

$$\left(\frac{\overline{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2}\right) \in \mathbb{H}_t^{\times},$$

if and only if $|a|^2 \neq t|b|^2$ in $\mathbb{R}_0^+ = \{r \in \mathbb{R} : r \geq 0\}$, if and only if (a,b) is contained in the subset,

$$\left\{(a,b) : |a|^2 \neq t|b|^2 \text{ in } \mathbb{R}_0^+\right\}, \tag{2.16}$$

of $\mathbb{H}_t^{\times}$. As application, if $t > 0$ in $\mathbb{R}$, then the all elements of

$$\left\{(a,0) \in \mathbb{H}_t : a \in \mathbb{C}^{\times}\right\} \cup \left\{(0,b) \in \mathbb{H}_t : b \in \mathbb{C}^{\times}\right\}, \tag{2.17}$$

are invertible in $\mathbb{H}_t$, where $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$.

Proof. Assume that $t > 0$ in $\mathbb{R}$, and $\mathbb{H}_t^{\times}$, the corresponding t-scaled hypercomplex monoid. Then $(a,b) \in \mathbb{H}_t^{\times}$ is invertible in $\mathbb{H}_t^{\times}$, if and only if the condition (2.13) does not hold, if and only if

$$|a|^2 \neq t|b|^2 \Longleftrightarrow \text{either } |a|^2 > t|b|^2, \text{ or } |a|^2 < t|b|^2,$$

in $\mathbb{R}_0^+$, since $t > 0$. Therefore, if $t > 0$ in $\mathbb{R}$, then an element (a,b) is invertible in $\mathbb{H}_t^{\times}$, if and only if

$$\text{either } |a|^2 > t|b|^2, \text{ or } |a|^2 < t|b|^2 \text{ in } \mathbb{R}_0^+,$$

if and only if (a,b) is contained in the subset (2.16) in $\mathbb{H}_t^{\times}$.

In particular, for $t > 0$ in $\mathbb{R}$, (i) if $(a,0) \in \mathbb{H}_t^{\times}$ with $a \in \mathbb{C}^{\times}$, then $|a|^2 > 0$; and (ii) if $(0,b) \in \mathbb{H}_t^{\times}$ with $b \in \mathbb{C}^{\times}$, then $0 < t|b|^2$. Therefore, the subset (2.17) is properly contained in the subset (2.16) in $\mathbb{H}_t^{\times}$, whenever $t > 0$. So, all elements, formed by $(a,0)$, or by $(0,b)$ with $a,b \in \mathbb{C}^{\times}$, are invertible in $\mathbb{H}_t^{\times}$. □

The above theorem characterizes the invertibility on the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$, where the scale t is positive in $\mathbb{R}$. Theorems 2.11, 2.12 and 2.13 refine Theorem 2.9, case-by-case. We again summarize the main results.

Corollary 2.14. *Let $\mathbb{H}_t^{\times}$ be the t-scaled hypercomplex monoid. If $t < 0$, then all nonzero elements of $\mathbb{H}_t^{\times}$ are invertible; and if $t = 0$, then*

$$\left\{(a,b) \in \mathbb{H}_0^{\times} : a \neq 0\right\}$$

is the invertible proper subset of $\mathbb{H}_0^{\times}$; and if $t > 0$, then

$$\left\{(a,b) : |a|^2 \neq t|b|^2 \text{ in } \mathbb{R}_0^+\right\}$$

is the invertible proper subset of $\mathbb{H}_t^{\times}$, where "invertible subset of $\mathbb{H}_t^{\times}$" means "a subset of $\mathbb{H}_t^{\times}$ containing of all invertible elements".

Proof. This corollary is nothing but a summary of Theorems 2.11, 2.12 and 2.13. □

2.5. DECOMPOSITIONS OF THE NONNEGATIVELY-SCALED HYPERCOMPLEX RINGS

In this section, we consider a certain decomposition of the t-scaled hypercomplex ring $\mathbb{H}_t$, for an arbitrary fixed "positive" scale $t > 0$ in $\mathbb{R}$. Let $t \geq 0$ and $\mathbb{H}_t$, the corresponding t-scaled hypercomplex ring. Partition $\mathbb{H}_t$ by

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}$$

with

$$\mathbb{H}_t^{inv} = \left\{(a,b) : |a|^2 \neq t|b|^2\right\}, \tag{2.18}$$

and

$$\mathbb{H}_t^{sing} = \left\{(a,b) : |a|^2 = t|b|^2\right\},$$

where $\sqcup$ is the disjoint union. By (2.15) and (2.16), $(a,b) \in \mathbb{H}_t^{inv}$, if and only if it is invertible, equivalently, $(a,b) \in \mathbb{H}_t^{sing}$, if and only if it is not invertible, in $\mathbb{H}_t$.

Recall-and-note that the determinant is a multiplicative map on $M_n(\mathbb{C})$, for all $n \in \mathbb{N}$, in the sense that:

$$\det(AB) = \det(A)\det(B), \quad \forall A, B \in M_n(\mathbb{C}). \tag{2.19}$$

Thus, by (2.19), one has

$$\xi, \eta \in \mathbb{H}_t^{inv} \Rightarrow \det\left([\xi \cdot_t \eta]_t\right) = \det\left([\xi]_t [\eta]_t\right) \neq 0. \tag{2.20}$$

Lemma 2.15. *Let $t \geq 0$ in $\mathbb{R}$. Then the subset $\mathbb{H}_t^{inv} \overset{\text{denote}}{=} \left(\mathbb{H}_t^{inv}, \cdot_t\right)$ of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ forms a non-Abelian group, i.e., $\mathbb{H}_t^{inv}$ is not only a sub-monoid, but also an embedded group in $\mathbb{H}_t^{\times}$.*

Proof. By (2.19), if $\xi, \eta \in \mathbb{H}_t^{inv}$, then $\xi \cdot_t \eta \in \mathbb{H}_t^{inv}$, too, i.e., the operation $(\cdot_t)$ is closed, and associative on $\mathbb{H}_t^{inv}$. Also, the $(\cdot_t)$-identity $(1,0)$ is contained in $\mathbb{H}_t^{inv}$ by (2.18). Therefore, the sub-structure $\left(\mathbb{H}_t^{inv}, \cdot_t\right)$ forms a sub-monoid of $\mathbb{H}_t^{\times}$. But, by (2.14) and (2.20), each element $\xi \in \mathbb{H}_t^{inv}$ has its $(\cdot_t)$-inverse ξ^{-1} contained in $\mathbb{H}_t^{inv}$. It shows that $\mathbb{H}_t^{inv}$ forms a non-Abelian group in the monoid $\mathbb{H}_t^{\times}$. $\square$

By the partition (2.18) and the multiplicativity (2.20), one can obtain the following equivalent result of the above theorem.

Lemma 2.16. *Let $t \geq 0$ in $\mathbb{R}$. Then the pair*

$$\mathbb{H}_t^{\times sing} \overset{\text{denote}}{=} \left(\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{\times},\ \cdot_t\right) = \left(\mathbb{H}_t^{sing} \setminus \{(0,0)\}\,,\ \cdot_t\right)$$

forms a semigroup without identity in the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$.

Proof. By (2.19) and (2.20), the operation $(\cdot_t)$ is closed and associative on the set,

$$\mathbb{H}_t^{\times sing} \overset{\text{def}}{=} \mathbb{H}_t^{\times} \cap \mathbb{H}_t^{sing} = \mathbb{H}_t^{sing} \setminus \{(0,0)\}\,.$$

However, the $(\cdot_t)$-identity $(1,0)$ is not contained in $\mathbb{H}_t^{\times sing}$, since $I_2 = [(1,0)]_t$ is in $\mathbb{H}_t^{inv}$. So, in the monoid $\mathbb{H}_t^{\times}$, the sub-structure $(\mathbb{H}_t^{\times sing},\ \cdot_t)$ forms a semigroup (without identity). $\square$

The above lemma definitely includes the fact that: $(\mathbb{H}_t^{sing}, \cdot_t)$ is just a semigroup (without identity), which is not a sub-monoid of $\mathbb{H}_t^{\times}$ (and hence, not a group).

The above two algebraic characterizations show that the set-theoretical decomposition (2.18) induces an algebraic decomposition of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$,

$$\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{inv},\ \cdot_t\right) \sqcup (\mathbb{H}_t^{\times sing},\ \cdot_t),$$

where

$$\mathbb{H}_t^{inv} = \left\{(a,b) \in \mathbb{H}_t^{\times} : |a|^2 \neq t|b|^2\right\}, \tag{2.21}$$

and

$$\mathbb{H}_t^{\times sing} = \left\{(a,b) \in \mathbb{H}_t^{\times} : |a|^2 = t|b|^2\right\},$$

whenever $t \geq 0$ in $\mathbb{R}$.

Theorem 2.17. *For $t \geq 0$ in $\mathbb{R}$, the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is algebraically decomposed to be*

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where $\mathbb{H}_t^{inv}$ is the group, and $\mathbb{H}_t^{\times sing}$ is the semigroup without identity in (2.21).

Proof. The algebraic decomposition,

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is obtained by the set-theoretic decomposition (2.18) of $\mathbb{H}_t^{\times}$, the above two lemmas, and (2.21). $\square$

By the above theorem, one can have the following concepts whenever a given scale t is nonnegative in $\mathbb{R}$.

Definition 2.18. Let $t \geq 0$ in $\mathbb{R}$, and $\mathbb{H}_t^{\times}$, the t-scaled hypercomplex monoid. The algebraic block,

$$\mathbb{H}_t^{inv} = \left(\left\{(a,b) \in \mathbb{H}_t^{\times} : |a|^2 \neq t|b|^2\right\}, \cdot_t\right),$$

is called the group-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$), and the other algebraic block,

$$\mathbb{H}_t^{\times sing} = \left(\left\{(a,b) \in \mathbb{H}_t^{\times} : |a|^2 = t|b|^2\right\}, \cdot_t\right),$$

is called the semigroup-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$).

By the above definition, Theorem 2.17 can be re-stated that: if a scale t is nonnegative in $\mathbb{R}$, then the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is decomposed to be the group-part $\mathbb{H}_t^{inv}$ and the semigroup-part $\mathbb{H}_t^{\times sing}$.

One may say that if $t < 0$ in $\mathbb{R}$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty in $\mathbb{H}_t^{\times}$. Indeed, for any scale $t \in \mathbb{R}$, the t-scaled hypercomplex monoid $\mathbb{H}_t$ is decomposed to be (2.21). As we have seen in this section, if $t \geq 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is nonempty, meanwhile, as we considered in Section 2.4, if $t < 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty, equivalently, the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is identified with its group-part $\mathbb{H}_t^{inv}$, i.e., $\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv}$ in $\mathbb{H}_t$, whenever $t < 0$.

Corollary 2.19. *For every $t \in \mathbb{R}$, the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is partitioned by*

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where the group-part $\mathbb{H}_t^{inv}$ and the semigroup-part $\mathbb{H}_t^{\times sing}$ are in the sense of (2.21). *In particular, if $t < 0$, then*

$$\mathbb{H}_t^{\times sing} = \emptyset \Longleftrightarrow \mathbb{H}_t^{\times} = \mathbb{H}_t^{inv};$$

meanwhile, if $t \geq 0$, then $\mathbb{H}_t^{\times sing}$ is a non-empty proper subset of $\mathbb{H}_t^{\times}$.

Proof. It is shown conceptually by the discussion of the very above paragraph. Also, see Theorems 2.11 and 2.17. □

3. SPECTRAL ANALYSIS ON $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ UNDER $\left\{\left(\mathbb{C}^2, \pi_t\right)\right\}_{t\in\mathbb{R}}$

Throughout this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding t-scaled hypercomplex ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2, +, \cdot_t\right),$$

containing its hypercomplex monoid $\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{\times}, \cdot_t\right)$. In Section 2, we showed that for a scale $t \in \mathbb{R}$, the monoid $\mathbb{H}_t^{\times}$ is partitioned by

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where $\mathbb{H}_t^{inv}$ is the group-part, and $\mathbb{H}_t^{\times sing}$ is the semigroup-part of $\mathbb{H}_t$. In particular, if $t < 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty in $\mathbb{H}_t^{\times}$, equivalently, $\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv}$ in $\mathbb{H}_t$, meanwhile, if $t \geq 0$, then $\mathbb{H}_t^{\times sing}$ is a non-empty proper subset of $\mathbb{H}_t^{\times}$.

Motivated by such an analysis of invertibility on $\mathbb{H}_t$, we here consider spectral analysis on $\mathbb{H}_t$.

3.1. HYPERCOMPLEX-SPECTRAL FORMS ON $\mathbb{H}_t$

For $t \in \mathbb{R}$, let $\mathbb{H}_t$ be the t-scaled hypercomplex ring realized to be

$$\mathcal{H}_2^t = \pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\},$$

in $M_2(\mathbb{C})$ under the Hilbert-space representation $\Pi_t = \left(\mathbb{C}^2, \pi_t\right)$ of $\mathbb{H}_t$.

Let $(a, b) \in \mathbb{H}_t$ be an arbitrary element with

$$\pi_t(a, b) = [(a, b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t.$$

Then, in a variable z on $\mathbb{C}$,

$$\begin{aligned}
\det\left([(a, b)]_t - z\,[(1, 0)]_t\right) &= \det \begin{pmatrix} a - z & tb \\ \overline{b} & \overline{a} - z \end{pmatrix} \\
&= (a - z)(\overline{a} - z) - t|b|^2 \\
&= |a|^2 - az - \overline{a}z + z^2 - t|b|^2 \\
&= z^2 - (a + \overline{a})\,z + \left(|a|^2 - t|b|^2\right) \\
&= z^2 - 2\mathrm{Re}\,(a)z + \det\left([(a, b)]_t\right),
\end{aligned}$$

where $\mathrm{Re}\,(a)$ is the real part of a in $\mathbb{C}$, and

$$\det\left([(a, b)]_t\right) = |a|^2 - t|b|^2,$$

by Theorem 2.9(i). Thus, the equation,

$$\det\left([(a, b)]_t - z\,[(1, 0)]_t\right) = 0,$$

in a variable z on $\mathbb{C}$, has its solutions,

$$z = \frac{2\mathrm{Re}\,(a) \pm \sqrt{4\mathrm{Re}\,(a)^2 - 4\det\left([(a, b)]_t\right)}}{2},$$

if and only if

$$z = \mathrm{Re}\,(a) \pm \sqrt{\mathrm{Re}\,(a)^2 - \det\left([(a, b)]_t\right)}. \tag{3.1}$$

Recall that a matrix $A \in M_n(\mathbb{C})$, for any $n \in \mathbb{N}$, has its spectrum

$$\mathrm{spec}\,(A) = \{\lambda \in \mathbb{C} : \det\,(A - \lambda I_n) = 0\},$$

equivalently,

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} : \text{ there exists } \eta \in \mathbb{C}^n \text{ such that } A\eta = \lambda\eta\}, \tag{3.2}$$

if and only if

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} : A - \lambda I_n \text{ is not invertible in } M_n(\mathbb{C})\},$$

as a nonempty discrete (compact) subset of $\mathbb{C}$, where I_n is the identity matrix of $M_n(\mathbb{C})$ (e.g., [9]). More generally, if $T \in B(H)$ is an operator on a Hilbert space H, then the spectrum $\sigma(T)$ of T is defined to be a nonempty compact subset,

$$\sigma(T) = \{z \in \mathbb{C} : T - zI_H \text{ is not invertible on } H\},$$

where I_H is the identity operator of $B(H)$. Remark that if H is infinite-dimensional, then $\sigma(T)$ is not a discrete subset of $\mathbb{C}$ as in (3.2), in general (e.g., [8]).

Theorem 3.1. *Let $(a, b) \in \mathbb{H}_t$ realized to be $[(a, b)]_t \in \mathcal{H}_2^t$. Then*

$$\operatorname{spec}([(a, b)]_t) = \left\{\operatorname{Re}(a) \pm \sqrt{\operatorname{Re}(a)^2 - \det([(a, b)]_t)}\right\},$$

in $\mathbb{C}$. More precisely, if

$$a = x + yi, \quad b = u + vi \in \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$ in $\mathbb{C}$, then

$$\operatorname{spec}([(a, b)]_t) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\} \text{ in } \mathbb{C}. \tag{3.3}$$

Proof. The realization $[(a, b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t$ of a hypercomplex number $(a, b) \in \mathbb{H}_t$ has its spectrum,

$$\operatorname{spec}([(a, b)]_t) = \left\{\operatorname{Re}(a) \pm \sqrt{\operatorname{Re}(a)^2 - (|a|^2 - t|b|^2)}\right\},$$

in $\mathbb{C}$, by (3.1) and (3.2). If

$$a = x + yi, \text{ and } b = u + vi \text{ in } \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$ in $\mathbb{C}$, then

$$\operatorname{Re}(a) = x,$$

and

$$|a|^2 - t|b|^2 = \left(x^2 + y^2\right) - t\left(u^2 + v^2\right),$$

in $\mathbb{R}$, and hence,

$$\operatorname{spec}([(a, b)]_t) = \left\{x \pm \sqrt{-y^2 + tu^2 + tv^2}\right\},$$

if and only if

$$\operatorname{spec}([(a, b)]_t) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\},$$

in $\mathbb{C}$. Therefore, the set-equality (3.3) holds. $\square$

From below, for our purposes, we let

$$a = x + yi \text{ and } b = u + vi \text{ in } \mathbb{C} \tag{3.4}$$

with

$$x, y, u, v \in \mathbb{R}, \text{ and } i = \sqrt{-1}.$$

The above theorem can be refined by the following result.

Corollary 3.2. *Let* $(a, b) \in \mathbb{H}_t$*, realized to be* $[(a, b)]_t \in \mathcal{H}_2^t$*, satisfy* (3.4)*. Then the following assertions hold.*

(i) *If* $\mathrm{Im}\,(a)^2 = t|b|^2$ *in* $\mathbb{R}$*, where* $\mathrm{Im}\,(a)$ *is the imaginary part of* a *in* $\mathbb{C}$*, then*

$$\mathrm{spec}\,([(a, b)]_t) = \{x\} = \{\mathrm{Re}\,(a)\} \ \textit{in}\ \mathbb{R}.$$

(ii) *If* $\mathrm{Im}\,(a)^2 < t|b|^2$ *in* $\mathbb{R}$*, then*

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm \sqrt{tu^2 + tv^2 - y^2}\right\} \ \textit{in}\ \mathbb{R}.$$

(iii) *If* $\mathrm{Im}\,(a)^2 > t|b|^2$ *in* $\mathbb{R}$*, then*

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\} \ \textit{in}\ \mathbb{C} \setminus \mathbb{R}.$$

Proof. For $(a, b) \in \mathbb{H}_t$, satisfying (3.4), one has

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\},$$

by (3.3). So, one can verify that: (a) if $y^2 - tu^2 - tv^2 = 0$, equivalently, if

$$\mathrm{Im}\,(a)^2 = t|b|^2 \text{ in } \mathbb{R},$$

then $\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm i\sqrt{0}\right\} = \{x\}$ in $\mathbb{R}$; (b) if $y^2 - tu^2 - tv^2 < 0$, equivalently, if

$$\mathrm{Im}\,(a)^2 < t|b|^2 \text{ in } \mathbb{R},$$

then

$$x \pm i\sqrt{y^2 - tu^2 - tv^2} = x \pm i\sqrt{-\left|y^2 - tu^2 - tv^2\right|},$$

implying that

$$x \pm i\sqrt{y^2 - tu^2 - tv^2} = x \pm i^2\sqrt{tu^2 + tv^2 - y^2},$$

and hence,

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \mp \sqrt{tu^2 + tv^2 - y^2}\right\} \text{ in } \mathbb{R};$$

and, finally, (c) if $y^2 - tu^2 - tv^2 > 0$, equivalently, if

$$\mathrm{Im}\,(a)^2 > t|b|^2 \text{ in } \mathbb{R},$$

then

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\},$$

contained in $\mathbb{C} \setminus \mathbb{R}$.

Therefore, the refined statements (i), (ii) and (iii) of the spectrum (3.3) of $[(a, b)]_t$ hold true. $\square$

By the above corollary, one immediately obtains the following result.

Corollary 3.3. *Suppose* $(a, b) \in \mathbb{H}_t$. *If* $\mathrm{Im}\,(a)^2 \leq t|b|^2$, *then*

$$\mathrm{spec}\left([(a,b)]_t\right) \subset \mathbb{R};$$

meanwhile, if $\mathrm{Im}\,(b)^2 > t|b|^2$, *then*

$$\mathrm{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R})\,, \text{ in } \mathbb{C}.$$

Proof. It is shown by (i)–(iii) of Corollary 3.2. □

Also, we have the following result.

Theorem 3.4. *Assume that the fixed scale* $t \in \mathbb{R}$ *is negative, i.e.,* $t < 0$ *in* $\mathbb{R}$. *If*

$$(a,b) \in \mathbb{H}_t, \text{ with } b \neq 0 \text{ in } \mathbb{C},$$

then

$$\mathrm{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C}. \tag{3.5}$$

Meanwhile, if $b = 0$ *in* $\mathbb{C}$ *for* $(a, b) \in \mathbb{H}_t$, *then*

$$a \in \mathbb{R} \Longrightarrow \mathrm{spec}\left([(a,0)]_t\right) = \{a\} \text{ in } \mathbb{R},$$

and

$$a \in \mathbb{C} \setminus \mathbb{R} \Longrightarrow \mathrm{spec}\left([(a,0)]_t\right) = \{a, \overline{a}\} \text{ in } \mathbb{C} \setminus \mathbb{R}. \tag{3.6}$$

Proof. Assume that the scale t is given to be negative in $\mathbb{R}$. Then, for any $(a, b) \in \mathbb{H}_t$, one immediately obtains that

$$\mathrm{Im}\,(a)^2 \geq t|b|^2,$$

because the left-hand side, $\mathrm{Im}\,(a)^2$, is nonnegative, but the right-hand side, $t|b|^2$ is either negative or zero in $\mathbb{R}$ by the negativity of t.

Suppose $b \neq 0$ in $\mathbb{C}$, equivalently, $|b|^2 > 0$, implying $t|b|^2 < 0$ in $\mathbb{R}$. Then

$$\mathrm{Im}\,(a)^2 > t|b|^2 \text{ in } \mathbb{R}.$$

Thus, by Corollary 3.2(iii), the spectra, $\mathrm{spec}\left([(a,b)]_t\right)$, of the realizations $[(a,b)]_t$ of $(a, b) \in \mathbb{H}_t$, with $b \neq 0$, is contained in $\mathbb{C} \setminus \mathbb{R}$. It proves the relation (3.5).

Meanwhile, if $a = \mathrm{Re}\,(a)$, and $b = 0$ in $\mathbb{C}$, then

$$0 = \mathrm{Im}\,(a)^2 \leq 0 = t \cdot 0 \text{ in } \mathbb{R},$$

implying that

$$\mathrm{spec}\left([(a,0)]_t\right) \subset \mathbb{R} \text{ in } \mathbb{C},$$

by Corollary 3.2(i). However, if $\mathrm{Im}\,(a) \neq 0$, and $b = 0$, then

$$\mathrm{Im}\,(a)^2 > 0 = t \cdot 0 \text{ in } \mathbb{R},$$

and hence,

$$\mathrm{spec}\left([(a,0)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C}.$$

So, the relation (3.6) is proven. □

The above theorem specifies Theorem 3.1 for the case where $t < 0$ in $\mathbb{R}$, by (3.5) and (3.6).

Theorem 3.5. *Assume that* $t = 0$ *in* $\mathbb{R}$. *If* $(a, b) \in \mathbb{H}_0$ *with* $\text{Im}\,(a) \neq 0$ *in* $\mathbb{C}$, *then*

$$\text{spec}\left(\left[(a,b)\right]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}) \;\; in\, \mathbb{C}. \tag{3.7}$$

Meanwhile, if $\text{Im}\,(a) = 0$, *then*

$$\text{spec}\left(\left[(a,b)\right]_t\right) \subset \mathbb{R} \; in\, \mathbb{C}. \tag{3.8}$$

Proof. Suppose the fixed scale t is zero in $\mathbb{R}$. Then, for any hypercomplex number $(a, b) \in \mathbb{H}_0$, one has

$$[(a,b)]_0 = \begin{pmatrix} a & 0 \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^0,$$

and hence,

$$\text{Im}\,(a)^2 \geq 0 = 0 \cdot |b|^2 \text{ in } \mathbb{R}.$$

In particular, if $\text{Im}\,(a) \neq 0$ in $\mathbb{C}$, then the above inequality becomes

$$\text{Im}\,(a)^2 > 0 \text{ in } \mathbb{R},$$

implying that

$$\text{spec}\left(\left[(a,b)\right]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}) \;\text{ in } \mathbb{C},$$

by Corollary 3.2(iii), i.e., for all $(a, b) \in \mathbb{H}_0$, with $a \in \mathbb{C}$ with $\text{Im}\,(a) \neq 0$, and $b \in \mathbb{C}$ arbitrary, the spectra of the realizations of such (a, b) are contained in $\mathbb{C} \setminus \mathbb{R}$. It shows that the relation (3.7) holds.

Meanwhile, if $\text{Im}\,(a) = 0$ in $\mathbb{C}$, then one has

$$\text{Im}\,(a)^2 = 0 \geq 0 = 0 \cdot |b|^2 \text{ in } \mathbb{R}.$$

So, by Corollary 3.2(i), we have

$$\text{spec}\left(\left[(a,b)\right]_t\right) \subset \mathbb{R} \text{ in } \mathbb{C}.$$

Therefore, the relation (3.8) holds true, too. □

The above theorem specifies Theorem 3.1 for the case where a scale t is zero in $\mathbb{R}$, by (3.7) and (3.8).

Theorem 3.6. *Assume that the fixed scale* t *is positive in* $\mathbb{R}$. *Then the* t*-scaled hypercomplex ring* $\mathbb{H}_t$ *is decomposed to be*

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

with

$$\mathbb{H}_t^+ = \left\{ (a,b) \in \mathbb{H}_t : \text{Im}\,(a)^2 > t|b|^2 \right\}, \tag{3.9}$$

and

$$\mathbb{H}_t^{-0} = \left\{ (a,b) \in \mathbb{H}_t : \text{Im}\,(a)^2 \leq t|b|^2 \right\},$$

where $\sqcup$ is the disjoint union. Moreover, if $(a,b) \in \mathbb{H}_t^{+}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}). \tag{3.10}$$

Meanwhile, if $(a,b) \in \mathbb{H}_t^{-0}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset \mathbb{R} \text{ in } \mathbb{C}. \tag{3.11}$$

Proof. Suppose that $t > 0$ in $\mathbb{R}$. Then one can decompose the t-scaled hypercomplex ring $\mathbb{H}_t$ by

$$\mathbb{H}_t = \mathbb{H}_t^{+} \sqcup \mathbb{H}_t^{-0},$$

with

$$\begin{aligned} \mathbb{H}_t^{+} &= \left\{(a,b) \in \mathbb{H}_t : \operatorname{Im}(a)^2 > t|b|^2\right\}, \\ \mathbb{H}_t^{-0} &= \left\{(a,b) \in \mathbb{H}_t : \operatorname{Im}(a)^2 \leq t|b|^2\right\}, \end{aligned} \tag{3.12}$$

set-theoretically. Thus, the partition (3.9) holds by (3.12).

If $(a,b) \in \mathbb{H}_t^{+}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}),$$

meanwhile, if $(a,b) \in \mathbb{H}_t^{-0}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset \mathbb{R}, \text{ in } \mathbb{C}.$$

So, the relations (3.10) and (3.11) are proven. □

The above theorem specifies Theorem 3.1 for the cases where a fixed scale t is positive in $\mathbb{R}$, by (3.10) and (3.11), up to the decomposition (3.9).

In fact, one can realize that, for "all" $t \in \mathbb{R}$, the corresponding t-scaled hypercomplex ring $\mathbb{H}_t$ is partitioned to be

$$\mathbb{H}_t = \mathbb{H}_t^{+} \sqcup \mathbb{H}_t^{-0},$$

where $\mathbb{H}_t^{+}$ and $\mathbb{H}_t^{-0}$ are in the sense of (3.9). Especially, Theorems 3.4, 3.5 and 3.6 characterize the above decomposition case-by-case, based on Theorem 3.1 and Corollary 3.2. So, we obtain the following universal spectral properties on $\mathbb{H}_t$.

Corollary 3.7. *Let $t \in \mathbb{R}$ be an arbitrarily fixed scale for $\mathbb{H}_t$. Then*

$$\mathbb{H}_t = \mathbb{H}_t^{+} \sqcup \mathbb{H}_t^{-0}, \text{ set-theoretically},$$

where $\left\{\mathbb{H}_t^{+}, \mathbb{H}_t^{-0}\right\}$ is a partition in the sense of (3.9) for t. Moreover, if $(a,b) \in \mathbb{H}_t^{+}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}),$$

meanwhile, if $(a,b) \in \mathbb{H}_t^{-0}$, then

$$\operatorname{spec}\left([(a,b)]_t\right) \subset \mathbb{R} \text{ in } \mathbb{C}.$$

Especially, if $t < 0$, then $\mathbb{H}_t^{-0} = \{(0,0)\}$, equivalently, $\mathbb{H}_t^{\times} = \mathbb{H}_t^{+}$.

Proof. This corollary is nothing but a summary of Theorems 3.4, 3.5 and 3.6. □

It is not hard to check the converses of the statements of Corollary 3.7 hold true, too.

Theorem 3.8. *Let $\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0}$ be the fixed t-scaled hypercomplex ring for $t \in \mathbb{R}$. Then the following assertions hold.*

(i) $(a, b) \in \mathbb{H}_t^+$, *if and only if* $\operatorname{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R})$.
(ii) $(a, b) \in \mathbb{H}_t^{-0}$, *if and only if* $\operatorname{spec}\left([(a,b)]_t\right) \subset \mathbb{R}$.

Proof. First, assume that $(a, b) \in \mathbb{H}_t^+$ in $\mathbb{H}_t$. Then, by Corollary 3.7,

$$\operatorname{spec}\left([a,b]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}).$$

Now, suppose that

$$\operatorname{spec}\left([a,b]_t\right) \subset \mathbb{R} \ \text{ in } \ \mathbb{C},$$

and assume that $(a, b) \in \mathbb{H}_t^+$. Then, (a, b) is contained in $\mathbb{H}_t^{-0}$, equivalently, it cannot be an element of $\mathbb{H}_t^+$, by Corollary 3.2(i)–(ii), (3.6), (3.8) and (3.11). It contradicts our assumption. Therefore,

$$(a, b) \in \mathbb{H}_t^+ \Longleftrightarrow \operatorname{spec}\left([(a,b)]_t\right) \subset (\mathbb{C} \setminus \mathbb{R}).$$

Thus, the statement (i) holds.

By the decomposition (3.9), the statement (ii) holds true, by (i). □

By the above theorem, we obtain the following result.

Corollary 3.9. *Let $\mathbb{H}_t$ be the t-scaled hypercomplex ring for an arbitrary $t \in \mathbb{R}$, and suppose it is decomposed to be*

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

as in (3.9)*. Assume that a given element (a, b) satisfies the condition* (3.4)*. Then the following assertions hold.*

(i) $(a, b) \in \mathbb{H}_t^+$, *if and only if*

$$\operatorname{spec}\left([(a,b)]_t\right) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\} \subset (\mathbb{C} \setminus \mathbb{R}).$$

(ii) $(a, b) \in \mathbb{H}_t^{-0}$, *if and only if either*

$$\operatorname{spec}\left([(a,b)]_t\right) = \begin{cases} \{x\} & \text{if } \operatorname{Im}(a)^2 = t|b|^2, \\ \left\{x \pm \sqrt{tu^2 + tv^2 - y^2}\right\} & \text{if } \operatorname{Im}(a)^2 < t|b|^2, \end{cases}$$

in $\mathbb{R}$.

Proof. The statement (i) holds by (3.5) and Theorem 3.8(i). Meanwhile, the statement (ii) holds by (3.6) and Theorem 3.8(ii). □

Recall that a Hilbert-space operator $T \in B(H)$ is self-adjoint, if $T^* = T$ in $B(H)$, where T^* is the adjoint of T (see Section 5 below). It is well-known that T is self-adjoint, if and only if its spectrum is contained in $\mathbb{R}$ in $\mathbb{C}$. So, one obtains the following result.

Proposition 3.10. *A hypercomplex number $(a,b) \in \mathbb{H}_t^{-0}$ in $\mathbb{H}_t$, if and only if the realization $[(a,b)]_t \in \mathcal{H}_2^t$ is self-adjoint "in $M_2(\mathbb{C})$".*

Proof. ($\Rightarrow$) Suppose $(a,b) \in \mathbb{H}_t^{-0}$ in $\mathbb{H}_t$. Then $\operatorname{spec}([(a,b)]_t) \subset \mathbb{R}$ in $\mathbb{C}$, implying that $[(a,b)]_t$ is self-adjoint in $M_2(\mathbb{C})$.

($\Leftarrow$) Suppose $[(a,b)]_t \in \mathcal{H}_2^t$ is self-adjoint in $M_2(\mathbb{C})$, and assume that $(a,b) \notin \mathbb{H}_t^{-0}$, equivalently, $(a,b) \in \mathbb{H}_t^{+}$ in $\mathbb{H}_t$. Then,

$$\operatorname{spec}([(a,b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C},$$

and hence, $[(a,b)]_t$ is not self-adjoint in $M_2(\mathbb{C})$. It contradicts our assumption that it is self-adjoint. $\square$

Equivalent to the above proposition, one can conclude that $(a,b) \in \mathbb{H}_t^{+}$ in $\mathbb{H}_t$, if and only if $[(a,b)]_t$ is not be self-adjoint in $M_2(\mathbb{C})$. The self-adjointness of realizations of hypercomplex numbers would be considered more in detail in Section 5.

3.2. THE SCALED-SPECTRALIZATIONS $\{\sigma_t\}_{t\in\mathbb{R}}$

In this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding hypercomplex ring $\mathbb{H}_t$, containing the t-scaled hypercomplex monoid

$$\mathbb{H}_t^{\times} = (\mathbb{H}_t \setminus \{(0,0)\},\ \cdot_t).$$

Recall that $\mathbb{H}_t^{\times}$ is algebraically decomposed to be

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

with

$$\mathbb{H}_t^{inv} = \left\{(a,b) : |a|^2 \neq t|b|^2\right\}, \text{ the group-part}, \tag{3.13}$$

and

$$\mathbb{H}_t^{\times sing} = \left\{(a,b) : |a|^2 = t|b|^2\right\}, \text{ the semigroup-part},$$

as in (2.21). Therefore, the t-scaled hypercomplex ring is set-theoretically decomposed to be

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \{(0,0)\} \sqcup \mathbb{H}_t^{\times sing} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}, \tag{3.14}$$

by (3.13), where

$$\mathbb{H}_t^{sing} \overset{\text{denote}}{=} \{(0,0)\} \sqcup \mathbb{H}_t^{\times sing} \text{ in } (3.2.2).$$

Also, the ring $\mathbb{H}_t$ is spectrally decomposed to be

$$\mathbb{H}_t = \mathbb{H}_t^{+} \sqcup \mathbb{H}_t^{-0},$$

with

$$\mathbb{H}_t^{+} = \left\{(a, b) : \mathrm{Im}\,(a)^2 > t|b|^2\right\}, \tag{3.15}$$

and

$$\mathbb{H}_t^{-0} = \left\{(a, b) : \mathrm{Im}\,(a)^2 \leq t|b|^2\right\},$$

satisfying that: $(a, b) \in \mathbb{H}_t^{+}$ if and only if $\mathrm{spec}\,([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R})$; meanwhile, $(a, b) \in \mathbb{H}_t^{-0}$ if and only if $\mathrm{spec}\,([(a, b)]_t) \subset \mathbb{R}$, by Corollary 3.9(i)–(ii).

Corollary 3.11. *Let $\mathbb{H}_t$ be the t-scaled hypercomplex ring for $t \in \mathbb{R}$. Then it is decomposed to be*

$$\begin{aligned} \mathbb{H}_t &= \left(\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{+}\right) \sqcup \left(\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{-0}\right) \\ &= (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{+}) \sqcup (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{-0}), \end{aligned} \tag{3.16}$$

set-theoretically.

Proof. It is proven by (3.14) and (3.15). □

Observe now that if $(a, 0) \in \mathbb{H}_t$, then

$$[(a, 0)]_t = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \text{ in } \mathcal{H}_2^t,$$

satisfying

$$\mathrm{spec}\,([(a, 0)]_t) = \{a, \ \overline{a}\} \ \text{ in } \mathbb{C}. \tag{3.17}$$

Indeed, by (3.3), if $(a, 0) \in \mathbb{H}_t$ satisfying $a = x + yi \in \mathbb{C}$ with $x, y \in \mathbb{R}$, then

$$\mathrm{spec}\,([(a, b)]_t) = \left\{x \pm i\sqrt{y^2}\right\} = \{x \pm |y|i\} = \{x \pm yi\},$$

implying (3.17), where $|y|$ is the absolute value of y in $\mathbb{R}$.

Motivated by (3.15), (3.16) and (3.17), we define a certain $\mathbb{C}$-valued function σ_t from $\mathbb{H}_t$. Define a function,

$$\sigma_t : \mathbb{H}_t \to \mathbb{C},$$

by

$$\sigma_t\,((a, b)) \overset{\text{def}}{=} \begin{cases} a = x + yi & \text{if } b = 0 \text{ in } \mathbb{C}, \\ x + i\sqrt{y^2 - tu^2 - tv^2} & \text{if } b \neq 0 \text{ in } \mathbb{C}, \end{cases} \tag{3.18}$$

for all $(a, b) \in \mathbb{H}_t$ satisfying the condition (3.4):

$$a = x + yi \text{ and } b = u + vi \text{ in } \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$.

Remark that such a morphism σ_t is indeed a well-defined function assigning all hypercomplex numbers of $\mathbb{H}_t$ to complex numbers of $\mathbb{C}$. Moreover, by (3.18), it is surjective. But it is definitely not injective. For instance, even though

$$\xi = (1 + 3i, \ -1 + i) \ \text{ and } \eta = (1 - 3\mathrm{i}, \ 1 - \mathrm{i})$$

are distinct in $\mathbb{H}_t$, one has

$$\sigma_t(\xi) = 1 + i\sqrt{9-2t} = \sigma_t(\eta),$$

by (3.18).

Definition 3.12. The surjection $\sigma_t : \mathbb{H}_t \to \mathbb{C}$ of (3.18) is called the t(-scaled)-spectralization on $\mathbb{H}_t$. The images $\{\sigma_t(\xi)\}_{\xi \in \mathbb{H}_t}$ are said to be t(-scaled)-spectral values. From below, we also understand each t-spectral value $\sigma_t(\xi) \in \mathbb{C}$ of a hypercomplex number $\xi \in \mathbb{H}_t$ as a hypercomplex number $(\sigma_t(\xi), 0)$ in $\mathbb{H}_t$, i.e., such an assigned hypercomplex number $(\sigma_t(\xi), 0)$ from the t-spectral value $\sigma_t(\xi)$ of ξ is also called the t-spectral value of ξ.

By definition, all t-spectral values are not only $\mathbb{C}$-quantities for many hypercomplex numbers of $\mathbb{H}_t$ whose realizations of $\mathcal{H}_2^t$ share the same eigenvalues, but also hypercomplex numbers of $\mathbb{H}_t$, whose first coordinates are the value and the second coordinates are 0.

Definition 3.13. Let $\xi \in \mathbb{H}_t$ be a hypercomplex number inducing its t-spectral value $w \overset{\text{denote}}{=} \sigma_t(\xi) \in \mathbb{C}$, also understood to be $\eta = (w, 0) \in \mathbb{H}_t$. The corresponding realization,

$$[\eta]_t = \begin{pmatrix} w & t \cdot 0 \\ 0 & \overline{w} \end{pmatrix} = \begin{pmatrix} \sigma_t(\xi) & 0 \\ 0 & \overline{\sigma_t(\xi)} \end{pmatrix} \in \mathcal{H}_2^t$$

is called the t(-scaled)-spectral form of ξ. By $\Sigma_t(\xi)$, we denote the t-spectral form of $\xi \in \mathbb{H}_t$.

Note that the conjugate-notation in Definition 3.13 is symbolic in the sense that: if $t > 0$, and

$$\sigma_t(\xi) = 1 + i\sqrt{1-5t} = 1 - \sqrt{5t-1},$$

(and hence, $\sigma_t(\xi) \in \mathbb{R}$), then the symbol,

$$\overline{\sigma_t(\xi)} \overset{\text{means}}{=} 1 - i\sqrt{1-5t} = 1 + \sqrt{5t-1},$$

in $\mathbb{R}$, i.e., the conjugate-notation in Definition 3.13 has a symbolic meaning containing not only the usual conjugate on $\mathbb{C}$, but also the above computational meaning on $\mathbb{R}$.

Remark 3.14. The conjugate-notation in Definition 3.13 is symbolic case-by-case. If the t-spectral value $\sigma_t(\xi)$ is in $\mathbb{C}$, then $\overline{\sigma_t(\xi)}$ means the usual conjugate. Meanwhile, if t-spectral value

$$\sigma_t(\xi) = x + \sqrt{tu^2 + tv^2 - y^2},$$

with

$$tu^2 + tv^2 - y^2 \geq 0, \text{ in } \mathbb{R},$$

then

$$\overline{\sigma_t(\xi)} = x - \sqrt{tu^2 + tv^2 - y^2} \text{ in } \mathbb{R},$$

where $\xi \in \mathbb{H}_t$ satisfies the condition (3.4).

For instance, if $\xi_1 = (-2-i, 0) \in \mathbb{H}_t$, then the t-spectral value is

$$\sigma_t(\xi_1) = -2 - i \text{ in } \mathbb{C},$$

inducing the t-spectral form,

$$\Sigma_t(\xi_1) = \begin{pmatrix} -2-i & 0 \\ 0 & -2+i \end{pmatrix} \text{ in } \mathcal{H}_2^t;$$

meanwhile, if $\xi_2 = (-2-i,\ 1+3i) \in \mathbb{H}_t$, then the t-spectral value is

$$w \overset{\text{denote}}{=} \sigma_t(\xi_2) = -2 + i\sqrt{1-10t},$$

inducing the t-spectral form,

$$\Sigma_t(\xi_2) = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} = \begin{pmatrix} -2+i\sqrt{1-10t} & 0 \\ 0 & -2-i\sqrt{1-10t} \end{pmatrix},$$

where $\overline{w}$ is symbolic in the sense of Remark 3.14; if $t \leq 0$, then

$$\Sigma_t(\xi_2) = \begin{pmatrix} -2+i\sqrt{1-10t} & 0 \\ 0 & -2-i\sqrt{1-10t} \end{pmatrix},$$

meanwhile, if $t > 0$, then

$$\Sigma_t(\xi_2) = \begin{pmatrix} -2+\sqrt{10t-1} & 0 \\ 0 & -2-\sqrt{10t-1} \end{pmatrix},$$

in $\mathcal{H}_2^t$.

Definition 3.15. Two hypercomplex numbers $\xi, \eta \in \mathbb{H}_t$ are said to be t(-scaled)-spectral-related, if

$$\sigma_t(\xi) = \sigma_t(\eta) \text{ in } \mathbb{C},$$

equivalently,

$$\Sigma_t(\xi) = \Sigma_t(\eta) \text{ in } \mathcal{H}_2^t.$$

On the t-hypercomplex ring $\mathbb{H}_t$, the t-spectral relation of Definition 3.15 is an equivalent relation. Indeed,

$$\sigma_t(\xi) = \sigma_t(\xi), \quad \forall \xi \in \mathbb{H}_t;$$

and if ξ and η are t-spectral related in $\mathbb{H}_t$, then

$$\sigma_t(\xi) = \sigma_t(\eta) \Longleftrightarrow \sigma_t(\eta) = \sigma_t(\xi),$$

and hence, η and ξ are t-spectral related in $\mathbb{H}_t$; and if ξ_1 and ξ_2 are t-spectral related, and if ξ_2 and ξ_3 are t-spectral related, then

$$\sigma_t(\xi_1) = \sigma_t(\xi_2) = \sigma_t(\xi_3) \text{ in } \mathbb{C},$$

and hence, ξ_1 and ξ_3 are t-spectral related.

Proposition 3.16. *The t-spectral relation on $\mathbb{H}_t$ is an equivalence relation.*

Proof. The t-spectral relation is reflexive, symmetric and transitive on $\mathbb{H}_t$, by the discussion of the very above paragraph. □

Since the t-spectral relation is an equivalence relation, each element ξ of $\mathbb{H}_t$ has its equivalence class,

$$\widetilde{\xi} \overset{\text{def}}{=} \{\eta \in \mathbb{H}_t : \eta \text{ is } t\text{-related to } \xi\},$$

and hence, the corresponding quotient set,

$$\widetilde{\mathbb{H}_t} \overset{\text{def}}{=} \left\{\widetilde{\xi} : \xi \in \mathbb{H}_t\right\}, \tag{3.19}$$

is well-defined to be the set of all equivalence classes.

Theorem 3.17. *Let $\widetilde{\mathbb{H}_t}$ be the quotient set* (3.19) *induced by the t-spectral relation on $\mathbb{H}_t$. Then*

$$\widetilde{\mathbb{H}_t} \text{ and } \mathbb{C} \text{ are equipotent.} \tag{3.20}$$

Proof. It is not difficult to check that, for any $z \in \mathbb{C}$, there exist $\xi \in \mathbb{H}_t$, such that $z = \sigma_t(\xi)$ by the surjectivity of the t-spectralization σ_t. It implies that there exists $(z, 0) \in \mathbb{H}_t$, such that

$$\widetilde{(z, 0)} = \widetilde{\xi} \text{ in } \widetilde{\mathbb{H}_t}, \text{ whenever } z = \sigma_t(\xi).$$

Thus, set-theoretically, we have

$$\widetilde{\mathbb{H}_t} = \left\{\widetilde{(z, 0)} : z \in \mathbb{C}\right\} \overset{\text{equip}}{=} \mathbb{C},$$

where "$\overset{\text{equip}}{=}$" means "being equipotent (or, bijective) to". Therefore, the relation (3.20) holds. □

The above equipotence (3.20) of the quotient set $\widetilde{\mathbb{H}_t}$ of (3.19) with the complex numbers $\mathbb{C}$ shows that the set $\mathbb{C}$ classifies $\mathbb{H}_t$, for "every" $t \in \mathbb{R}$, up to the t-spectral relation.

3.3. SIMILARITY ON $M_2(\mathbb{C})$ AND THE t-SCALED-SPECTRAL RELATION ON $\mathbb{H}_t$

In Section 3.2, we defined the t-spectralization σ_t on the t-scaled hypercomplex ring $\mathbb{H}_t$, for a fixed scale $t \in \mathbb{R}$, and it induces the t-spectral forms $\{\Sigma_t(\xi)\}_{\xi \in \mathbb{H}_t}$ in $\mathcal{H}_2^t$ as complex diagonal matrices whose main diagonals are the eigenvalues of the realizations $\{[\xi]_t\}_{\xi \in \mathbb{H}_t}$, under the symbolic understanding of Remark 3.14. Moreover, σ_t lets the set $\mathbb{C}$ classify $\mathbb{H}_t$ by (3.20) under the t-spectral relation.

Independently, we showed in [2] and [3] that: on the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, the (-1)-spectral relation, called the quaternion-spectral relation in [2] and [3], is equivalent to the similarity "on $\mathcal{H}_2^{-1}$", as equivalence relations. Here, the similarity "on $\mathcal{H}_2^{-1}$"

means that the realizations $[q_1]_{-1}$ and $[q_2]_{-1}$ of two quaternions $q_1, q_2 \in \mathbb{H}_{-1}$ are similar "in $\mathcal{H}_2^{-1}$", if there exists invertible element U "in $\mathcal{H}_2^{-1}$", such that

$$[q_2]_{-1} = U^{-1} [q_1]_{-1} U \text{ in } \mathcal{H}_2^{-1}.$$

Here, we consider such property for an arbitrary scale $t \in \mathbb{R}$. Recall that, we showed in [2] and [3] that: the (-1)-spectral form $\Sigma_{-1}(\eta)$ and the realization $[\eta]_{-1}$ are similar "in $\mathcal{H}_2^{-1}$", for "all" quaternions which are the (-1)-scaled hypercomplex numbers $\eta \in \mathbb{H}_{-1} = \mathbb{H}$. Are the t-spectral relation on $\mathbb{H}_t$ and the similarity on $\mathcal{H}_2^t$ same as equivalence relations? In conclusion, the answer is negative in general.

Two matrices A and B of $M_n(\mathbb{C})$, for any $n \in \mathbb{N}$, are said to be similar, if there exists an invertible matrix $U \in M_n(\mathbb{C})$, such that

$$B = U^{-1} A U \text{ in } M_n(\mathbb{C}).$$

Remember that if two matrices A and B are similar, then (i) they share the same eigenvalues, (ii) they have the same traces, and (iii) their determinants are same (e.g., [9] and [8]). We here focus on the fact (iii): the similarity of matrices implies their identical determinants, equivalently, if

$$\det(A) \neq \det(B),$$

then A and B are not similar in $M_n(\mathbb{C})$.

Definition 3.18. Let $A, B \in \mathcal{H}_2^t$ be realizations of certain hypercomplex numbers of $\mathbb{H}_t$, for $t \in \mathbb{R}$. They are said to be similar "in $\mathcal{H}_2^t$", if there exists an invertible $U \in \mathcal{H}_2^t$, such that

$$B = U^{-1} A U \quad \text{in} \quad \mathcal{H}_2^t.$$

By abusing notation, we say that two hypercomplex numbers ξ and η are similar in $\mathbb{H}_t$, if their realizations $[\xi]_t$ and $[\eta]_t$ are similar in $\mathcal{H}_2^t$.

Let $(a, b) \in \mathbb{H}_t$ be a hypercomplex number satisfying the condition (3.4) and $(a, b) \neq (0, 0)$. Then it has

$$[(a, b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t,$$

$$\sigma_t((a, b)) = x + i\sqrt{y^2 - tu^2 - tv^2} \overset{\text{let}}{=} w \in \mathbb{C},$$

and

$$\Sigma_t((a, b)) = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \in \mathcal{H}_2^t, \tag{3.21}$$

where $\overline{w}$ is symbolic in the sense of Remark 3.14. Observe that

$$\det([(a, b)]_t) = |a|^2 - t|b|^2 = \left(x^2 + y^2\right) - t\left(u^2 + v^2\right),$$

and

$$det(\Sigma_t((a, b))) = |w|^2 = x^2 + \left|y^2 - tu^2 - tv^2\right|, \tag{3.22}$$

by (3.21). These computations in (3.22) show that, in general, $[(a,b)]_t$ and $\Sigma_t((a,b))$ are "not" similar "as matrices of $M_2(\mathbb{C})$", and hence, not similar in $\mathcal{H}_2^t$. Indeed, for instance, if

$$t > 0, \text{ and } |a|^2 < t|b|^2,$$

then $\det([(a,b)]_t) < 0$, but $\det(\Sigma_t((a,b))) > 0$ in $\mathbb{R}$, by (3.22), implying that

$$\det([(a,b)]_t) \neq \det(\Sigma_t((a,b))) \text{ in general,}$$

showing that $[(a,b)]_t$ and $\Sigma_t((a,b))$ are not similar in $M_2(\mathbb{C})$, and hence, they are not similar in $\mathcal{H}_2^t$, in general.

Proposition 3.19. *Let $(a,b) \in \mathbb{H}_t$ be "nonzero" hypercomplex number satisfying $|a|^2 < t|b|^2$ in $\mathbb{R}$. Then the realization $[(a,b)]_t$ and the t-spectral form $\Sigma_t((a,b))$ are not similar "in $\mathcal{H}_2^t$".*

Proof. Suppose $(a,b) \in \mathbb{H}_t$ satisfies $(a,b) \neq (0,0)$ and $|a|^2 < t|b|^2$, for $t > 0$. And assume that $[(a,b)]_t$ and $\Sigma_t((a,b))$ are similar in $\mathcal{H}_2^t$. Since they are assumed to be similar, their determinants are identically same. However,

$$\det([(a,b)]_t) < 0 \text{ and } \det(\Sigma_t((a,b))) > 0,$$

by (3.22). It contradicts our assumption that they are similar in $\mathcal{H}_2^t$. □

The above proposition confirms that the realizations and the corresponding t-spectral forms of a t-scaled hypercomplex number are not similar in $\mathcal{H}_2^t$, in general.

Consider that, in the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, since the scale is $t = -1 < 0$ in $\mathbb{R}$,

$$\det\left([\xi]_{-1}\right) = \det(\Sigma_{-1}(\xi)) \geq 0, \quad \forall \xi \in \mathbb{H}_{-1},$$

and it is proven that $[\xi]_{-1}$ and $\Sigma_{-1}(\xi)$ are indeed similar in $\mathcal{H}_2^{-1}$, for "all" $\xi \in \mathbb{H}_{-1}$ in [2] and [3], which motivates a question: if a scale $t < 0$ in $\mathbb{R}$, then

$$\det([\eta]_t) = \det(\Sigma_t(\eta)) \geq 0, \quad \forall \eta \in \mathbb{H}_t,$$

by (3.22); so, are the realizations $[\eta]_t$ and the corresponding t-spectral forms $\Sigma_t(\eta)$ similar in $\mathcal{H}_2^t$ as in the case of $t = -1$?

First of all, we need to recall that if $t < 0$, then the t-scaled hypercomplex ring $\mathbb{H}_t$ forms a noncommutative field, since the t-scaled hypercomplex monoid $\mathbb{H}_t^\times$ is a non-Abelian group, by (2.14). It allows us to use similar techniques of [2] and [3].

In the rest part of this section, a given scale $t \in \mathbb{R}$ is automatically assumed to be negative in $\mathbb{R}$.

Assume that $(a,0) \in \mathbb{H}_t$, where $t < 0$. Then

$$[(a,0)]_t = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \Sigma_t((a,0)),$$

in $\mathcal{H}_2^t$, since $\sigma_t((a,0)) = a$ in $\mathbb{C}$. So, clearly, $[(a,0)]_t$ and $\Sigma_t((a,0))$ are similar in $\mathcal{H}_2^t$, because they are equal in $\mathcal{H}_2^t$. Indeed, there exist diagonal matrices with nonzero real entries,

$$X = [(x,0)]_t \in \mathcal{H}_2^t, \text{ with } x = x + 0i \in \mathbb{C}, \ x \neq 0,$$

such that

$$[(a,0)]_t = X^{-1}\left(\Sigma_t(a,0)\right)X \text{ in } \mathcal{H}_2^t.$$

Thus, we are interested in the cases where $(a,b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$.

Lemma 3.20. *Let $t < 0$ in $\mathbb{R}$, and $(a,0) \in \mathbb{H}_t$, a hypercomplex number. Then the realization $[(a,0)]_t$ and the t-spectral form $\Sigma_t((a,0))$ are identically same in $\mathcal{H}_2^t$, and hence, they are similar in $\mathcal{H}_2^t$. (Remark that, in fact, the scale t is not necessarily negative in $\mathbb{R}$ here.)*

Proof. It is proven by the discussion of the very above paragraph. Indeed, one has

$$[(a,0)]_t = \Sigma_t((a,0)) \text{ in } \mathcal{H}_2^t,$$

since $\sigma_t((a,0)) = a$ in $\mathbb{C}$. □

Let $h = (a,b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^{\times}$, satisfying the condition (3.4), where $t < 0$, having its realization,

$$[h]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} x+yi & t(u+vi) \\ u-vi & x-yi \end{pmatrix},$$

and its t-spectral form,

$$\Sigma_t(h) = \begin{pmatrix} x + i\sqrt{y^2 - tu^2 - tv^2} & 0 \\ 0 & x - i\sqrt{y^2 - tu^2 - tv^2} \end{pmatrix} \overset{\text{let}}{=} \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix},$$

in $\mathcal{H}_2^t$. Since $t < 0$ and $b \neq 0$ (by assumption), the t-spectral value $w = \sigma_t(h)$ is a $\mathbb{C}$-quantity with its conjugate $\overline{w}$. Define now a matrix,

$$Q_h \overset{\text{def}}{=} \begin{pmatrix} 1 & t\left(\overline{\frac{w-a}{tb}}\right) \\ \frac{w-a}{tb} & 1 \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

Remark that, by the assumption that $t < 0$ and $b \neq 0$, this matrix is well-defined. Furthermore, one can immediately recognize that $Q_h \in \mathcal{H}_2^t$, i.e.,

$$Q_h = \left[\left(1, \overline{\left(\frac{w-a}{tb}\right)}\right)\right]_t \in \mathcal{H}_2^t. \tag{3.23}$$

One can find that the element $Q_h \in \mathcal{H}_2^t$ of (3.23) is indeed invertible by our negative-scale assumption, since

$$\det(Q_h) = 1 - t\left|\frac{w-a}{tb}\right|^2 \geq 1, \text{ since } t < 0,$$

implying that

$$\det(Q_h) \neq 0 \Longleftrightarrow Q_h \text{ is invertible in } \mathcal{H}_2^t.$$

Observe now that

$$Q_h \Sigma_t(h) = \begin{pmatrix} w & t\left(\overline{\frac{w^2-aw}{tb}}\right) \\ \frac{w^2-aw}{tb} & \overline{w} \end{pmatrix}$$

and

$$[h]_t Q_h = \begin{pmatrix} \dfrac{w}{a\left(\frac{w-a}{tb}\right)+b} & t\left(a\left(\overline{\frac{w-a}{tb}}\right)+b\right) \\ & \overline{w} \end{pmatrix}, \tag{3.24}$$

in $\mathcal{H}_2^t$. Now, let us compare the $(1,2)$-entries of resulted matrices in (3.24). The $(1,2)$-entry of the element $Q_h\Sigma_t(h)$ is

$$\begin{aligned} t\left(\overline{\frac{w^2-aw}{tb}}\right) &= \overline{\frac{w(w-a)}{b}} \\ &= \overline{\frac{\left(x+i\sqrt{y^2-tu^2-tv^2}\right)\left(i\sqrt{y^2-tu^2-tv^2}-yi\right)}{u+vi}} \\ &= \overline{\frac{ix\sqrt{R}-xyi-R+y\sqrt{R}}{u+vi}}, \end{aligned}$$

where

$$R \overset{\text{denote}}{=} y^2 - tu^2 - tv^2 \text{ in } \mathbb{R}, \tag{3.25}$$

and the $(1,2)$-entry of the matrix $[h]_t Q_h$ is

$$\begin{aligned} & t\left(a\left(\overline{\frac{w-a}{tb}}\right)+b\right) \\ &= t\left(\overline{\overline{a}\left(\frac{w-a}{tb}\right)+\overline{b}}\right) = t\left(\overline{\frac{\overline{a}w-|a|^2+t|b|^2}{tb}}\right) = \overline{\frac{\overline{a}w-|a|^2+t|b|^2}{b}} \\ &= \overline{\frac{(x-yi)\left(x+i\sqrt{y^2-tu^2-tv^2}\right)-(x^2+y^2)-t\left(u^2+v^2\right)}{u+vi}} \\ &= \overline{\frac{x^2+ix\sqrt{R}-xyi+y\sqrt{R}-x^2-y^2-tu^2-tv^2}{u+vi}} \\ &= \overline{\frac{x^2+ix\sqrt{R}-xyi+y\sqrt{R}-x^2-R}{u+vi}} \\ &= \overline{\frac{ix\sqrt{R}-xyi-R+y\sqrt{R}}{u+vi}}, \end{aligned} \tag{3.26}$$

where the $\mathbb{R}$-quantity R is in the sense of (3.25). As one can see in (3.25) and (3.26), the $(1,2)$-entries of $[h]_t Q_h$ and $Q_h\Sigma_t(h)$ are identically same, i.e.,

$$Q_h\Sigma_t(h) = [h]_t Q_h \ \text{ in } \ \mathcal{H}_2^t, \tag{3.27}$$

where the matrix $Q_h \in \mathcal{H}_2^t$ is in the sense of (3.23).

Lemma 3.21. *Let $t < 0$ in $\mathbb{R}$, and let $h = (a,b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^\times$. Then the realization $[h]_t$ and the t-spectral form $\Sigma_t(h)$ are similar in $\mathcal{H}_2^t$. In particular, there exists*

$$q_h = \left(1,\ t\left(\overline{\frac{w-a}{tb}}\right)\right) \in \mathbb{H}_t,$$

having its realization,

$$Q_h = [q_h]_t = \begin{pmatrix} 1 & t\left(\overline{\frac{w-a}{tb}}\right) \\ \frac{w-a}{tb} & 1 \end{pmatrix} \in \mathcal{H}_2^t,$$

such that

$$\Sigma_t(h) = Q_h^{-1}[h]_t Q_h \text{ in } \mathcal{H}_2^t. \tag{3.28}$$

Proof. Under the hypothesis, one obtains that

$$Q_h \Sigma_t(h) = [h]_t Q_b \text{ in } \mathcal{H}_2^t,$$

by (3.27). By the invertibility of Q_h, we have

$$\Sigma_t(h) = Q_h^{-1}[h]_t Q_h \text{ in } \mathcal{H}_2^t,$$

implying the relation (3.28). □

The above lemma shows that if a scale t is negative in $\mathbb{R}$, then the realization $[h]_t$ and the t-spectral form $\Sigma_t(h)$ are similar in $\mathcal{H}_2^t$, whenever $h = (a, b) \in \mathbb{H}_t$ satisfies $b \neq 0$ in $\mathbb{C}$.

Theorem 3.22. *If $t < 0$ in $\mathbb{R}$, then every hypercomplex number $h \in \mathbb{H}_t$ is similar to its t-spectral value $(\sigma_t(h), 0) \in \mathbb{H}_t$, in the sense that:*

$$[h]_t \text{ and } \Sigma_t(h) \text{ are similar in } \mathcal{H}_2^t. \tag{3.29}$$

Proof. Let $h = (a, b) \in \mathbb{H}_t$, for $t < 0$. If $b = 0$ in $\mathbb{C}$, then $[(a, 0)]_t$ and $\Sigma_t((a, 0))$ are similar in $\mathcal{H}_2^t$, by the above lemma. Indeed, if $b = 0$, then these matrices are identically same in $\mathcal{H}_2^t$. Meanwhile, if $b \neq 0$ in $\mathbb{C}$, then $[h]_t$ and $\Sigma_t(h)$ are similar in $\mathcal{H}_2^t$ by Lemma 3.20. In particular, if $b \neq 0$, then there exists

$$q_h = \left(1, \overline{\frac{w-a}{tb}}\right) \in \mathbb{H}_t,$$

such that

$$\Sigma_t(h) = [q_h]_t^{-1} [h]_t [q_h]_t,$$

in $\mathcal{H}_2^t$, by (3.28). Therefore, if $t < 0$, then $[h]_t$ and $\Sigma_t(h)$ are similar in $\mathcal{H}_2^t$, equivalently, two hypercomplex numbers h and $(\sigma_t(h), 0)$ are similar in $\mathbb{H}_t$, for all $h \in \mathbb{H}_t$. □

The above theorem guarantees that the negative-scale condition on hypercomplex numbers implies the similarity of the realizations and the scaled-spectral forms of them, just like the quaternionic case (whose scale is -1), shown in [2] and [3].

Theorem 3.23. *If $t < 0$ in $\mathbb{R}$, then the t-spectral relation on $\mathbb{H}_t$ and the similarity on $\mathbb{H}_t$ are same as equivalence relations on $\mathbb{H}_t$, i.e.,*

$$t < 0 \Longrightarrow t\text{-spectral relation} \overset{\text{equi}}{=} \text{similarity on } \mathbb{H}_t, \tag{3.30}$$

where "$\overset{\text{equi}}{=}$" means "being equivalent to, as equivalence relations".

Proof. Suppose a negative scale $t < 0$ is fixed, and let $\mathbb{H}_t$ be the corresponding t-scaled hypercomplex ring. Assume that two hypercomplex numbers h_1 and h_2 are t-spectral related. Then their t-spectral values are identical in $\mathbb{C}$, i.e.,

$$\sigma_t(h_1) = \sigma_t(h_2) \overset{\text{let}}{=} w \text{ in } \mathbb{C}.$$

Thus the realizations $[h_1]_t$ and $[h_2]_t$ are similar to

$$\Sigma_t(h_1) = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} = \Sigma_t(h_2) \overset{\text{let}}{=} W,$$

in $\mathcal{H}_2^t$, by (3.29), i.e., there exist $q_1, q_2 \in \mathbb{H}_t$ such that

$$[q_1]_t^{-1} [h_1]_t [q_1]_t = W = [q_2]_t^{-1} [h_2]_t [q_2]_t,$$

in $\mathcal{H}_2^t$. So, one obtains that

$$[h_1]_t = \left([q_1]_t [q_2]_t^{-1}\right) [h_2]_t \left([q_2]_t [q_1]_t^{-1}\right)$$

if and only if

$$[h_1]_t = \left([q_2]_t [q_1]_t^{-1}\right)^{-1} [h_2]_t \left([q_2]_t [q_1]_t^{-1}\right),$$

in $\mathcal{H}_2^t$, implying that $[h_1]_t$ and $[h_2]_t$ are similar in $\mathcal{H}_2^t$. Thus, if h_1 and h_2 are t-spectral related, then they are similar in $\mathbb{H}_t$.

Conversely, suppose $T_1 \overset{\text{denote}}{=} [h_1]_t$ and $T_2 \overset{\text{denote}}{=} [h_2]_t$ are similar in $\mathcal{H}_2^t$. Then there exists $U \in \mathcal{H}_2^t$, such that

$$T_1 = U^{-1} T_2 U \text{ in } \mathcal{H}_2^t.$$

Since the realizations T_l and the corresponding t-spectral forms $S_l \overset{\text{denote}}{=} \Sigma_t(h_l)$ are similar by (3.29), say,

$$T_l = V_l^{-1} S_l V_l \text{ in } \mathcal{H}_2^t, \text{ for some } V_l \in \mathcal{H}_2^t,$$

for all $l = 1, 2$. Thus,

$$\begin{aligned} & T_1 = U^{-1} T_2 U = U^{-1} \left(V_2^{-1} S_2 V_2\right) U \\ & \Longleftrightarrow V_1 S_1 V_1^{-1} = T_1 = (V_2 U)^{-1} S_2 (V_2 U) \\ & \Longleftrightarrow S_1 = V_1^{-1} (V_2 U)^{-1} S_2 (V_2 U) V_1 \\ & \Longleftrightarrow S_1 = (V_2 U V_1)^{-1} S_2 (V_2 U V_1), \end{aligned}$$

and hence, two matrices S_1 and S_2 are similar in $\mathcal{H}_2^t$. It means that S_1 and S_2 share the same eigenvalues. So, it ie either

$$S_1 = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} = S_2,$$

for some $w \in \mathbb{C}$, or

$$S_1 = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix}, \text{ and } S_2 = \begin{pmatrix} \overline{w} & 0 \\ 0 & w \end{pmatrix},$$

in $\mathcal{H}_2^t$. However, by the assumption that $t < 0$, we have

$$S_1 = S_2 \text{ in } \mathcal{H}_2^t,$$

by Corollary 3.2(iii). It shows that, if the realizations T_1 and T_2 are similar, then the t-spectral forms S_1 and S_2 are identically same in $\mathcal{H}_2^t$, implying that

$$\sigma_t(h_1) = \sigma_t(h_2) \text{ in } \mathbb{C},$$

thus h_1 and h_2 are t-spectral related in $\mathbb{H}_t$.

Therefore, the equivalence (3.30) between the t-spectral relation and the similarity on $\mathbb{H}_t$ holds, whenever $t < 0$ in $\mathbb{R}$. □

The above theorem generalizes the equivalence between the quaternion-spectral relation, which is the (-1)-spectral relation, and the similarity on the quaternions $\mathbb{H}_{-1} = \mathbb{H}$ (e.g., [2] and [3]).

How about the cases where given scale t are nonnegative in $\mathbb{R}$, i.e., $t \geq 0$? One may need to consider the decomposition (3.16),

$$\begin{aligned} \mathbb{H}_t &= \left(\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{+}\right) \sqcup \left(\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{-0}\right) \\ &= (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{+}) \sqcup (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{-0}), \end{aligned}$$

of $\mathbb{H}_t$, for $t \geq 0$, where

$$\begin{aligned} \mathbb{H}_t^{inv} &= \left\{(a,b) : |a|^2 \neq t|b|^2\right\}, \\ \mathbb{H}_t^{sing} &= \left\{(a,b) : |a|^2 = t|b|^2\right\}, \\ \mathbb{H}_t^{+} &= \left\{(a,b) : \operatorname{Im}(a)^2 > t|b|^2\right\}, \end{aligned}$$

and

$$\mathbb{H}_t^{-0} = \left\{(a,b) : \operatorname{Im}(a)^2 \leq t|b|^2\right\},$$

block-by-block. In particular, if

$$h \in \mathbb{H}_t^{inv} \cap \mathbb{H}_t^{+},$$

then it "seems" that the realization $[h]_t$ and the t-spectral form $\Sigma_t(h)$ are similar in $\mathcal{H}_2^t$. The proof "may" be similar to the above proofs for negative scales. We leave this problem for a future project.

3.4. THE t-SPECTRAL MAPPING THEOREM

In this section, we let a scale t be arbitrary in $\mathbb{R}$, and let $\mathbb{H}_t$ be the t-scaled hypercomplex ring. Let $h = (a,b) \in \mathbb{H}_t$ satisfy the condition (3.4), and suppose it has its t-spectral value,

$$\sigma_t(h) = x + i\sqrt{y^2 - tu^2 - tv^2} \overset{\text{let}}{=} w,$$

and hence, its t-spectral form

$$\Sigma_t(h) = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \text{ in } \mathcal{H}_2^t$$

(see Remark 3.14).

Now recall that if $n \in \mathbb{N}$, and $A \in M_n(\mathbb{C})$, and if

$$f \in \mathbb{C}[z] \overset{\text{def}}{=} \left\{ g : g = \sum_{k=0}^{m} z_k z^k, \text{ with } z_1, \ldots, z_m \in \mathbb{C}, \text{ for } m \in \mathbb{N} \right\},$$

then

$$\operatorname{spec}(f(A)) = \{f(w) : w \in \operatorname{spec}(A)\}, \tag{3.31}$$

in $\mathbb{C}$, where $\mathbb{C}[z]$ is the polynomial ring in a variable z over $\mathbb{C}$, consisting of all polynomials in z whose coefficients are in $\mathbb{C}$, and

$$f(A) = \sum_{k=0}^{N} s_k A^k, \text{ with } A^0 = I_n,$$

whenever

$$f(z) = \sum_{k=0}^{N} s_k z^k \in \mathbb{C}[z], \text{ with } s_1, \ldots, s_N \in \mathbb{C},$$

where I_n is the identity matrix of $M_n(\mathbb{C})$, by the spectral mapping theorem (e.g., [9] and [8]). By (3.31), if $\mathbb{R}[x]$ is the polynomial ring in a variable x over the real field $\mathbb{R}$, then

$$\operatorname{spec}(g(A)) = \{g(w) : w \in \operatorname{spec}(A)\} \text{ in } \mathbb{C}, \tag{3.32}$$

for all $g \in \mathbb{R}[x]$, because $\mathbb{R}[z]$ is a subring of $\mathbb{C}[z]$ if we identify x to z.

It is shown in [2] and [3] that, for $f \in \mathbb{C}[z]$,

$$\operatorname{spec}\left(f\left([\xi]_{-1}\right)\right) = \left\{ f\left(\sigma_{-1}(\xi)\right), \ f\left(\overline{\sigma_{-1}(\xi)}\right) \right\}$$

in $\mathbb{C}$, by (3.31), but

$$f\left(\overline{\sigma_{-1}(\xi)}\right) \neq \overline{f\left(\sigma_{-1}(\xi)\right)}, \text{ in general},$$

and hence, even though the relation (3.31) holds "on $M_2(\mathbb{C})$, for $[\xi]_{-1} \in \mathcal{H}_2^{-1}$", it does not hold "on $\mathcal{H}_2^{-1}$", in general. It demonstrates that, in a similar manner, the spectral mapping theorem (3.31) holds "on $M_2(\mathbb{C})$," but it does not hold "on the t-scaled realization $\mathcal{H}_2^t$ of $\mathbb{H}_t$", for $t \in \mathbb{R}$, because the spectra of hypercomplex numbers satisfy

$$\operatorname{spec}\left([\eta]_t\right) = \{w, \overline{w}\}, \text{ with } w = \sigma_t(\eta),$$

by (3.3), for all $\eta \in \mathbb{H}_t$ in the sense of Remark 3.14, just like the quaternionic case of [2] and [3].

However, in [2] and [3], it is proven that, for all $g \in \mathbb{R}[x]$, one has

$$\mathrm{spec}\left(g\left([\xi]_{-1}\right)\right) = \left\{g\left(\sigma_t(\xi)\right), \overline{g\left(\sigma_t(\xi)\right)}\right\},$$

in $\mathbb{C}$, by (3.32), since

$$g \in \mathbb{R}[x] \Longrightarrow g\left(\overline{w}\right) = \overline{g\left(w\right)}, \ \forall w \in \mathbb{C}.$$

It means that the "restricted" spectral mapping theorem of (3.32) holds "on the realization $\mathcal{H}_2^{-1}$ of the quaternions $\mathbb{H}_{-1}$". Similarly, we obtain the following result.

Theorem 3.24. *Let $\xi \in \mathbb{H}_t$, realized to be $[\xi]_t \in \mathcal{H}_2^t$. Then, for any $g \in \mathbb{R}[x]$,*

$$\mathrm{spec}\left(g\left([\xi]_t\right)\right) = \left\{g\left(\sigma_t(\xi)\right), \overline{g\left(\sigma_t(\xi)\right)}\right\},$$

i.e.,

$$\mathrm{spec}\left(g\left([\xi]_t\right)\right) = \left\{g\left(w\right) : w \in \mathrm{spec}\left([\xi]_t\right)\right\} \ \text{in}\ \mathbb{C}, \ \forall t \in \mathbb{R}. \tag{3.33}$$

Proof. By (3.3) and (3.18), if $\xi \in \mathbb{H}_t$, then

$$\mathrm{spec}\left([\xi]_t\right) = \left\{w, \overline{w}\right\}, \ \text{with } w = \sigma_t(\xi),$$

in $\mathbb{C}$ (under the symbolic understanding of Remark 3.14). For any $g = \sum_{k=1}^{N} s_k x^k \in \mathbb{R}[x]$, with $s_1, \ldots, s_N \in \mathbb{R}$, and $N \in \mathbb{N}$, one has that

$$g\left(\overline{w}\right) = \sum_{k-1}^{N} s_k \overline{w}^k = \sum_{k=1}^{N} \overline{s_k w^k} = \overline{\sum_{k=1}^{N} s_k w^k} = \overline{g\left(w\right)}, \tag{3.34}$$

in $\mathbb{C}$. It implies that

$$\mathrm{spec}\left(g\left([\xi]_t\right)\right) = \left\{g\left(w\right), g\left(\overline{w}\right)\right\} = \left\{g\left(w\right), \overline{g\left(w\right)}\right\},$$

in $\mathbb{C}$, by (3.32) and (3.34). Therefore, the relation (3.33) holds true. □

One may call the relation (3.33), the hypercomplex-spectral mapping theorem, since it holds for all scales $t \in \mathbb{R}$.

4. THE USUAL ADJOINT ON $\mathcal{H}_2^t$ IN $M_2(\mathbb{C})$

In this section, we consider how the usual adjoint on $M_2(\mathbb{C}) = B\left(\mathbb{C}^2\right)$ acts on the t-scaled realization $\mathcal{H}_2^t$ of the t-scaled hypercomplex numbers. Throughout this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding t-scaled hypercomplex ring $\mathbb{H}_t$ realized to be $\mathcal{H}_2^t$ in $M_2(\mathbb{C})$ under the representation $\Pi_t = \left(\mathbb{C}^2, \pi_t\right)$. Recall that every Hilbert-space operator T acting on a Hilbert space H has its unique adjoint T^* on H.

Especially, if $T \in M_n(\mathbb{C}) = B(\mathbb{C}^n)$, for $n \in \mathbb{N}$, is a matrix which is an operator on $\mathbb{C}^n$, then its adjoint T^* is determined to be the conjugate-transpose of T in $M_n(\mathbb{C})$. For instance,

$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C}) \iff T^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} \in M_2(\mathbb{C}).$$

It says that, if we understand our t-scaled realization $\mathcal{H}_2^t$ as a sub-structure of $M_2(\mathbb{C})$, then each hypercomplex number $(a,b) \in \mathbb{H}_t$ assigns a unique adjoint $[(a,b)]_t^*$ of the realization $[(a,b)]_t$ "in $M_2(\mathbb{C})$".

Let $(a,b) \in \mathbb{H}_t$ realized to be

$$[(a,b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t.$$

Then, as a matrix of $M_2(\mathbb{C})$, this realization has its adjoint,

$$[(a,b)]_t^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

It shows that the usual adjoint (conjugate-transpose) of $[(a,b)]_t$ is not contained "in $\mathcal{H}_2^t$", in general. In particular, if

$$t^2 \neq 1 \iff \text{either } t \neq 1 \text{ or } t \neq -1, \text{ in } \mathbb{R},$$

then

$$[(a,b)]_t \notin \mathcal{H}_2^t \text{ in general.}$$

Theorem 4.1. *The scale $t \in \mathbb{R}$ satisfies that $t^2 = 1$ in $\mathbb{R}$, if and only if the adjoint of every realization of a hypercomplex number $\mathbb{H}_t$ is contained in $\mathcal{H}_2^t$, i.e.,*

$$\textit{either } t = 1, \textit{ or } t = -1 \iff [\xi]_t^* \in \mathcal{H}_2^t, \quad \forall \xi \in \mathbb{H}_t. \tag{4.1}$$

Proof. For an arbitrary scale $t \in \mathbb{R}$, if $(a,b) \in \mathbb{H}_t$, then

$$[(a,b)]_t^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

($\Rightarrow$) Assume that either $t = 1$, or $t = -1$, equivalently, suppose $t^2 = 1$ in $\mathbb{R}$. Then

$$[(a,b)]_t^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} = \begin{pmatrix} \overline{a} & t\left(\frac{b}{t}\right) \\ t^2\overline{\left(\frac{b}{t}\right)} & a \end{pmatrix} = \begin{pmatrix} \overline{a} & t\left(\frac{b}{t}\right) \\ \overline{\left(\frac{b}{t}\right)} & a \end{pmatrix},$$

contained in $\mathcal{H}_2^t$. So, if either $t = 1$, or $t = -1$, then $[(a,b)]_t^* \in \mathcal{H}_2^t$, for all $(a,b) \in \mathbb{H}_t$. Moreover, in such a case,

$$[(a,b)]_t^* = \left[\left(\overline{a}, \frac{b}{t}\right)\right]_t \text{ in } \mathcal{H}_2^t. \tag{4.2}$$

($\Leftarrow$) Assume now that $t^2 \neq 1$ in $\mathbb{R}$. Then the adjoint $[(a,b)]_t^*$ of $[(a,b)]_t$ is identical to the matrix,

$$[(a,b)]_t^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

which "can" be

$$\begin{pmatrix} \overline{a} & t\left(\frac{b}{t}\right) \\ t^2\left(\overline{\frac{b}{t}}\right) & a \end{pmatrix} \text{ in } \mathcal{H}_2^t.$$

However, by the assumption that $t^2 \neq 1$, the adjoint $[(a,b)]_t^*$ is not contained in $\mathcal{H}_2^t$, in general. In particular, if $b \neq 0$ in $\mathbb{C}$, then the adjoint $[(a,b)]_t^* \notin \mathcal{H}_2^t$ in $M_2(\mathbb{C})$, i.e.,

$$t^2 \neq 1 \text{ and } b \neq 0 \text{ in } \mathbb{C} \implies [(a,b)]_t^* \in \left(M_2(\mathbb{C}) \setminus \mathcal{H}_2^t\right). \tag{4.3}$$

Therefore, the characterization (4.1) holds by (4.2) and (4.3). $\square$

Note that, if $t = -1$, then $\mathbb{H}_{-1}$ is the quaternions; and if $t = 1$, then $\mathbb{H}_1$ is the bicomplex numbers. The above theorem shows that, only when the scaled hypercomplex ring $\mathbb{H}_t$ is either the quaternions $\mathbb{H}_{-1}$, or the bicomplex numbers $\mathbb{H}_1$, the usual adjoint $(*)$ is closed on $\mathcal{H}_2^t$, as a well-defined unary operation, by (4.1).

5. FREE PROBABILITY ON $\mathbb{H}_t$

In this section, we establish a universal free-probabilistic model on our t-scaled hypercomplex ring $\mathbb{H}_t$, for "every" scale $t \in \mathbb{R}$. First, recall that, on $M_2(\mathbb{C})$, we have the usual trace tr, defined by

$$tr\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = a_{11} + a_{22},$$

for all $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$; and the normalized trace τ,

$$\tau = \frac{1}{2}\text{tr} \text{ on } M_2(\mathbb{C}).$$

i.e., we have two typical free-probabilistic models,

$$(M_2(\mathbb{C}),\, tr) \text{ and } (M_2(\mathbb{C}),\, \tau).$$

5.1. FREE PROBABILITY

For more about free probability theory, see e.g., [20] and [22]. Let A be an noncommutative algebra over $\mathbb{C}$, and $\varphi : A \to \mathbb{C}$, a linear functional on A. Then the pair (A, φ) is called a (noncommutative) free probability space. By definition, free probability spaces are the noncommutative version of classic measure spaces (X, μ) consisting of a set X and a measure μ on the σ-algebra of X. As in measure theory, the (noncommutative) free probability on (A, φ) is dictated by the linear functional φ. Meanwhile, if (A, φ) is unital in the sense that (i) the unity 1_A of A exists, and (ii) $\varphi(1_A) = 1$, then it is called a unital free probability space. These unital free probability spaces are the noncommutative analogue of classical probability spaces (Y, ρ) where the given measures ρ are the probability measures satisfying $\rho(Y) = 1$.

If A is a topological algebra, and if φ is bounded (and hence, continuous under linearity), then the corresponding free probability space (A, φ) is said to be a topological free probability space. Similarly, if A is a topological $*$-algebra equipped with the adjoint $(*)$, then the topological free probability space (A, φ) is said to be a topological (free) $*$-probability space. More in detail, if A is a C^*-algebra, or a von Neumann algebra, or a Banach $*$-algebra, we call (A, φ), a C^*-probability space, respectively, a W^*-probability space, respectively, a Banach $*$-probability space, etc. For our main purposes, we focus on C^*-probability spaces from below.

If (A, φ) is a C^*-probability space, and $a \in A$, then the algebra-element a is said to be a free random variable of (A, φ). For any arbitrarily fixed free random variables $a_1, \ldots, a_s \in (A, \varphi)$ for $s \in \mathbb{N}$, one can get the corresponding free distribution of $a_1, \ldots, a_s$, characterized by the joint free moments,

$$\varphi\left(\prod_{l=1}^{n} a_{i_l}^{r_i}\right) = \varphi\left(a_{i_1}^{r_1} a_{i_2}^{r_2} \ldots a_{i_n}^{r_n}\right),$$

for all $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$ and $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where a_l^* are the adjoints of a_l, for all $l = 1, \ldots, s$. For instance, if $a \in (A, \varphi)$ is a free random variable, then the free distribution of a is fully characterized by the joint free moments of $\{a, a^*\}$,

$$\varphi\left(\prod_{l=1}^{n} a^{r_l}\right) = \varphi\left(a^{r_1} a^{r_2} \ldots a^{r_n}\right),$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$ (e.g., [20] and [22]). So, if a free random variable $a \in (A, \varphi)$ is self-adjoint in the sense that: $a^* = a$ in A, then the free distribution of a is determined by the free-moment sequence,

$$(\varphi(a^n))_{n=1}^{\infty} = \left(\varphi(a), \varphi\left(a^2\right), \varphi\left(a^3\right), \ldots\right)$$

(e.g., [20] and [22]).

5.2. FREE-PROBABILISTIC MODELS INDUCED BY $\mathbb{H}_t$

By identifying the t-scaled hypercomplex ring $\mathbb{H}_t$ and its realization $\mathcal{H}_2^t$ as the same ring, we identify the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ and its realization $\mathcal{H}_2^{t\times}$ as the same monoid. As a subset in $M_2(\mathbb{C})$, we define a subset,

$$\mathcal{H}_2^{t\times}(*) \stackrel{\text{def}}{=} \left\{[\xi]_t^* \in M_2(\mathbb{C}) : \xi \in \mathbb{H}_t^{\times}\right\},$$

i.e.,

$$\mathcal{H}_2^{t\times}(*) = \left\{\begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t^{\times}\right\}, \tag{5.1}$$

by the subset of all adjoints of realizations in $\mathcal{H}_2^{\times t}$. Indeed,

$$[(a,b)]_t^* = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

As we have seen in Section 4, the adjoint is not closed on $\mathcal{H}_2^t$ in general, and hence,

$$\mathcal{H}_2^{t\times}(*) \neq \mathcal{H}_2^{t\times} \text{ in } M_2(\mathbb{C}),$$

in general. In particular, the scale t satisfies $t^2 \neq 1$ in $\mathbb{R}$, if and only if the above non-equality holds in $M_2(\mathbb{C})$, by (4.1). Now, let

$$\mathcal{H}_2^{t\times}(1,*) \overset{\text{denote}}{=} \mathcal{H}_2^{t\times} \cup \mathcal{H}_2^{t\times}(*),$$

i.e.,

$$\mathcal{H}_2^{t\times}(1,*) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}, \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} : (a,b) \in \mathbb{H}_t^\times \right\}, \tag{5.2}$$

in $M_2(\mathbb{C})$, set-theoretically. By (4.1), (5.1) and (5.2),

$$\mathcal{H}_2^{t\times}(1,*) \supsetneq \mathcal{H}_2^{t\times} \text{ in } M_2(\mathbb{C}), \text{ in general.}$$

Define now the C^*-algebra $\mathfrak{H}_2^t$ by the C^*-subalgebra of $M_2(\mathbb{C})$ generated by the set $\mathcal{H}_2^{t\times}(1,*)$ of (5.2), i.e.,

$$\mathfrak{H}_2^t \overset{\text{denote}}{=} C^*\left(\mathcal{H}_2^{t\times}\right) \overset{\text{def}}{=} \overline{\mathbb{C}\left[\mathcal{H}_2^{t\times}(1,*)\right]}, \tag{5.3}$$

in $M_2(\mathbb{C})$, where $C^*(Z)$ means the C^*-subalgebra of $B\left(\mathbb{C}^2\right)$ generated by the subset Z and their adjoints, and $\mathbb{C}[X]$ is the (pure-algebraic) algebra (over $\mathbb{C}$) generated by a subset X of $M_2(\mathbb{C})$, and $\overline{Y}$ means the operator-norm-topology closure of a subset Y of the operator algebra $M_2(\mathbb{C}) = B\left(\mathbb{C}^2\right)$, which is a C^*-algebra over $\mathbb{C}$.

Definition 5.1. The C^*-algebra $\mathfrak{H}_2^t$ of (5.3), generated by the t-scaled hypercomplex monoid $\mathbb{H}_t^\times \overset{\text{monoid}}{=} \mathcal{H}_2^{t\times}$, is called the t-scaled(-hypercomplex)-monoidal C^*-algebra of $\mathbb{H}_t^\times$ (or, of $\mathbb{H}_t$).

Clearly, by the definition (5.3), the t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$ is well-determined in $M_2(\mathbb{C})$. So, the usual trace tr and the normalized trace τ on $M_2(\mathbb{C})$ are well-defined on $\mathfrak{H}_2^t$, i.e., we have two trivial free-probabilistic models of $\mathfrak{H}_2^t$,

$$\left(\mathfrak{H}_2^t, \text{ tr}\right) \text{ and } \left(\mathfrak{H}_2^t, \tau\right),$$

as C^*-probability spaces (e.g., see Section 5.1). Note that such free-probabilistic structures are independent from the choice of the scales $t \in \mathbb{R}$.

Observe that, if $\begin{pmatrix} \overline{a_l} & b_l \\ t\overline{b_l} & a_l \end{pmatrix} \in \mathcal{H}_2^{t\times}(*)$ in $\mathfrak{H}_2^t$, for $l = 1, 2$, then

$$\begin{pmatrix} \overline{a_1} & b_1 \\ t\overline{b_1} & a_1 \end{pmatrix} \begin{pmatrix} \overline{a_2} & b_2 \\ t\overline{b_2} & a_2 \end{pmatrix} = \begin{pmatrix} \overline{a_1 a_2} + tb_1\overline{b_2} & \overline{a_1}b_2 + b_1 a_2 \\ t\left(\overline{b_1 a_2} + a_1\overline{b_2}\right) & t\overline{b_1}b_2 + a_1 a_2 \end{pmatrix},$$

identifying to be

$$\begin{pmatrix} \overline{a_1 a_2 + t\overline{b_1} b_2} & b_1 a_2 + \overline{a_1} b_2 \\ t\left(\overline{b_1 a_2 + \overline{a_1} b_2}\right) & a_1 a_2 + t\overline{b_1} b_2 \end{pmatrix} \text{ in } \mathfrak{H}_2^t. \tag{5.4}$$

Therefore,

$$\begin{pmatrix} \overline{a_1} & b_1 \\ t\overline{b_1} & a_1 \end{pmatrix} \begin{pmatrix} \overline{a_2} & b_2 \\ t\overline{b_2} & a_2 \end{pmatrix} \in \mathcal{H}_2^{t\times}(*), \text{ too.}$$

i.e., the matricial multiplication is closed on the set $\mathcal{H}_2^{t\times}(*)$ of (5.2), by (5.4). In fact, under the closed-ness (5.4), the algebraic pair,

$$\mathcal{H}_2^{t\times}(*) \overset{\text{denote}}{=} \left(\mathcal{H}_2^{t\times}(*),\ \cdot\right),$$

forms a monoid with its identity I_2. So, the generating set $\mathcal{H}_2^{t\times}(1,*)$ of the t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$ is the set-theoretical union of two monoids $\mathcal{H}_2^{t\times}$ and $\mathcal{H}_2^{t\times}(*)$, under the matricial multiplication. Note, however, that the matricial multiplication is not closed on the generating set $\mathcal{H}_2^{t\times}(1,*)$ of (5.2). Indeed, if

$$\begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} \in \mathcal{H}_2^{t\times}, \begin{pmatrix} \overline{a_2} & b_2 \\ t\overline{b_2} & a_2 \end{pmatrix} \in \mathcal{H}_2^{t\times}(*)$$

in $\mathfrak{H}_2^t$, then

$$\begin{aligned} \begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} \begin{pmatrix} \overline{a_2} & b_2 \\ t\overline{b_2} & a_2 \end{pmatrix} &= \begin{pmatrix} a_1\overline{a_2} + t^2 b_1 \overline{b_2} & a_1 b_2 + t a_2 b_1 \\ \overline{a_2 b_1} + t\overline{a_1 b_2} & \overline{b_1} b_2 + \overline{a_1} a_2 \end{pmatrix}, \\ \begin{pmatrix} \overline{a_2} & b_2 \\ t\overline{b_2} & a_2 \end{pmatrix} \begin{pmatrix} a_1 & tb_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} &= \begin{pmatrix} a_1\overline{a_2} + \overline{b_1} b_2 & t b_1 \overline{a_2} + \overline{a_1} b_2 \\ t a_1 \overline{b_2} + \overline{b_1} a_2 & t^2 b_1 \overline{b_2} + \overline{a_1} a_2 \end{pmatrix}, \end{aligned} \tag{5.5}$$

in $\mathfrak{H}_2^t$. However, the resulted products of (5.5), contained in $\mathfrak{H}_2^t$, are not contained in $\mathcal{H}_2^{t\times}(1,*)$, in general.

Observation 5.2. *By* (5.4) *and* (5.5), *one can realize that:*

(i) *if* $A, B \in \mathcal{H}_2^{t\times}$, *then* $AB \in \mathcal{H}_2^{t\times}$,
(ii) *if* $C, D \in \mathcal{H}_2^{t\times}(*)$, *then* $CD \in \mathcal{H}_2^{t\times}(*)$,
(iii) *if* $T, S \in \mathcal{H}_2^{t\times}(1,*)$, *then* $TS \notin \mathcal{H}_2^{t\times}(1,*)$, *in general, as elements of the t-scaled-monoidal C^*-algebra* $\mathfrak{H}_2^t$.

Even though the non-closed rule (iii) *is satisfied "on* $\mathcal{H}_2^t(1,*)$*", at least, we have a multiplication rule* (5.5) *"in the C^*-algebra* $\mathfrak{H}_2^t$*".*

Assume that $[(a,b)]_t \in \mathcal{H}_2^{t\times}$ in $\mathfrak{H}_2^t$. Then

$$\operatorname{tr}\left([(a,b)]_t\right) = a + \overline{a} = 2\operatorname{Re}(a),$$

and

$$\tau\left([(a,b)]_t\right) = \frac{1}{2}\operatorname{tr}\left([(a,b)]_t\right) = \operatorname{Re}(a), \tag{5.6}$$

where $\mathrm{Re}\,(a)$ is the real part of a in $\mathbb{C}$. Similarly, if $[(a,b)]_t^* \in \mathcal{H}_2^{t\times}(*)$ in $\mathfrak{H}_2^t$, then we have

$$\mathrm{tr}\,\left([(a,b)]_t^*\right) = \mathrm{tr}\,\begin{pmatrix}\overline{a} & b\\ t\overline{b} & a\end{pmatrix} = \overline{a} + a = 2\mathrm{Re}\,(a),$$

and

$$\tau\left([(a,b)]_t^*\right) = \frac{1}{2}\left(2\mathrm{Re}\,(a)\right) = \mathrm{Re}\,(a). \tag{5.7}$$

Remark that, since tr and τ are well-defined linear functional on the C^*-algebra $\mathfrak{H}_2^t$, they satisfy

$$\mathrm{tr}\,(T^*) = \overline{\mathrm{tr}\,(T)}, \text{ and } \tau\,(T^*) = \overline{\tau\,(T)},$$

for all $T \in \mathfrak{H}_2^t$. So, the relation (5.7) is well-verified, too.

Also, if $[(a_1,b_1)]_t$, $[(a_2,b_2)]_t^* \in \mathcal{H}_2^{t\times}(1,*)$ in $\mathfrak{H}_2^t$, then

$$\mathrm{tr}\,\left([(a_1,b_1)]_t\,[(a_2,b_2)]_t^*\right) = \mathrm{tr}\,\left(\begin{pmatrix}a_1\overline{a_2} + t^2 b_1\overline{b_2} & a_1 b_2 + t a_2 b_1\\ \overline{a_2 b_1 + t a_1 b_2} & \overline{b_1}b_2 + \overline{a_1}a_2\end{pmatrix}\right)$$

by (5.5)

$$\begin{aligned} &= a_1\overline{a_2} + t^2 b_1\overline{b_2} + \overline{b_1}b_2 + \overline{a_1}a_2\\ &= 2\mathrm{Re}\,\left(a_1\overline{a_2}\right) + t^2 b_1\overline{b_2} + \overline{b_1}b_2,\end{aligned}$$

and similarly,

$$\mathrm{tr}\,\left([(a_1,b_1)]_t^*\,[(a_2,b_2)]_t\right) = 2\mathrm{Re}\,\left(\overline{a_1}a_2\right) + t^2\overline{b_1}b_2 + b_1\overline{b_2}, \tag{5.8}$$

and hence,

$$\tau\left([(a_1,b_1)]_t\,[(a_2,b_2)]_t^*\right) = \mathrm{Re}\,\left(a_1\overline{a_2}\right) + \frac{t^2 b_1\overline{b_2} + \overline{b_1}b_2}{2},$$

and

$$\tau\left([(a_1,b_1)]_t^*\,[(a_2,b_2)]_t\right) = \mathrm{Re}\,\left(\overline{a_1}a_2\right) + \frac{t^2\overline{b_1}b_2 + b_1\overline{b_2}}{2}, \tag{5.9}$$

by (5.8).

Proposition 5.3. *Let $(a,b),(a_l,b_l) \in \mathbb{H}_t$, for $l = 1,2$, and let $A = [(a,b)]_t$ and $A_l = [(a_l,b_l)]_t$ be the corresponding realizations of $\mathcal{H}_2^t$, regarded as elements of the t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$. Then*

$$\tau\,(A) = \frac{1}{2}\mathrm{tr}\,(A) = \mathrm{Re}\,(a) = \frac{1}{2}\mathrm{tr}\,(A^*) = \tau\,(A^*)\,,$$

and

$$\tau\,(A_1A_2^*) = \frac{1}{2}\mathrm{tr}\,(A_1A_2^*) = \mathrm{Re}\,\left(a_1\overline{a_2}\right) + \frac{t^2 b_1\overline{b_2} + \overline{b_1}b_2}{2}, \tag{5.10}$$

and

$$\tau\,(A_1^*A_2) = \frac{1}{2}\mathrm{tr}\,(A_1^*A_2) = \mathrm{Re}\,\left(\overline{a_1}a_2\right) + \frac{t^2\overline{b_1}b_2 + b_1\overline{b_2}}{2}.$$

Proof. The joint free moments in (5.10) are proven by (5.6), (5.7), (5.8) and (5.9). □

The above computations in (5.10) provide a general way to compute free-distributional data, in particular, the joint free moments of matrices in the t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$, up to the trace tr, and up to the normalized trace τ. And, they demonstrate that computing such free-distributional data is not easy. So, we will restrict our interests to a certain specific case.

5.3. FREE PROBABILITY ON $(\mathfrak{H}_2^t, \mathrm{tr})$

In this section, we fix a scale $t \in \mathbb{R}$, and the corresponding t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$ generated by the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$. Let $(\mathfrak{H}_2^t, \mathrm{tr})$ be the C^*-probability space with respect to the usual trace tr on $\mathfrak{H}_2^t$.

Recall that if a scale t is negative, then the realization $[\xi]_t$ and the t-spectral form $\Sigma_t(\xi)$ are similar "in $\mathcal{H}_2^t$" by (3.29), for all $\xi \in \mathbb{H}_t$. It implies that the similarity "on $\mathcal{H}_2^t$" is equivalent to the t-spectral relation on $\mathbb{H}_t$ by (3.30). Also, recall that if two matrices A and B are similar in $M_n(\mathbb{C})$, for any $n \in \mathbb{N}$,

$$\mathrm{tr}\ (A) = \mathrm{tr}\ (B)\,.$$

So, if the realization $[\xi]_t$ and the t-spectral form $\Sigma_t(\xi)$ are similar in $\mathcal{H}_2^t$, then the free-moment computations would be much simpler than the computations of (5.10). Note again that if $(a, b) \in \mathbb{H}_t$ satisfies the condition (3.4), then

$$\mathrm{tr}\ ([(a,b)]_t) = 2\mathrm{Re}\,(a) = 2x = \left(x + i\sqrt{R}\right) + \left(x - i\sqrt{R}\right) = \mathrm{tr}\ (\Sigma_t(a,b))\,,$$

where

$$R = y^2 - tu^2 - tv^2 \text{ in } \mathbb{R}, \tag{5.11}$$

in the sense of Remark 3.14. Even though the identical results hold in (5.11) (without similarity), if $[(a,b)]_t$ and $\Sigma_t(a,b)$ are not similar in $\mathcal{H}_2^t$, then

$$\mathrm{tr}\ ([(a,b)]_t^n) \neq \mathrm{tr}\ ((\Sigma_t(a,b))^n)\,,$$

for some $n \in \mathbb{N}$, by (5.5). It implies that some (joint) free-moments of $[(a,b)]_t$ and those of $\Sigma_t(a,b)$ are not identical, and hence, the free distributions of them are distinct.

Lemma 5.4. *Suppose the realization $[(a,b)]_t$ and the t-spectral form $\Sigma_t(a,b)$ are similar in $\mathcal{H}_2^t$ for $(a,b) \in \mathbb{H}_t$. Then*

$$\mathrm{tr}\ ([(a,b)]_t^n) = 2\mathrm{Re}\ (\sigma_t(a,b)^n) = \mathrm{tr}\ \left(\left([(a,b)]_t^*\right)^n\right) \tag{5.12}$$

for all $n \in \mathbb{N}$, where $\sigma_t(a,b)$ is the t-spectral value of (a,b).

Proof. Suppose $(a,b) \in \mathbb{H}_t$ satisfies the condition (3.4). Then

$$[(a,b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \quad \text{and} \quad \Sigma_t\left((a,b)\right) = \begin{pmatrix} \sigma_t(a,b) & 0 \\ 0 & \overline{\sigma_t(a,b)} \end{pmatrix},$$

in $\mathcal{H}_2^t$, where

$$\sigma_t(a,b) = x + i\sqrt{y^2 - tu^2 - tv^2},$$

in the sense of Remark 3.14. Assume that $[(a,b)]_t$ and $\Sigma_t((a,b))$ are similar in $\mathcal{H}_2^t$. Then the matrices $[(a,b)]_t^n$ and $\Sigma_t((a,b))^n$ are similar in $\mathcal{H}_2^t$, for all $n \in \mathbb{N}$. Indeed, if two elements A and B are similar in $\mathcal{H}_2^t$, satisfying $B = U^{-1}AU$ in $\mathcal{H}_2^t$, for an invertible element $U \in \mathcal{H}_2^t$, then

$$B^n = \left(U^{-1}AU\right)^n = U^{-1}A^nU \text{ in } \mathcal{H}_2^t,$$

implying the similarity of A^n and B^n, for $n \in \mathbb{N}$. Thus,

$$\text{tr}\left([(a,b)]_t^n\right) = \text{tr}\left(\Sigma_t((a,b))^n\right),$$

and

$$\text{tr}\left(\Sigma_t((a,b))^n\right) = \text{tr}\left(\begin{pmatrix} \sigma_t(a,b)^n & 0 \\ 0 & \overline{\sigma_t(a,b)^n} \end{pmatrix}\right),$$

implying that

$$\text{tr}\left([(a,b)]_t^n\right) = \text{tr}\left(\Sigma_t((a,b))^n\right) = 2\text{Re}\left(\sigma_t(a,b)^n\right),$$

for all $n \in \mathbb{N}$. Therefore, the first equality in (5.12) holds.

Since tr is a well-defined linear functional on the C^*-algebra $\mathfrak{H}_2^t$, one has

$$\text{tr}\,(A^*) = \overline{\text{tr}\,(A)}, \text{ for all } A \in \mathfrak{H}_2^t.$$

Since

$$\text{tr}\left(\left([(a,b)]_t^*\right)^n\right) = \text{tr}\left(\left([(a,b)]_t^n\right)^*\right) = \overline{\text{tr}\left([(a,b)]_t^n\right)},$$

one has

$$\text{tr}\left(\left([(a,b)]_t^*\right)^n\right) = \overline{2\text{Re}\left(\sigma_t(a,b)^n\right)} = 2\text{Re}\left(\sigma_t(a,b)^n\right),$$

for all $n \in \mathbb{N}$. So, the second equality in (5.12) holds, too. □

Note that the formula (5.12) holds true under the similarity assumption of the realization and the t-spectral form.

Remark that every complex number $w \in \mathbb{C}$ is polar-decomposed to be

$$w = |w|w_o \text{ with } w_o \in \mathbb{T},$$

uniquely, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in $\mathbb{C}$. So, all our t-spectral values $\sigma_t(\xi)$ are polar-decomposed to be

$$\sigma_t(\xi) = |\sigma_t(\xi)|\,\sigma_t(\xi)_o \text{ with } \sigma_t(\xi)_o \in \mathbb{T},$$

for all $\xi \in \mathbb{H}_t$. In such a sense, we have that

$$tr\left([\xi]_t^n\right) = 2\,|\sigma_t(\xi)|^n\,\text{Re}\left(\sigma_t(\xi)_o^n\right),$$

for all $n \in \mathbb{N}$, by (5.12).

Corollary 5.5. *Suppose the realization $[\xi]_t$ and the t-spectral form $\Sigma_t(\xi)$ are similar in $\mathcal{H}_2^t$ for $\xi \in \mathbb{H}_t$. Then*

$$tr\left([\xi]_t^n\right) = 2\left|\sigma_t(\xi)\right|^n \mathrm{Re}\left(\sigma_t(\xi)_o^n\right) = \mathrm{tr}\left(\left([\xi]_t^*\right)^n\right), \tag{5.13}$$

for all $n \in \mathbb{N}$, where $\sigma_t(\xi) = |\sigma_t(\xi)|\,\sigma_t(\xi)_o$ is the polar decomposition of $\sigma_t(\xi)$, with $\sigma_t(\xi)_o \in \mathbb{T}$.

Proof. The free-distributional data (5.13) is immediately obtained by (5.12) under the polar decomposition of the t-spectral value $\sigma_t(\xi)$ in $\mathbb{C}$. $\square$

Assume again that a hypercomplex number $(a,b) \in \mathbb{H}_t$ satisfies our similarity assumption, i.e., $T \overset{\text{denote}}{=} [(a,b)]_t$ and $S \overset{\text{denote}}{=} \Sigma_t\left((a,b)\right)$ are similar in $\mathcal{H}_2^t$. Then, for any

$$(r_1, \ldots, r_n) \in \{1, *\}^n, \text{ for } n \in \mathbb{N},$$

the matrix $\prod_{l=1}^{n} T^{r_l}$ is similar to $\prod_{l=1}^{n} S^{r_l}$ in $\mathcal{H}_2^t$ (and hence, in $\mathfrak{H}_2^t$).

Theorem 5.6. *Let $(a,b) \in \mathbb{H}_t$ satisfy the similarity assumption that: $T \overset{\text{denote}}{=} [(a,b)]_t$ and $S \overset{\text{denote}}{=} \Sigma_t\left((a,b)\right)$ are similar in $\mathcal{H}_2^t$. If*

$$\sigma_t(a,b) = rw_o, \textit{ polar decomposition},$$

with

$$r = |\sigma_t(a,b)| \textit{ and } w_o \in \mathbb{T}, \tag{5.14}$$

then

$$\mathrm{tr}\left(\prod_{l=1}^{n} T^{r_l}\right) = 2r^n \mathrm{Re}\left(w_o^{\sum_{l=1}^{n} e_l}\right), \tag{5.15}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where

$$e_l = \begin{cases} 1 & \textit{if } r_l = 1, \\ -1 & \textit{if } r_l = *, \end{cases}$$

for all $l = 1, \ldots, n$.

Proof. Since the realization T and the t-spectral form S are assumed to be similar in $\mathcal{H}_2^t$, their adjoints T^* and S^* are similar in $\mathcal{H}_2^{t\times}(*) \cup \{[(0,0)]_t\}$; and hence, the matrix $\prod_{l=1}^{n} T^{r_l}$ and $\prod_{l=1}^{n} S^{r_l}$ are similar "in $\mathfrak{H}_2^t$". Consider that

$$S = \begin{pmatrix} \sigma_t(a,b) & 0 \\ 0 & \overline{\sigma_t(a,b)} \end{pmatrix} = \begin{pmatrix} rw_o & 0 \\ 0 & r\overline{w_o} \end{pmatrix} = r\begin{pmatrix} w_o & 0 \\ 0 & w_o^{-1} \end{pmatrix},$$

under hypotheses, because $\overline{z} = \frac{1}{z} = z^{-1}$ in $\mathbb{T}$, whenever $z \in \mathbb{T}$ in $\mathbb{C}$. It shows that

$$S^j = r^j \begin{pmatrix} w_o^j & 0 \\ 0 & w_o^{-j} \end{pmatrix}, \text{ for all } j \in \mathbb{N} \cup \{0\},$$

and

$$S^{*}=\overline{r}\begin{pmatrix}\overline{w_o} & 0\\ 0 & w_o\end{pmatrix}=r\begin{pmatrix}w_o^{-1} & 0\\ 0 & w_o\end{pmatrix},$$

satisfying that

$$\left(S^{*}\right)^{j}=\left(S^{j}\right)^{*},\ \text{for all } j\in\mathbb{N}.$$

It implies that, for any $(r_1,\ldots,r_n)\in\{1,*\}^n$, for $n\in\mathbb{N}$, there exists $(e_1,\ldots,e_n)\in\{\pm1\}^n$, such that

$$e_l=\begin{cases}1 & \text{if } r_l=1,\\ -1 & \text{if } r_l=*,\end{cases}$$

for all $l=1,\ldots,n$, and

$$\prod_{l=1}^{n}S^{r_l}=r^n\begin{pmatrix}w_o^{\sum\limits_{l=1}^{n}e_l} & 0\\ 0 & w_o^{-\left(\sum\limits_{l=1}^{n}e_l\right)}\end{pmatrix}, \tag{5.16}$$

in $\mathfrak{H}_2^t$. Thus, under our similarity assumption,

$$\mathrm{tr}\left(\prod_{l=1}^{n}T^{r_l}\right)=\mathrm{tr}\left(\prod_{l=1}^{n}S^{r_l}\right)=r^n\left(w_o^{\sum\limits_{l=1}^{n}e_l}+w_o^{-\left(\sum\limits_{l=1}^{n}e_l\right)}\right),$$

implying that

$$\mathrm{tr}\left(\prod_{l=1}^{n}T^{r_l}\right)=r^n\left(2\mathrm{Re}\left(w_o^{\sum\limits_{l=1}^{n}e_l}\right)\right),$$

for all $(r_1,\ldots,r_n)\in\{1,*\}^n$, for all $n\in\mathbb{N}$, where $(e_1,\ldots,e_n)\in\{\pm1\}^n$ satisfies (5.16).

Therefore, under our similarity assumption and the polar decomposition (5.14), the free-distributional data (5.15) holds. $\square$

By the above theorem, one immediately obtain the following result.

Corollary 5.7. *Let $(a,b)\in\mathbb{H}_t$ satisfy the similarity assumption that: $T\overset{\text{denote}}{=}[(a,b)]_t$ and $S\overset{\text{denote}}{=}\Sigma_t\left((a,b)\right)$ are similar in $\mathcal{H}_2^t$. If*

$$\sigma_t(a,b)=rw_o,\ \textit{polar decomposition,}$$

with

$$r=|\sigma_t(a,b)|\ \textit{and}\ w_o\in\mathbb{T}, \tag{5.17}$$

then

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = r^n \mathrm{Re}\left(w_o^{\sum_{l=1}^{n} e_l}\right), \tag{5.18}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, *for all* $n \in \mathbb{N}$, *where*

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$.

Proof. By (5.15), the free-distributional data (5.18) holds up to the normalized trace $\tau = \frac{1}{2}\mathrm{tr}$ on $\mathfrak{H}_2^t$, under (5.17). □

Under our similarity assumption and the condition (5.17), the free-distributional data (5.18) fully characterizes the free distribution of $[(a,b)]_t \in \mathcal{H}_2^t$ in the C^*-probability space $(\mathfrak{H}_2^t, \tau)$.

Corollary 5.8. *Suppose a given scale* t *is negative in* $\mathbb{R}$. *Let* $(a,b) \in \mathbb{H}_t$, *and let* $T \overset{\text{denote}}{=} [(a,b)]_t$ *and* $S \overset{\text{denote}}{=} \Sigma_t((a,b))$ *in* $\mathcal{H}_2^t$. *If*

$$\sigma_t(a,b) = rw_o, \text{ polar decomposition,}$$

with

$$r = |\sigma_t(a,b)| \text{ and } w_o \in \mathbb{T}, \tag{5.19}$$

then

$$\mathrm{tr}\left(\prod_{l=1}^{n} T^{r_l}\right) = 2r^n \mathrm{Re}\left(w_o^{\sum_{l=1}^{n} e_l}\right) = 2\tau\left(\prod_{l=1}^{n} T^{r_l}\right), \tag{5.20}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, *for all* $n \in \mathbb{N}$, *where*

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$.

Proof. In Theorem 5.6 and Corollary 5.7, we showed that if T and S are similar in $\mathcal{H}_2^t$, then the free-distributional data (5.20) holds under the condition (5.19), by (5.15) and (5.18), respectively. So, it suffices to show that the realization T and the t-spectral form S are similar in $\mathcal{H}_2^t$. However, since $t < 0$ in $\mathbb{R}$, the matrices T and S are similar in $\mathcal{H}_2^t$ by (3.29). □

The above corollary shows that, if a given scale t is negative in $\mathbb{R}$, then the free-distributional data (5.20) fully characterizes the free distributions of the realizations $[\xi]_t$ in the t-scaled-monoidal C^*-algebra $\mathfrak{H}_2^t$ up to the usual trace tr, and the normalized trace τ, for "all" $\xi \in \mathbb{H}_t$. In other words, it illustrates that, if $t < 0$ in $\mathbb{R}$, then the free-distributional data on the C^*-probability spaces,

$$\left(\mathfrak{H}_2^t,\ \mathrm{tr}\right) \text{ and } \left(\mathfrak{H}_2^t,\ \tau\right),$$

are fully characterized by the spectra of hypercomplex numbers of $\mathbb{H}_t$, by (5.19) and (5.20).

But, if $t \geq 0$, and hence, there are some hypercomplex numbers η of $\mathbb{H}_t$ whose realization and spectral form are not similar in $\mathcal{H}_2^t$, then computing joint free moments of $[\eta]_t$ in $\mathfrak{H}_2^t$ would not be easy, e.g., see (5.10).

5.4. MORE FREE-DISTRIBUTIONAL DATA ON $(\mathfrak{H}_2^t,\ \tau)$ FOR $t < 0$

In this section, a fixed scale t is automatically assumed to be negative, i.e., $t < 0$ in $\mathbb{R}$. At this moment, we emphasize that most main results of this section would hold even though t is not negative in $\mathbb{R}$. However, we assume a given scale t is negative for convenience (e.g., see (5.20)). Let $\mathfrak{H}_2^t$ be the t-scaled-monoidal C^*-algebra inducing a C^*-probability space $(\mathfrak{H}_2^t, \tau)$, where τ is the normalized trace on $\mathfrak{H}_2^t$. Since t is assumed to be negative in $\mathbb{R}$, the realizations $T = [\eta]_t$ and the t-spectral forms $S = \Sigma_t(\eta)$ are similar in $\mathcal{H}_2^t$ by (3.29), and hence,

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = r^n \mathrm{Re}\left(w_o^{\sum\limits_{l=1}^{n} e_l}\right) = \tau\left(\prod_{l=1}^{n} S^{r_l}\right),$$

by (5.15), where

$$\sigma_t(\eta) = r w_o \in \mathbb{C}, \quad \text{polar decomposition,} \tag{5.21}$$

with $r = |\sigma_t(\eta)|$ and $w_o \in \mathbb{T}$, for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, where $(e_1, \ldots, e_n) \in \{\pm 1\}^n$ satisfies (5.16), for all $n \in \mathbb{N}$, for "all" $\eta \in \mathbb{H}_t$. And the free-distributional data (5.21) fully characterizes the free distribution of $[\eta]_t \in (\mathfrak{H}_2^t, \tau)$, for all $\eta \in \mathbb{H}_t$.

In this section, we refine (5.21) case-by-case, up to operator-theoretic properties of elements of $(\mathfrak{H}_2^t, \tau)$.

Definition 5.9. Let $\mathcal{A}$ be a unital C^*-algebra with its unity $1_{\mathcal{A}}$, and let $T \in \mathcal{A}$, and $T^* \in \mathcal{A}$, the adjoint of T.

(1) T is said to be self-adjoint, if $T^* = T$ in $\mathcal{A}$.
(2) T is a projection, if $T^* = T = T^2$ in $\mathcal{A}$.
(3) T is normal, if $T^*T = TT^*$ in $\mathcal{A}$.
(4) T is a unitary, if $T^*T = 1_{\mathcal{A}} = TT^*$ in $\mathcal{A}$.

Let $(a, b) \in \mathbb{H}_t$, satisfying the condition (3.4), and $T \overset{\text{denote}}{=} [(a, b)]_t \in \mathcal{H}_2^t$, as an element of $(\mathfrak{H}_2^t, \tau)$. Then its adjoint,

$$T^* = \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} \in \mathcal{H}_2^t(*),$$

is well-defined in $(\mathfrak{H}_2^t, \tau)$, and the corresponding t-spectral form,

$$S \overset{\text{denote}}{=} \Sigma_t\left((a,b)\right) = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \in \mathcal{H}_2^t,$$

is contained in $(\mathfrak{H}_2^t, \tau)$, where $\overline{w}$ is determined by Remark 3.14, and

$$w = \sigma_t(a,b) = x + i\sqrt{y^2 - tu^2 - tv^2}$$

is the t-spectral value, uniquely polar-decomposed to be

$$w = rw_o \text{ with } r = |\sigma_t(a,b)| \text{ and } w_o \in \mathbb{T}.$$

For a given hypercomplex number $(a,b) \in \mathbb{H}_t$, let us assume that

> it has its realization denoted by T, its t-spectral form denoted by S, determined by the t-spectral value denoted by w, which is polar-decomposed to be $w = rw_o$, as indicated in the very above paragraph. (5.22)

From now on, if we say that "a given hypercomplex number $(a,b) \in \mathbb{H}_t$ satisfies (5.22)", we understand that the above properties hold.

Let $(a,b) \in \mathbb{H}_t$ satisfy (5.22). Then the self-adjointness of the realization $T \in \mathcal{H}_2^t$ in $\mathfrak{H}_2^t$ says that

$$T^* = T \Longleftrightarrow \begin{pmatrix} \overline{a} & b \\ t\overline{b} & a \end{pmatrix} = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix},$$

if and only if

$$\overline{a} = a \;\text{ and }\; tb = b \text{ in } \mathbb{C},$$

if and only if

$$a \in \mathbb{R} \text{ and } b = 0. \tag{5.23}$$

Especially, the equality $b = 0$ in (5.23) is obtained by our negative-scale assumption: $t < 0$ in $\mathbb{R}$.

Proposition 5.10. *Let $(a,b) \in \mathbb{H}_t$ satisfy (5.22). Then the realization $T \in \mathcal{H}_2^t$ is self-adjoint in $\mathfrak{H}_2^t$, if and only if*

$$a \in \mathbb{R} \text{ and } b = 0 \Longleftrightarrow (a,b) = (\operatorname{Re}(a), 0) \text{ in } \mathbb{H}_t. \tag{5.24}$$

Proof. The self-adjointness (5.24) is shown by (5.23). □

The self-adjointness (5.24) illustrates that the self-adjoint generating elements $T \in \mathcal{H}_2^t$ of $(\mathfrak{H}_2^t, \tau)$ have their forms,

$$T = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathcal{H}_2^t\,(1,*) \text{ with } x \in \mathbb{R}.$$

Remark 5.11. The above self-adjointness characterization (5.24) is obtained for the case where $t < 0$ in $\mathbb{R}$. How about the other cases? Generally, one has T is self-adjont in $\mathfrak{H}_2^t$, if and only if

$$\overline{a} = a \quad \text{and} \quad tb = b,$$

like (5.23). Thus one can verify that: (i) if $t = 0$, then T is self-adjoint, if and only if $a \in \mathbb{R}$ and $b = 0$, just like (5.24); (ii) if $t > 0$ and $t \neq 1$, then T is self-adjoint, if and only if $a \in \mathbb{R}$ and $b = 0$, just like (5.24); meanwhile, (iii) if $t = 1$ (equivalently, if (a, b) is a bicomplex number of $\mathbb{H}_1$), then T is self-adjoint in $\mathfrak{H}_2^1$, if and only if $a \in \mathbb{R}$, if and only if $(a, b) = (\mathrm{Re}\,(a), b)$ in $\mathbb{H}_1$. In summary,

$$T \text{ is self-adjoint in } \mathfrak{H}_2^t \Longleftrightarrow (a, b) = (\mathrm{Re}\,(a), 0) \ \text{ in } \mathbb{H}_t,$$

like (5.24), whenever $t \in \mathbb{R} \setminus \{1\}$, meanwhile,

$$T \text{ is self-adjoint in } \mathfrak{H}_2^1 \Longleftrightarrow (a, b) = (\mathrm{Re}\,(a), b) \in \mathbb{H}_1.$$

Now, let $(a, b) \in \mathbb{H}_t$, under (5.22) and our negative-scale assumption, satisfy the self-adjointness (5.24), i.e., it is actually $(a, 0)$ with $a \in \mathbb{R}$. Then

$$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = S \text{ in } \mathcal{H}_2^t\,(1, *)\,,$$

as an element of $\mathfrak{H}_2^t$.

Theorem 5.12. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22), and assume that the realization T is self-adjoint in $(\mathfrak{H}_2^t, \tau)$. Then*

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = \tau\left(T^n\right) = a^n \quad \textit{in } \mathbb{R} \tag{5.25}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. By the self-adjointness (5.24) of the realization T of $(a, b) \in \mathbb{H}_t$, one has $(a, b) = (a, 0)$ in $\mathbb{H}_t$, with $a \in \mathbb{R}$, and

$$T = S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = S^* = T^* \text{ in } \mathfrak{H}_2^t.$$

So,

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = \tau\left(T^n\right) = \tau\left(S^n\right) = \tau\left(\begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix}\right),$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. Therefore, the free-distributional data (5.25) holds true. □

Remark 5.13. Similar to the above theorem, one can verify that: if $t \in \mathbb{R} \setminus \{1\}$, then the free-distributional data (5.25) holds for self-adjoint realizations $T \in (\mathfrak{H}_2^t, \tau)$ of $(a, 0) \in \mathbb{H}_t$ with $a \in \mathbb{R}$.

By (5.24), the realization T of a hypercomplex number $(a, b) \in \mathbb{H}_t$, satisfying (5.22), is self-adjoint, if and only if $(a, b) = (a, 0)$ with $a \in \mathbb{R}$. And, by definition, such a self-adjoint matrix T can be a projection, if and only if it is idempotent in the sense that

$$T^2 = T \text{ in } \mathfrak{H}_2^t.$$

Observe that a self-adjoint realization T satisfies the above idempotence, if and only if

$$T^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = T,$$

if and only if

$$a^2 = a \Longleftrightarrow a = 0, \text{ or } a = 1, \text{ in } \mathbb{R}. \tag{5.26}$$

Proposition 5.14. *Let* $(a, b) \in \mathbb{H}_t$ *satisfy* (5.22)*. Then the realization* T *is a projection, if and only if*

$$\text{either } T = I_2, \text{ or } T = O_2 \text{ in } \mathcal{H}_2^t, \tag{5.27}$$

where $I_2 = [(1, 0)]_t$ *is the identity matrix, and* $O_2 = [(0, 0)]_t$ *is the zero matrix of* $\mathfrak{H}_2^t$*.*

Proof. The operator-equality (5.27) holds in $\mathcal{H}_2^t$ (and hence, in $\mathfrak{H}_2^t$) by (5.26). □

Remark 5.15. Like in the above proposition, one can conclude that: if $t \in \mathbb{R} \setminus \{1\}$, then the realization T is a projection in $\mathfrak{H}_2^t$, if and only if it is either the identity matrix I_2, or the zero matrix O_2 of $\mathfrak{H}_2^t$. How about the case where $t = 1$? As we discussed above, $T \in \mathfrak{H}_2^1$ is self-adjoint, if and only if $(a, b) = (\mathrm{Re}\,(a), b)$ in $\mathbb{H}_1$, if and only if

$$T = \begin{pmatrix} x & b \\ \overline{b} & x \end{pmatrix} \in \mathcal{H}_2^1, \quad \text{and} \quad S = \begin{pmatrix} x + i\sqrt{-u^2 - v^2} & 0 \\ 0 & x - i\sqrt{-u^2 - v^2} \end{pmatrix},$$

implying that

$$S = \begin{pmatrix} x - |b| & 0 \\ 0 & x + |b| \end{pmatrix} \text{ in } \mathfrak{H}_2^1,$$

under (5.22). Such a self-adjoint T is a projection, if and only if $T^2 = T$ in $\mathfrak{H}_2^1$, if and only if

$$x^2 + |b|^2 = x \quad \text{and} \quad 2xb = b.$$

Thus if $b = 0$, then $x \in \{0, 1\}$, meanwhile, if $b \neq 0$, then

$$x^2 + |b|^2 = x \text{ and } x = \frac{1}{2},$$

if and only if

$$x = \frac{1}{2} \quad \text{and} \quad \frac{1}{4} + |b|^2 = \frac{1}{2},$$

if and only if

$$x = \frac{1}{2} \quad \text{and} \quad |b|^2 = \frac{1}{4},$$

if and only if

$$(a,b)=\left(\frac{1}{2},b\right) \text{ with } |b|^2=\frac{1}{4}.$$

It implies that T is a projection in $\mathfrak{H}_2^1$, if and only if

$$(a,b)=(0,0), \text{ or } (a,b)=(1,0),$$

or

$$(a,b)=\left(\frac{1}{2},\ b\right) \text{ with } |b|^2=\frac{1}{4},$$

in $\mathbb{H}_1$.

The above proposition says that, under our negative-scale assumption, the only projections of $\mathfrak{H}_2^t$ induced by hypercomplex numbers of $\mathbb{H}_t$ are the identity element $I_2=[(1,0)]_t$, and the zero element $O_2=[(0,0)]_t$ in $\mathfrak{H}_2^t$. For any unital C^*-probability spaces $(\mathcal{A},\varphi)$, the unity $1_{\mathcal{A}}$ has its free distributions characterized by its free-moment sequence,

$$\left(\varphi\left(1_{\mathcal{A}}^n\right)=\varphi\left(1_{\mathcal{A}}\right)\right)_{n=1}^{\infty}=(1,1,1,1,1,\dots);$$

and the free distribution of the zero element $0_{\mathcal{A}}$ is nothing but the zero-free distribution, characterized by the free-moment sequence,

$$\left(\varphi\left(0_{\mathcal{A}}^n\right)=\varphi\left(0_{\mathcal{A}}\right)\right)_{n=1}^{\infty}=(0,0,0,0,\dots).$$

Theorem 5.16. *Let $(a,b)\in\mathbb{H}_t$, satisfying (5.22), have its realization $T\in\mathcal{H}_2^t$, which is a "non-zero" projection in $\mathfrak{H}_2^t$. Then*

$$\tau\left(T^n\right)=1,\quad \forall n\in\mathbb{N}.$$

(*In fact, this result holds true for all $t\in\mathbb{R}\setminus\{1\}$.*)

Proof. Under hypothesis, the realization $T\in\mathcal{H}_2^t$ is a projection in $\mathfrak{H}_2^t$, if and only if $(a,b)=(1,0)$, or $(0,0)$ in $\mathbb{H}_t$, by (5.27). Since $T\in\mathcal{H}_2^t$ is assumed to a non-zero projection in $\mathfrak{H}_2^t$, we have

$$(a,b)=(1,0) \text{ in } \mathbb{H}_{\mathrm{t}} \iff T=I_2=S \text{ in } \mathfrak{H}_2^t.$$

Therefore,

$$\tau\left(T^n\right)=\tau\left(I_2^n\right)=1,\quad \forall n\in\mathbb{N}.$$

(Note that it holds true for all $t\in\mathbb{R}\setminus\{1\}$.) □

Let $(a,b)\in\mathbb{H}_t$ satisfy (5.22), and let $T\in\mathcal{H}_2^t$ be the realization in $\mathfrak{H}_2^t$. Observe that

$$T^*T=\begin{pmatrix}\overline{a} & b\\ t\overline{b} & a\end{pmatrix}\begin{pmatrix}a & tb\\ \overline{b} & \overline{a}\end{pmatrix}=\begin{pmatrix}|a|^2+|b|^2 & (t+1)\,\overline{a}b\\ (t+1)\,a\overline{b} & t^2|b|^2+|a|^2\end{pmatrix},$$

and

$$TT^*=\begin{pmatrix}a & tb\\ \overline{b} & \overline{a}\end{pmatrix}\begin{pmatrix}\overline{a} & b\\ t\overline{b} & a\end{pmatrix}=\begin{pmatrix}|a|^2+t^2|b|^2 & (t+1)\,ab\\ (t+1)\,\overline{ab} & |b|^2+|a|^2\end{pmatrix}, \tag{5.28}$$

in $\mathfrak{H}_2^t$. So, the realization T of (a, b) is normal in $\mathfrak{H}_2^t$, if and only if

$$|a|^2 + t^2|b|^2 = |a|^2 + |b|^2 \text{ and } (t+1)\,\overline{a}b = (t+1)\,ab, \tag{5.29}$$

in $\mathbb{C}$, by (5.28).

Proposition 5.17. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22). Then the realization $T \in \mathcal{H}_2^t$ is normal in $\mathfrak{H}_2^t$, if and only if*

$$t^2|b|^2 = |b|^2 \text{ and } (t+1)\,\overline{a}b = (t+1)\,ab, \tag{5.30}$$

in $\mathbb{C}$. In particular, if $t = -1$ (equivalently, if $(a, b) \in \mathbb{H}_{-1}$ is a quaternion), then T is normal in $\mathfrak{H}_2^{-1}$; if $t = 1$, (equivalently, if $(a, b) \in \mathbb{H}_1$ is a bicomplex number), then T is normal in $\mathfrak{H}_2^1$, if and only if

$$\textit{either } (a, b) = (\mathrm{Re}\,(a), b) \;\; \textit{or } (a, b) = (a, 0) \;\; \textit{in } \mathbb{H}_1; \tag{5.31}$$

meanwhile, if $t \in \mathbb{R} \setminus \{\pm 1\}$, then T is normal in $\mathfrak{H}_2^t$, if and only if

$$b = 0 \text{ in } \mathbb{C} \Longleftrightarrow (a, b) = (a, 0) \in \mathbb{H}_t. \tag{5.32}$$

Proof. By (5.29), the normality characterization (5.30) holds.

By (5.30), if $t = -1$ in $\mathbb{R}$, and hence, if $(a, b) \in \mathbb{H}_{-1}$ is a quaternion, then the condition (5.30) is identified with

$$|b|^2 = |b|^2, \text{ and } 0 = 0,$$

which are the identities on $\mathbb{C}$. These identities demonstrate that the realization of every quaternion is automatically normal in $\mathfrak{H}_2^{-1}$.

Suppose $t = 1$ in $\mathbb{R}$. Then the condition (5.30) is equivalent to

$$|b|^2 = |b|^2 \text{ and } 2\overline{a}b = 2ab,$$

if and only if either

$$\overline{a} = a \text{ in } \mathbb{C} \iff (a, b) = (\mathrm{Re}\,(a), b) \in \mathbb{H}_1 \text{ (if } b \neq 0),$$

or

$$(a, b) = (a, 0) \in \mathbb{H}_1 \quad (\text{if } b = 0).$$

Thus, if $t = 1$, then T is normal, if and only if the condition (5.31) holds.

Assume now that both $t \neq 1$ and $t \neq -1$, i.e., suppose $t^2 \neq 1$ in $\mathbb{R}$. So, the first condition of (5.30) is identified with

$$t^2|b|^2 = |b|^2 \iff b = 0 \text{ in } \mathbb{C}.$$

So, the second condition of (5.30) automatically holds, since

$$(t+1)\,\overline{a} \cdot 0 = (t+1)\,a \cdot 0 \iff 0 = 0.$$

Therefore, the realization $T \in \mathcal{H}_2^t$ of $(a, b) \in \mathbb{H}_t$ is normal in $\mathfrak{H}_2^t$, if and only if $(a, b) = (a, 0)$ in $\mathbb{H}_t$, whenever $t \in \mathbb{R} \setminus \{\pm 1\}$, i.e., the normality (5.32) holds. □

The above proposition illustrates that: (i) the realizations of "all" quaternions are normal in $\mathfrak{H}_2^{-1}$, (ii) the realizations of bicomplex numbers are normal in $\mathfrak{H}_2^1$, if and only if either $(a,b) = (\mathrm{Re}\,(a), b)$, or $(a,b) = (a,0)$ in $\mathbb{H}_1$, by (5.31), and (iii) the only realizations $[(a,0)]_t$ are normal in $\mathfrak{H}_2^t$, whenever $t \in \mathbb{R} \setminus \{\pm 1\}$, by (5.32).

Theorem 5.18. *Let $(a,b) \in \mathbb{H}_t$ satisfy (5.22).*

(i) *Suppose $t = -1$. Then T is normal in $\mathfrak{H}_2^{-1}$, and its free distribution is characterized by the formula (5.20).*
(ii) *Let $t \in \mathbb{R} \setminus \{\pm 1\}$. If T is "non-zero" normal in $\mathfrak{H}_2^t$, then*

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = R^n \mathrm{Re}\left(W_o^{\sum\limits_{l=1}^{n} e_l}\right),$$

with

$$R = |a| \text{ and } W_o = \frac{a}{|a|} \in \mathbb{T}, \tag{5.33}$$

where

$$e_l = \begin{cases} 1 & \textit{if } r_l = 1, \\ -1 & \textit{if } r_l = *, \end{cases}$$

for $l = 1, \dots, n$, for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The statement (i) holds by (5.20).

Of course, if $t < 0$, and if $T \in \mathcal{H}_2^t$, then the free-distributional data (5.33) holds by (5.20), because T and the t-spectral form S are similar in $\mathcal{H}_2^t$ as elements of $(\mathfrak{H}_2^t, \tau)$. However, in the statement (ii), the normality works for all the scales $t \in \mathbb{R} \setminus \{\pm 1\}$. Assume that the realization T is a "non-zero", "normal" element of $\mathfrak{H}_2^t$. Then

$$(a,b) = (a,0) \in \mathbb{H}_t, \text{ with } a \neq 0,$$

by (5.32). Therefore,

$$T = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = S,$$

because $\sigma_t(a,0) = a$ in $\mathbb{C}$, i.e., the realization T and the t-spectral form S are identical in $\mathfrak{H}_2^t$, implying the similarity of them. So, under (5.22),

$$a = w \overset{\text{denote}}{=} \sigma_t(a,0),$$

polar-decomposed to be

$$w = a = |a|\left(\frac{a}{|a|}\right) \in \mathbb{C},$$

i.e., $r = |a|$ and $w_o = \frac{a}{|a|}$ under (5.22). Therefore, similar to (5.20), the free-distributional data (5.33) holds. $\square$

Note that, in the proof of the statement (ii) of Theorem 5.18, we did not use our negative-scale assumption for the cases where $t < 0$, but $t \neq -1$. Indeed, even though $t \geq 0$, but $t \neq 1$, the normality (5.32) shows that the realization T is a diagonal matrix not affected by the scale t. So, whatever scales t are given in $\mathbb{R} \setminus \{\pm 1\}$, the free-distributional data (5.33) holds in $(\mathfrak{H}_2^t, \tau)$, under normality. Then, how about the case where $t = 1$? Recall that if $t = 1$, then the realization T of $(a, b) \in \mathbb{H}_1$ is normal in $\mathfrak{H}_2^1$, if and only if either

$$(a, b) = (\mathrm{Re}\,(a), b)\,, \text{ if } b \neq 0,$$

or

$$(a, b) = (a, 0), \text{ if } b = 0,$$

in $\mathbb{H}_1$, by (5.31). So, if $(a, b) = (a, 0)$ in $\mathbb{H}_1$, the joint free moments of T are determined similarly by the formula (5.33), by the identity (and hence, the similarity) of T and S (under (5.22)). However, if $(a, b) = (\mathrm{Re}\,(a), b)$ with $b \neq 0$, then we need a better tool than (5.10) to compute the corresponding free-distributional data, because we cannot use our similarity technique (of Theorem 5.6) here.

By the definition of the unitarity, if an element U of a C^*-algebra $\mathcal{A}$ is a unitary, then it is automatically normal, i.e., the unitarity implies the normality. Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22) with its realization $T \in \mathcal{H}_2^t$ in $(\mathfrak{H}_2^t, \tau)$, and suppose it is a unitary in $\mathfrak{H}_2^t$. By the assumption that T is a unitary in $\mathfrak{H}_2^t$, it is normal.

Assume first that $t = -1$ in $\mathbb{R}$, and hence, $(a, b) \in \mathbb{H}_{-1}$ is a quaternion. Then the realization T is automatically normal in $\mathfrak{H}_2^t$ by Theorem 5.18(i). Indeed, in this case,

$$T = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \quad \text{with} \quad T^* = \begin{pmatrix} \overline{a} & b \\ -\overline{b} & a \end{pmatrix} = [(\overline{a}, -b)]_{-1}\,,$$

in $\mathcal{H}_2^{-1}$, as elements of $\mathfrak{H}_2^{-1}$. So, the normality is guaranteed;

$$T^*T = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = TT^*,$$

in $\mathcal{H}_2^{-1}$, as elements of $\mathfrak{H}_2^{-1}$. It shows that T is a unitary in $\mathfrak{H}_2^{-1}$, if and only if

$$|a|^2 + |b|^2 = 1. \tag{5.34}$$

Meanwhile, if $t \in \mathbb{R} \setminus \{\pm 1\}$ in $\mathbb{R}$, then T is normal, if and only if $(a, b) = (a, 0)$ in $\mathbb{H}_t$ by (5.32), if and only if

$$T = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t,$$

which is identical (and hence, similar) to the t-spectral form S of $(a, 0)$ in $\mathfrak{H}_2^t$. This normal element T is a unitary in $\mathfrak{H}_2^t$, if and only if

$$T^*T = I_2 = TT^* \Longleftrightarrow \begin{pmatrix} |a|^2 & 0 \\ 0 & |a|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

if and only if

$$|a|^2 = 1 \text{ in } \mathbb{C}. \tag{5.35}$$

Proposition 5.19. *Let* $(a, b) \in \mathbb{H}_t$ *satisfy* (5.22).

(i) *Let* $t = -1$. *Then* T *is a unitary in* $\mathfrak{H}_2^t$, *if and only if* $|a|^2 + |b|^2 = 1$.
(ii) *Let* $t \in \mathbb{R} \setminus \{\pm 1\}$. *Then* T *is a unitary in* $\mathfrak{H}_2^t$, *if and only if* $|a|^2 = 1$ *and* $b = 0$.

Proof. The statements (i) and (ii) hold by (5.34) and (5.35), respectively, because a unitary realization T of (a, b) automatically satisfies the normality (5.30). □

Observation 5.20. *Now, assume that* $t = 1$, *and let* $(a, b) \in \mathbb{H}_1$ *be a bicomplex number satisfying* (5.22). *By* (5.31), *the realization* $T \in \mathcal{H}_2^1$ *is normal in* $\mathfrak{H}_2^1$, *if and only if either*

$$(a, b) = (\mathrm{Re}\,(a), b)\,, \text{ or } (a, b) = (a, 0),$$

in $\mathbb{H}_1$. *So, if* $(a, b) = (a, 0)$ *in* $\mathbb{H}_1$, *then one obtains the unitarity that:* T *is a unitary in* $\mathfrak{H}_2^1$, *if and only if* $|a|^2 = 1$, *just like Proposition 5.19* (ii). *However, if*

$$(a, b) = (\mathrm{Re}\,(a), b) = (x, b) \text{ in } \mathbb{H}_1,$$

with $b \neq 0$ *in* $\mathbb{C}$, *then* T *is a unitary in* $\mathfrak{H}_2^1$, *if and only if*

$$\begin{pmatrix} x & \overline{b} \\ b & x \end{pmatrix} \begin{pmatrix} x & b \\ \overline{b} & x \end{pmatrix} = \begin{pmatrix} x^2 + \overline{b^2} & 2x\mathrm{Re}\,(b) \\ 2x\mathrm{Re}\,(b) & x^2 + b^2 \end{pmatrix} = I_2,$$

and

$$\begin{pmatrix} x & b \\ \overline{b} & x \end{pmatrix} \begin{pmatrix} x & \overline{b} \\ b & x \end{pmatrix} = \begin{pmatrix} x^2 + b^2 & 2x\mathrm{Re}\,(b) \\ 2x\mathrm{Re}\,(b) & x^2 + \overline{b^2} \end{pmatrix} = I_2,$$

in $\mathfrak{H}_2^1$, *if and only if*

$$x^2 + \overline{b^2} = x^2 + b^2 = 1 \quad \textit{and} \quad 2x\mathrm{Re}\,(b) = 0,$$

if and only if

$$b^2 = \overline{b^2} = 1 - x^2 \quad \textit{and} \quad 2x\mathrm{Re}\,(b) = 0,$$

if and only if

$$b^2 = 1 - x^2 \in \mathbb{R} \quad \textit{and} \quad x = 0,$$

because b *is assumed not to be zero in* $\mathbb{C}$, *if and only if*

$$x = 0 \quad \textit{and} \quad b = \pm 1 \quad \text{in} \quad \mathbb{R},$$

if and only if

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \text{ or } T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ in } \mathcal{H}_2^1,$$

if and only if

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1) \text{ in } \mathbb{H}_1.$$

i.e., if $(a, b) = (\mathrm{Re}\,(a), b)$ *in* $\mathbb{H}_1$, *then* T *is a unitary in* $\mathfrak{H}_2^1$, *if and only if*

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1),$$

in $\mathbb{H}_1$. In summary, the realization $T \in \mathcal{H}_2^1$ of a bicomplex number $(a,b) \in \mathbb{H}_1$ is a unitary in $\mathfrak{H}_2^t$, if and only if either

$$(a,b) = (a,0) \;\text{ with }\; |a|^2 = 1,$$

or

$$(a,b) = (0,1), \text{ or } (a,b) = (0,-1),$$

in $\mathbb{H}_1$.

By Proposition 5.19, one has the following result.

Theorem 5.21. *Let $(a,b) \in \mathbb{H}_t$ satisfy (5.22).*

(i) *Suppose $t = -1$. If T is a unitary in $\mathfrak{H}_2^t$, then its free distribution is characterized by the formula (5.20) with $r = 1$.*
(ii) *Let $t \in \mathbb{R} \setminus \{\pm 1\}$. If T is a unitary in $\mathfrak{H}_2^t$, then*

$$\tau\left(\prod_{l=1}^{n} T^{r_l}\right) = \operatorname{Re}\left(a^{\sum_{l=1}^{n} e_l}\right), \text{ with } a \in \mathbb{T} \text{ in } \mathbb{C},$$

where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases} \tag{5.36}$$

for $l = 1, \ldots, n$, for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The statement (i) holds by (5.20). In particular, by the unitarity characterization in Proposition 5.19(i), the free-distributional data in (5.20) must have $r = 1$, since

$$|\sigma_t(a,b)| = |w| \overset{\text{denote}}{=} r = 1,$$

under the similarity of T and S, by Proposition 5.19(i).

By Theorem 5.18(ii), if $t \neq \pm 1$, then the free-distributional data (5.36) holds by (5.33). Indeed, under the unitarity of T, the formula (5.33) satisfies

$$R = |a| = 1 \text{ and } W_o = a \in \mathbb{T}.$$

Therefore, the joint free moments (5.36) holds. □

The above theorem characterizes the free distributions of unitary elements of $(\mathfrak{H}_2^t, \tau)$ induced by $\mathbb{H}_t$, where $t \in \mathbb{R} \setminus \{1\}$.

Suppose $t = 1$, and $(a,b) \in \mathbb{H}_1$ satisfies (5.22). In the above observation, we showed that the realization $T \in \mathcal{H}_2^1$ of (a,b) is a unitary, if and only if either

$$(a,b) = (a,0) \text{ with } a \in \mathbb{T},$$

or

$$(a,b) = (0,1), \text{ or } (a,b) = (0,-1),$$

in $\mathbb{H}_1$, equivalently, either

$$T = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \text{ with } a \in \mathbb{T},$$

or

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ or } T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

in $\mathcal{H}_2^1$ (as an element of $\mathfrak{H}_2^1$). Thus, if $(a,b) = (a,0) \in \mathbb{H}_1$ with $|a|^2 = 1$, then the free distribution of T is similarly characterized by the formula (5.36). Meanwhile, if $T = [(0,1)]_1$, then

$$T^* = T \in \mathcal{H}_2^1 \subset \mathcal{H}_2^1(1,*) \text{ in } \mathfrak{H}_2^1,$$

and

$$T^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

in $\mathfrak{H}_2^1$, satisfying that

$$(T^n)_{n=1}^{\infty} = (T, I_2, T, I_2, T, I_2, \ldots); \tag{5.37}$$

and, if $T = [(0,-1)]_1$, then

$$T^* = T \in \mathcal{H}_2^1 \subset \mathcal{H}_2^1(1,*) \text{ in } \mathfrak{H}_2^1,$$

and

$$T^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

in $\mathfrak{H}_2^1$, satisfying that

$$(T^n)_{n=1}^{\infty} = (T, I_2, T, I_2T, I_2, \ldots). \tag{5.38}$$

Therefore, one obtains the following result in addition with Theorem 5.21.

Theorem 5.22. *Let $(a,b) \in \mathbb{H}_1$ be a bicomplex number satisfying* (5.22). *Then the realization T is a unitary in $(\mathfrak{H}_2^1, \tau)$, if and only if either*

$$(a,b) = (a,0), \text{ with } |a|^2 = 1,$$

or

$$(a,b) = (0,1), \text{ or } (a,b) = (0,-1) \text{ in } \mathbb{H}_1. \tag{5.39}$$

(i) *If $(a,b) = (a,0)$, with $|a|^2 = 1$, in $\mathbb{H}_1$, then the free distribution of T is characterized by the formula* (5.36).

(ii) *If either $(a,b) = (0,1)$, or $(a,b) = (0,-1)$ in $\mathbb{H}_1$, then the free distribution of the unitary realization T is fully characterized by the free-moment sequence,*

$$(\tau(T^n))_{n=1}^{\infty} = (0,1,0,1,0,1,0,1,\ldots). \tag{5.40}$$

Proof. By Observation 5.20, it is shown that the realization $T \in \mathcal{H}_2^1$ of a bicomplex number $(a,b) \in \mathbb{H}_1$ is a unitary in $\mathfrak{H}_2^1$, if and only if the condition (5.39) holds true.

The statement (i) is shown similarly by the proof of the statement Theorem 5.21(ii). So, the free-distributional data (5.36) holds.

Now, if either $T = [(0,1)]_1$, or $T = [(0,-1)]_1$ in $\mathcal{H}_2^1$, it is not only a unitary, but also a self-adjoint element of $\left(\mathfrak{H}_2^1, \tau\right)$, and hence, the free distribution of T is fully characterized by the free-moment sequence $\left(\tau\left(T^n\right)\right)_{n=1}^{\infty}$. However, by (5.37) and (5.38), one immediately obtain the free-moment sequence (5.40). Therefore, the statement (ii) holds. $\square$

The above theorem fully characterizes the free distributions of the unitaries of $\left(\mathfrak{H}_2^1, \tau\right)$ induced by bicomplex numbers of $\mathbb{H}_1$.

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Received: October 11, 2022.
Revised: April 14, 2023.
Accepted: April 17, 2023.