Opuscula Math. 43, no. 3 (2023), 455–467
https://doi.org/10.7494/OpMath.2023.43.3.455 Opuscula Mathematica

NEW OSCILLATION CONSTRAINTS FOR EVEN-ORDER DELAY DIFFERENTIAL EQUATIONS

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Communicated by Josef Diblík

Abstract. The purpose of this paper is to study the oscillatory properties of solutions to a class of delay differential equations of even order. We focus on criteria that exclude decreasing positive solutions. As in this paper, this type of solution emerges when considering the noncanonical case of even equations. By finding a better estimate of the ratio between the Kneser solution with and without delay, we obtain new constraints that ensure that all solutions to the considered equation oscillate. The new findings improve some previous findings in the literature.

Keywords: delay differential equations, even-order, Kneser solutions, oscillation.

Mathematics Subject Classification: 34C10, 34K11.

1. INTRODUCTION

Since the inception of calculus and the emergence of differential equations, they have been used to model and describe various life and technological problems that appear in engineering, physics, chemistry, biology and other sciences. Delay differential equations (DDE) are a preferred method for describing phenomena, due to the fact that they take into account the temporal memory of phenomena. Understanding and investigating these phenomena, however, is hampered by the difficulty of solving the equations that come from the modeling of these processes. As a result, the qualitative theory makes a significant contribution to finding a solution to this issue and enabling the study of the qualitative aspects of equations. Oscillation theory, which deals with oscillatory, non-oscillatory, asymptotic behavior and the distribution of zeros for solutions of differential equations, is one of the disciplines of qualitative theory.

Several vital applications of DDEs have appeared in the various natural sciences, population dynamics and technology, see [9]. As a result of the applications of these equations, the asymptotic and oscillatory behavior of these equations has sparked significant research work; see, for instance, the monographs [1, 4, 10, 19]. In particular, the qualitative properties of different classes of Emden–Fowler DDEs have many applications in engineering and physics (e.g., many real-world problems involve Emden–Fowler DDEs, such as the study of porous medium difficulties, p-Laplace equations, and so on); see [5, 11, 12].

In this study, we consider the following class of DDEs of even-order:

$$\left(b\cdot(v^{(n-1)})^{\kappa}\right)' + q\cdot(\phi\circ v\circ\sigma) = 0, \quad t\ge t_0, \tag{1.1}$$

where $n\ge 4$ is an even integer, $\kappa\in\mathbb{Q}^{+}_{odd} := \{x/y : x \text{ and } y \text{ are odd integers}\}$ and $\kappa^* = (\kappa/\kappa+1)^{\kappa+1}$. Throughout this study, we assume that:

(H1) $b\in\mathbf{C}^1\big([t_0,\infty),(0,\infty)\big)$, $q,\sigma\in\mathbf{C}\big([t_0,\infty),\mathbb{R}\big)$, $b'(t)\ge 0$, $q(t)\ge 0$, $\sigma(t)\le t$, and $\lim_{t\to\infty}\sigma(t)=\infty$,
(H2) the function $\phi\in\mathbf{C}(\mathbb{R},\mathbb{R})$ satisfies $\phi(v)/v^{\beta}\ge k>0$ for $v\ne 0$ and $\beta\in\mathbb{Q}^{+}_{odd}$,
(H3) $\eta_m(t_0)<\infty$, where

$$\eta_0(t) := \int_{t_0}^{\infty} b^{-1/\kappa}(h)\mathrm{d}h$$

and

$$\eta_m(t) := \int_{t}^{\infty} \eta_{m-1}(h)\mathrm{d}h, \quad \text{for} \quad m=1,2,\dots,n-2.$$

For a solution of (1.1) we denote a function v in $\mathbf{C}^{n-1}([t_*,\infty))$ for some $t_*\ge t_0$, which $b(v^{(n-1)})^{\kappa}\in\mathbf{C}^1([t_*,\infty))$ and satisfies (1.1) on $[t_*,\infty)$. We take into account these solutions v of (1.1) such that $\sup\{|v(s)| : s\ge t_v\}>0$ for every t_v in $[t_*,\infty)$.

A solution v of (1.1) is said to be *nonoscillatory* if it is eventually positive or eventually negative; otherwise, it is said to be *oscillatory*. We define the class $\Bbbk$ as

$$\Bbbk := \Big\{v(t) : \text{there exists } t_1\ge t_0 \text{ such that } v^{(i)}(t)v^{(i+1)}(t)<0 \\ \text{for } i=0,1,\dots,n-2, \text{ and } t\ge t_1\Big\}.$$

In the following, we review some of the previous results that were the motivation for this study.

Zhang *et al.* [21] studied the asymptotic behavior of (1.1) in the noncanonical case, that is when

$$\int_{t_0}^{\infty} b^{-1/\kappa}(h)\mathrm{d}h<\infty.$$

Without taking into account the sign of first derivative v', the results in [21] ensured that all nonoscillatory solutions of equation (1.1) tend to zero. As an improvement of results in [21], Zhang *et al.* [20] proved that if $\beta\le\kappa$ and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(M^{\beta-\kappa}q(h)b_{n-3}^{\kappa}(h) - \frac{\kappa^{\kappa+1}}{(\kappa+1)^{\kappa+1}}\frac{\left(b'_{n-3}(h)\right)^{\kappa+1}}{b_{n-3}(h)\,b_{n-4}^{\kappa}(h)}\right)\mathrm{d}h=\infty,$$

then $\Bbbk = \varnothing$, where $M > 0$ and $b_p(t) := \frac{1}{p!}\int_t^\infty (s-t)^p \eta_0(s)\mathrm{d}s$. Combining the results obtained in [20, 21], it is easy to attain oscillation of all solutions of (1.1).

On the other hand, by imposing the following conditions

$$\phi'(\upsilon) \geq 0 \quad \text{and} \quad -\phi(-x\upsilon) \geq \phi(x\upsilon) \geq \phi(x)\phi(\upsilon), \quad \text{for } x\upsilon > 0.$$

Baculíková *et al.* [3] studied the oscillatory properties of (1.1). They used the comparison with first-order DDEs and proved that if that there exists $\zeta_1 \in \mathbf{C}([t_0,\infty))$ such that $\zeta'(t) > 0$, $\zeta(t) > t$, $\zeta_{n-2}(\sigma(t)) < t$ and

$$\liminf_{t\to\infty} \int_{\zeta_{n-2}(\sigma(t))}^{t} b^{-1/\kappa}(h) \left(\int_{t_0}^{h} q(s)\mathrm{d}s \right)^{1/\kappa} \phi^{1/\kappa}\left(J_{n-2}(\sigma(h))\right)\mathrm{d}h > \frac{1}{\mathrm{e}},$$

then $\Bbbk = \varnothing$, where

$$\zeta_{i+1}(t) := \zeta\left(\zeta_i(t)\right), \quad J_{i+1}(t) := \int_t^{\zeta(t)} J_i(h)\mathrm{d}h \quad \text{and} \quad J_1(t) := \zeta_1(t) - t.$$

For the works that focused on studying the oscillation of the even-order equations in the canonical case, that is,

$$\int_{t_0}^{\infty} b^{-1/\kappa}(h)\mathrm{d}h = \infty,$$

see, for example, [8, 14, 16].

While proving the main results, we will need the following lemma.

Lemma 1.1 ([17]). *Let $F \in C^n([t_0,\infty),(0,\infty))$. If $F^{(n)}$ is eventually of one sign, then there exist a t_* such that $t_* \geq t_0$ and a $\gamma \in \mathbb{Z}$, $0 \leq \gamma \leq n$, with $n+\gamma$ even for $F^{(n)}(t) \geq 0$, or $n+\gamma$ odd for $F^{(n)}(t) \leq 0$ such that*

$$\gamma > 0 \quad \textit{implies} \quad F^{(\kappa)}(t) > 0 \quad \textit{for} \quad 0 \leq \kappa \leq \gamma - 1,$$

and

$$\gamma \leq n-1 \quad \textit{implies} \quad (-1)^{\gamma+\kappa} F^{(\kappa)}(t) > 0 \quad \textit{for} \quad \gamma \leq \kappa \leq n-1,$$

eventually.

In this study, we provide new criteria that ensure that $\Bbbk = \varnothing$. Our results take into account the influence of the delay argument $\sigma(t)$ that has been neglected in related results. Furthermore, the technique we have used does not rely on the traditional form $(\limsup(\cdot) = \infty)$ and consider all cases of α and β. The criteria reported in this study improve, generalize, and complement those in [3, 15, 20, 21].

2. MAIN RESULTS

The following lemma is obtained by applying Lemma 1.1, and taking into consideration that n is even and the fact that $\left(b\cdot\left(\upsilon^{(n-1)}\right)^{\kappa}\right)' \leq 0$ for every positive solution υ. Moreover, for $n = 4$, the following lemma is explained in detail in [18].

Lemma 2.1. *Suppose that υ is a positive solution of* (1.1)*. Then there are three possible classes for υ:*

(i) $\upsilon' > 0$, $\upsilon^{(n-1)} > 0$ *and* $\upsilon^{(n)} < 0$,
(ii) $\upsilon' > 0$, $\upsilon^{(n-2)} > 0$ *and* $\upsilon^{(n-1)} < 0$,
(iii) $(-1)^{\iota}\upsilon^{(\iota)} > 0$ *for* $\iota = 1, 2, \ldots, n-1$.

Lemma 2.2. *Assume that $\upsilon \in \mathbb{k}$. Then $\upsilon^{\beta-\kappa}(t) \geq \theta(t)$, where*

$$\theta(t) = \begin{cases} m_1^{\beta-\kappa}, & \text{if } \kappa \geq \beta, \\ m_2\eta_{n-2}^{\beta-\kappa}(t), & \text{if } \kappa < \beta, \end{cases}$$

and m_1, m_2 are positive constants. Moreover, if

$$\int_{t_0}^{\infty}\left(\frac{1}{b(s)}\int_{t_1}^{s} q(h)\mathrm{d}h\right)^{1/\kappa}\mathrm{d}s = \infty, \tag{2.1}$$

then $\lim_{t\to\infty}\upsilon(t) = 0$.

Proof. Let $\upsilon \in \mathbb{k}$. Suppose that $\kappa \geq \beta$. Taking into account the facts that $\upsilon > 0$ and $\upsilon' < 0$, there exists an $m_1 > 0$ such that $\upsilon(t) \leq m_1$, and hence $\upsilon^{\beta-\kappa}(t) \geq m_1^{\beta-\kappa}$.

Next, let $\kappa < \beta$. From (1.1), we get $\left(b\left(\upsilon^{(n-1)}\right)^{\kappa}\right)' \leq 0$, and so $b(\upsilon^{(n-1)})^{\kappa} \leq -M < 0$. Thus, by integrating the last inequality from t to ∞, we have that

$$\upsilon^{(n-2)}(t) \geq M^{1/\kappa}\eta_0(t). \tag{2.2}$$

Integrating (2.2) $n-2$ times from t to ∞ and using (iii), we get

$$\upsilon^{\beta-\kappa}(t) \geq M^{(\beta-\kappa)/\kappa}\eta_{n-2}^{\beta-\kappa}(t).$$

Now, since $\upsilon > 0$ and $\upsilon' < 0$, we obtain that $\lim_{t\to\infty}\upsilon(t) = c \geq 0$. We claim that $\lim_{t\to\infty}\upsilon(t) = 0$. Assuming the contrary, we let $c > 0$. Thus, there exists an $t_1 \geq t_0$ with $\upsilon(\sigma(t)) \geq c$ for $t \geq t_1$, and then

$$-\left(b(\upsilon^{(n-1)})^{\kappa}\right)' \geq kq\cdot\left(\upsilon^{\beta}\circ\sigma\right) \geq kc^{\beta}\cdot q, \tag{2.3}$$

for $t \geq t_1$. Integrating (2.3), twice, from t_1 to t and using (c), we obtain

$$\upsilon^{(n-1)}(t) \leq -(kc^{\beta})^{1/\kappa}\left(\frac{1}{b(t)}\int_{t_1}^{t} q(h)\mathrm{d}h\right)^{1/\kappa}$$

and

$$\upsilon^{(n-2)}(t) \leq \upsilon^{(n-2)}(t_1) - (kc^\beta)^{1/\kappa} \int_{t_1}^{t} \left(\frac{1}{b(s)} \int_{t_1}^{s} q(h)\mathrm{d}h \right)^{1/\kappa} \mathrm{d}s.$$

From (2.1), we see that $\lim_{t\to\infty} \upsilon^{(n-2)}(t) = -\infty$, which contradicts $\upsilon^{(n-2)}(t) > 0$. Thus, the proof is complete. □

Lemma 2.3. *Assume that* $\upsilon \in \Bbbk$ *and* (2.1) *holds. If there exists a constant* $\delta \geq 0$ *such that*

$$\theta^{1/\kappa}(t)\eta_{n-2}(t) \left(k \int_{t_0}^{t} q(h)\mathrm{d}h \right)^{1/\kappa} \geq \delta, \tag{2.4}$$

then

$$\upsilon'(t) \leq b^{1/\kappa}(t)\upsilon^{(n-1)}(t)\eta_{n-3}(t) \tag{2.5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\upsilon(t)}{\eta_{n-2}^{\delta}(t)} \right) \leq 0. \tag{2.6}$$

Proof. Let $\upsilon \in \Bbbk$. From (1.1), we see that $b(t)\left(\upsilon^{(n-1)}(t)\right)^\kappa$ is a decreasing function, and so

$$\upsilon^{(n-2)}(s) \leq \upsilon^{(n-2)}(t) + b^{1/\kappa}(t)\upsilon^{(n-1)}(t) \int_{t}^{s} \frac{1}{b^{1/\kappa}(h)}\mathrm{d}h. \tag{2.7}$$

Taking $\lim_{s\to\infty}$ on (2.7), we get

$$\upsilon^{(n-2)}(t) \geq -b^{1/\kappa}(t)\upsilon^{(n-1)}(t)\eta_0(t). \tag{2.8}$$

Integrating (2.8) $n-3$ times from t to ∞ and taking into account (iii), we arrive at

$$\upsilon'(t) \leq b^{1/\kappa}(t)\upsilon^{(n-1)}(t)\eta_{n-3}(t).$$

Integrating (1.1) from t_1 to t, we find

$$\begin{aligned} b(t)\left(\upsilon^{(n-1)}(t)\right)^\kappa &\leq b(t_1)\left(\upsilon^{(n-1)}(t_1)\right)^\kappa - k\int_{t_1}^{t} q(h)\upsilon^\beta(\sigma(h))\mathrm{d}h \\ &\leq b(t_1)\left(\upsilon^{(n-1)}(t_1)\right)^\kappa - k\upsilon^\beta(\sigma(t))\int_{t_0}^{t} q(h)\mathrm{d}h \\ &\quad + k\upsilon^\beta(\sigma(t))\int_{t_0}^{t_1} q(h)\mathrm{d}h. \end{aligned} \tag{2.9}$$

Using Lemma 2.2, we get that $\lim_{t\to\infty} \upsilon(t) = 0$. Thus, there is $t_2 \geq t_1$ such that

$$b(t_1)\left(\upsilon^{(n-1)}(t_1)\right)^{\kappa} + k\upsilon^{\beta}(\sigma(t))\int_{t_0}^{t_1} q(h)\mathrm{d}h < 0, \quad \text{for every } t \geq t_2.$$

It follows from (2.9) that

$$\begin{aligned}\left(b(\upsilon^{(n-1)})^{\kappa}\right)(t) &\leq -k\upsilon^{\beta}(\sigma(t))\int_{t_0}^{t} q(h)\mathrm{d}h \\ &\leq -k\upsilon^{\beta-\kappa}(t)\upsilon^{\kappa}(t)\int_{t_0}^{t} q(h)\mathrm{d}h \\ &\leq -k\theta(t)\upsilon^{\kappa}(t)\int_{t_0}^{t} q(h)\mathrm{d}h.\end{aligned} \tag{2.10}$$

Next, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\upsilon}{\eta_{n-2}^{\delta}}\right) = \frac{\eta_{n-2}^{\delta}\upsilon' + \delta\eta_{n-2}^{\delta-1}\eta_{n-3}\upsilon}{\eta_{n-2}^{2\delta}}. \tag{2.11}$$

Combining (2.5) and (2.10), we get

$$\upsilon' \leq -k\theta^{1/\kappa}\eta_{n-3}\upsilon\left(\int_{t_0}^{t} q(h)\mathrm{d}h\right)^{1/\kappa}.$$

This implies

$$\begin{aligned}\eta_{n-2}^{\delta}\upsilon' + \delta\eta_{n-2}^{\delta-1}\eta_{n-3}\upsilon &\leq -\theta^{1/\kappa}\eta_{n-3}\eta_{n-2}^{\delta}\upsilon\left(k\int_{t_0}^{t} q(h)\mathrm{d}h\right)^{1/\kappa} + \delta\eta_{n-2}^{\delta-1}\eta_{n-3}\upsilon \\ &= \left(-\theta^{1/\kappa}\eta_{n-2}\left(k\int_{t_0}^{t} q(h)\mathrm{d}h\right)^{1/\kappa} + \delta\right)\eta_{n-2}^{\delta-1}\eta_{n-3}\upsilon.\end{aligned}$$

It follows from (2.4) that

$$\eta_{n-2}^{\delta}\upsilon' + \delta\eta_{n-2}^{\delta-1}\eta_{n-3}\upsilon \leq 0,$$

which with (2.11) implies that $\left(\upsilon/\eta_{n-2}^{\delta}\right)' \leq 0$. Hence, the proof is complete. □

Theorem 2.4. *Suppose that* (2.1) *holds. If there exist a function*

$$\rho \in \mathbf{C}^1\left([t_0,\infty),(0,\infty)\right)$$

and a constant $\delta \geq 0$ *such that* (2.4) *holds and*

$$\limsup_{t\to\infty} \frac{\eta_{n-2}^{\kappa}(t)}{\rho(t)} \int\limits_{t_0}^{t} \left(\rho(h)\Phi(h) - \frac{1}{(\kappa+1)^{\kappa+1}} \frac{(\rho'(h))^{\kappa+1}}{\rho^{\kappa}(h)\eta_{n-3}^{\kappa}(h)}\right) \mathrm{d}h > 1, \tag{2.12}$$

then $\Bbbk = \varnothing$, *where*

$$\Phi(t) := k\theta(t)q(t)\left(\frac{\eta_{n-2}(\sigma(t))}{\eta_{n-2}(t)}\right)^{\delta\beta}.$$

Proof. Assuming the contrary, we let $\upsilon \in \Bbbk$. Using Lemma 2.3, we have that (2.5) and (2.6) hold. Integrating (2.6) from σ to t yields

$$\upsilon(\sigma(t)) \geq \left(\frac{\eta_{n-2}(\sigma(t))}{\eta_{n-2}(t)}\right)^{\delta} \upsilon(t),$$

which with (1.1) gives

$$\left(b(\upsilon^{(n-1)})^{\kappa}\right)' \leq -kq\left(\frac{\eta_{n-2}(\sigma)}{\eta_{n-2}}\right)^{\delta\beta} \upsilon^{\beta}. \tag{2.13}$$

Next, we define the function

$$\psi := \rho\left(b\left(\frac{\upsilon^{(n-1)}}{\upsilon}\right)^{\kappa} + \frac{1}{\eta_{n-2}^{\kappa}}\right). \tag{2.14}$$

Thus, $\psi(t) > 0$. From (2.5), (2.13) and (2.14), we obtain

$$\psi' \leq \frac{\rho'}{\rho}\psi - k\left(\frac{\eta_{n-2}(\sigma)}{\eta_{n-2}}\right)^{\delta\beta}\theta\rho q - \frac{\kappa\eta_{n-3}}{\rho^{1/\kappa}}\left(\psi - \frac{\rho}{\eta_{n-2}^{\kappa}}\right)^{\frac{\kappa+1}{\kappa}} + \rho\frac{\kappa\eta_{n-3}}{\eta_{n-2}^{\kappa+1}}.$$

Using Lemma 1.2 in [15] with $D = \rho'/\rho$, $E = \kappa\eta_{n-3}\rho^{-1/\kappa}$, $\phi = \rho/\eta_{n-2}^{\kappa}$ $\alpha = \kappa$ and $\nu = \psi$, we find

$$\psi' \leq \frac{(\rho')^{\kappa+1}}{(\kappa+1)^{\kappa+1}\rho^{\kappa}\eta_{n-3}^{\kappa}} - \rho\Phi + \left(\frac{\rho}{\eta_{n-2}^{\kappa}}\right)'. \tag{2.15}$$

Integrating (2.15) from t_1 to t, we are led to

$$\int\limits_{t_1}^{t} \left(\rho(h)\Phi(h) - \frac{(\rho'(h))^{\kappa+1}}{(\kappa+1)^{\kappa+1}\rho^{\kappa}(h)\eta_{n-3}^{\kappa}(h)}\right) \mathrm{d}h \leq \left(\frac{\rho(h)}{\eta_{n-2}^{\kappa}(h)} - \psi(h)\right)\Bigg|_{t_1}^{t},$$

which follows from (2.14) that

$$\int_{t_1}^{t}\left(\rho(h)\Phi(h)-\frac{(\rho'(h))^{\kappa+1}}{(\kappa+1)^{\kappa+1}\rho^{\kappa}(h)\eta_{n-3}^{\kappa}(h)}\right)\mathrm{d}h\le\left(\rho(h)b(h)\left(\frac{\upsilon^{(n-1)}(h)}{\upsilon(h)}\right)^{\kappa}\right)\Bigg|_{t}^{t_1}. \tag{2.16}$$

Integrating (2.5) from t to ∞ provides

$$\upsilon(t)\ge -b^{1/\kappa}(t)\upsilon^{(n-1)}(t)\eta_{n-2}(t), \tag{2.17}$$

which with (2.16), gives

$$\int_{t_1}^{t}\left(\rho(h)\Phi(h)-\frac{(\rho'(h))^{\kappa+1}}{(\kappa+1)^{\kappa+1}\rho^{\kappa}(h)\eta_{n-3}^{\kappa}(h)}\right)\mathrm{d}h\le\frac{\rho(t)}{\eta_{n-2}^{\kappa}(t)}.$$

Multiplying thus inequality by η_{n-2}^{κ}/ρ and then taking $\limsup_{t\to\infty}$, we are led to a contradiction. Hence, the proof is complete. □

Theorem 2.5. *Suppose that there is a* $\delta\ge 0$ *such that* (2.1) *and* (2.4) *hold. If the differential equation*

$$\left(\frac{1}{\eta_{n-3}^{\kappa}(t)}\left(\upsilon'(t)\right)^{\kappa}\right)'+\Phi(t)\upsilon^{\kappa}(t)=0 \tag{2.18}$$

is oscillatory, then $\Bbbk=\varnothing$.

Proof. Assuming the contrary, we let $\upsilon\in\Bbbk$. Proceeding as in the proof of Theorem 2.4, we get that (2.5), (2.13) and (2.17) hold. Next, we set

$$w:=b\left(\frac{\upsilon^{(n-1)}}{\upsilon}\right)^{\kappa}<0. \tag{2.19}$$

From (2.13) and (2.19), we conclude that

$$w'\le -k\left(\frac{\eta_{n-2}(\sigma)}{\eta_{n-2}}\right)^{\delta\beta}\theta q-\kappa\frac{b(\upsilon^{(n-1)})^{\kappa}}{\upsilon^{\kappa+1}}\upsilon',$$

which, in view of (2.5), gives

$$w'+\Phi+\kappa\eta_{n-3}w^{(\kappa+1)/\kappa}\le 0. \tag{2.20}$$

From [2], if there is a $w\in\mathbf{C}\left([t_1,\infty),\mathbb{R}\right)$ which satisfies (2.20) for $t\ge t_1\ge t_0$, then equation (2.18) is non-oscillatory, which is a contradiction. Hence, the proof is complete. □

Under the assumption $\eta_{n-2}(t_0)<\infty$, in view of [6, Theorem 3] and [7, Theorem 2.3], we obtain the following criteria for oscillation of (1.1)

Corollary 2.6. *Suppose that there is a* $\delta \geq 0$ *such that* (2.1) *and* (2.4) *hold. If* $\eta_{n-2}(t_0) < \infty$ *and*

$$\limsup_{t\to\infty} \eta_{n-2}^{\kappa}(t) \int_{t_0}^{t} \Phi(\tau)\mathrm{d}\tau > 1 \tag{2.21}$$

or

$$\liminf_{t\to\infty} \frac{1}{\eta_{n-2}(t)} \int_{t}^{\infty} \eta_{n-2}^{\kappa+1}(h)\Phi(h)\mathrm{d}h > \kappa^* \tag{2.22}$$

hold, then $\mathbb{k} = \varnothing$.

Remark 2.7. Combining Theorem 2.5 and the results reported in the paper [5], one can derive various oscillation criteria for equation (1.1). The details are left to the reader.

Remark 2.8. Since m_1 and m_2 are arbitrary, the conditions that include them must be fulfilled for all their values.

3. APPLICATIONS

Combining the results obtained in [21] with existing in the previous section, we provide a new criterion for oscillation of all solutions of equation (1.1).

Theorem 3.1. *If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(h) \frac{\left(\sigma^{n-1}(h)\right)^{\kappa}}{b(\sigma(h))} \mathrm{d}h > \frac{((n-2)!)^{\kappa}}{\mathrm{e}} \tag{3.1}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(q(h) \left(\frac{\mu_1 \eta_0(h) \sigma^{n-2}(h)}{(n-2)!} \right)^{\kappa} - \frac{\kappa^{\kappa+1}}{(\kappa+1)^{\kappa+1}} \frac{1}{\eta_0(h) b^{1/\kappa}(h)} \right) \mathrm{d}h = \infty, \tag{3.2}$$

for some $\mu_1 \in (0,1)$, *then every non-oscillatory solution of* (1.1) *tends to zero.*

Proof. The proof of this theorem is the same as that of [21, Corollary 2.1] when $\phi(\upsilon) = \upsilon^{\kappa}$, and so we omit it. □

Theorem 3.2. *Suppose that* $\phi(\upsilon) := \upsilon^{\kappa}$, (3.1) *and* (3.2) *hold. If there exist a function* $\sigma \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ *and a constant* $\delta \geq 0$ *such that* (2.4) *and* (2.12) *hold, then every solution of* (1.1) *is oscillatory.*

Proof. Assuming the contrary, υ is a non-oscillatory solution of (1.1). Therefore, that there exists an $t_1 \in [t_0,\infty)$ with υ and $\upsilon \circ \sigma$ are positive for $t \geq t_1$. It follows from Lemma 2.1 that υ satisfies one of the cases (i)–(iii). Now, from Theorem 3.1, we see that the cases (i) and (ii) contradict conditions (3.1) and (3.2), respectively. Then, we have that (iii) holds, that is, $\upsilon \in \Bbbk$. From Theorem 2.4, we arrive at a contradiction with (2.12). Hence, the proof is complete. □

Theorem 3.3. *Suppose that* $\phi(\upsilon) := \upsilon^{\kappa}$, (3.1) *and* (3.2) *hold. If there exist a function* $\sigma \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ *and a constant* $\delta \geq 0$ *such that* (2.4) *and* (2.22) *hold, then all solutions of equation* (1.1) *are oscillatory.*

Example 3.4. Consider the delay differential equation

$$(\mathrm{e}^{\kappa t}(\upsilon'''(t))^{\kappa})' + q_0\mathrm{e}^{\kappa t}\upsilon^{\kappa}(t-\sigma_0) = 0, \quad t \geq t_0, \tag{3.3}$$

where $q_0, \sigma_0 > 0$. We note that $\beta = \kappa$, $b(t) = \mathrm{e}^{\kappa t}$, $q(t) = q_0\mathrm{e}^{\kappa t}$, $\phi(\upsilon) = \upsilon^{\kappa}$ and $\sigma(t) = t - \sigma_0$. Thus, we conclude that

$$\eta_{\iota}(t) = \mathrm{e}^{-t} \quad \text{for all } \iota = 0, 1, 2.$$

Then, (3.1) and (3.2) are satisfied. Now, if we choose $\delta := (q_0/\kappa)^{1/\kappa}$ and $\rho(t) = \mathrm{e}^{-\kappa t}$, then we see that (2.4) is satisfied. Moreover, (2.12) holds if

$$q_0\mathrm{e}^{\sigma_0\delta\kappa} > \frac{\kappa^{\kappa+1}}{(\kappa+1)^{\kappa+1}}. \tag{3.4}$$

Thus, from Theorem 3.2, (3.3) is oscillatory if (3.4) holds. Moreover, we can obtain the same criterion (3.4) by using Theorem 3.3.

Remark 3.5. By using [21, Corollary 2.1], (3.3) is oscillatory when $q_0 \in (\kappa^*, \infty)$. Whereas, condition (3.4) becomes $q_0 \in \left(\kappa^*\mathrm{e}^{-\sigma_0\delta\kappa}, \infty\right)$. Since $\mathrm{e}^{\sigma_0\delta\kappa} \geq 1$, we have that Theorem 3.2 presents an improved result for oscillation of equation (3.3). Furthermore, condition (3.4) considers the impact of $\sigma(t)$, which has been neglected in the previous studies [15, 21].

Remark 3.6. Set $\kappa = 1$ and $\sigma_0 = 1$ in (3.3), [3, Corollary 4] showed that equation (3.3) is oscillatory when $q_0 > 2^5/\mathrm{e}$. However, condition (3.4) reduces to $q_0 > 0.20389$, which is better for testing the oscillation of (3.3).

4. CONCLUSION

This paper is concerned with creating improved criteria that ensure that there are no decreasing solutions to a class of even-order delay differential equations. Using these criteria, we obtained new oscillation constraints for the considered equation. Through an example and appropriate remarks, we also noted that the new criterion provides sharper results for oscillation than the related previous results in [3, 15, 20, 21]. It will be interesting to extend these results to the neutral delay differential equations of even-order; see [5, 13] for more details.

REFERENCES

[1] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Marcel Dekker: Kluwer Academic, Dordrecht, 2000.

[2] R.P. Agarwal, S.L. Shieh, C.C. Yeh, *Oscillation criteria for second order retarded differential equations*, Math. Comput. Model. **26** (1997), 1–11.

[3] B. Baculíková, J. Džurina, J.R. Graef, *On the oscillation of higher-order delay differential equations*, J. Math. Sci. (N.Y.) **187** (2012), no. 4, 387–400.

[4] D.D. Bainov, D.P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, New York, NY, USA, 1991.

[5] J. Džurina, S.R. Grace, I. Jadlovská, T. Li, *Oscillation criteria for second-order Emden–Fowler delay differential equations with a sublinear neutral term*, Math. Nachr. **293** (2020), no. 5, 910–922.

[6] J. Džurina, I. Jadlovská, *A note on oscillation of second-order delay differential equations*, Appl. Math. Lett. **69** (2017), 126–132.

[7] J. Džurina, I. Jadlovská, I.P. Stavroulakis, *Oscillatory results for second-order noncanonical delay differential equations*, Opuscula Math. **39** (2019), no. 4, 483–495.

[8] J. Graef, S. Grace, E. Tunç, *Oscillation criteria for even-order differential equations with unbounded neutral coefficients and distributed deviating arguments*, Funct. Differ. Equ. **25** (2018), 143–153.

[9] J.K. Hale, *Theory of Functional Differential Equations*, Spring-Verlag, New York, 1977.

[10] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.

[11] T. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys. **70** (2019), no. 86, 1–18.

[12] T. Li, Yu.V. Rogovchenko, *On asymptotic behavior of solutions to higher-order sublinear Emden–Fowler delay differential equations*, Appl. Math. Lett. **67** (2017), 53–59.

[13] T. Li, Yu.V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett. **105** (2020), 106293, 1–7.

[14] O. Moaaz, E.M. Elabbasy, A. Muhib, *Oscillation criteria for even-order neutral differential equations with distributed deviating arguments*, Adv. Difference Equ. **2019** (2019), no. 297.

[15] O. Moaaz, A. Muhib, *New oscillation criteria for nonlinear delay differential equations of fourth-order*, Appl. Math. Comput. **377** (2020), 125192.

[16] O. Moaaz, C. Park, A. Muhib, O. Bazighifan, *Oscillation criteria for a class of even-order neutral delay differential equations*, J. Appl. Math. Comput. **63** (2020), 607–617.

[17] Ch.G. Philos, *A new criterion for the oscillatory and asymptotic behavior of delay differential equations*, Bull. Acad. Pol. Sci., Ser. Sci. Math. **39** (1981), 61–64.

[18] H. Ramos, O. Moaaz, A. Muhib, J. Awrejcewicz, *More effective results for testing oscillation of non-canonical NDDEs*, Mathematics **9** (2021), 111.

[19] S.H. Saker, *Oscillation Theory of Delay Differential and Difference Equations*, VDM Verlag Dr: Muller Saarbrucken, Saarbrucken, Germany, 2010.

[20] C. Zhang, R.P. Agarwal, M. Bohner, T. Li, *New results for oscillatory behavior of even-order half-linear delay differential equations*, Appl. Math. Lett. **26** (2013), 179–183.

[21] C. Zhang, T. Li, B. Sun, E. Thandapani, *On the oscillation of higher-order half-linear delay differential equations*, Appl. Math. Lett. **24** (2011), 1618–1621.

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Received: June 1, 2022.
Revised: March 4, 2023.
Accepted: March 4, 2023.