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GENERALIZED DERIVATIONS AND GENERALIZED EXPONENTIAL MONOMIALS ON HYPERGROUPS

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Abstract. In one of our former papers *Endomorphisms of the measure algebra of commutative hypergroups* we considered exponential monomials on hypergroups and higher order derivations of the corresponding measure algebra. Continuing with this, we are now looking for the connection between the generalized exponential polynomials of a commutative hypergroup and the higher order derivations of the corresponding measure algebra.

Keywords: moment function, moment sequence, exponential monomial, exponential polynomial, derivation, higher order derivation, hypergroup.

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1. MOMENT FUNCTIONS AND EXPONENTIAL MONOMIALS

In this paragraph we are going to discuss moment functions and exponential monomials in the hypergroup settings. Concerning hypergroups, we will follow the notation and terminology of the monograph [1]. More about moment functions and exponentials on hypergroups one can find in [3] and references therein.

Given a commutative hypergroup X we denote by $\mathscr{C}(X)$ the space of all continuous complex valued functions on X. Equipped with the pointwise linear operations (addition of functions and multiplication by scalars) and with the topology of uniform convergence on compact sets $\mathscr{C}(X)$ is a locally convex topological vector space. The space $\mathscr{C}(X)$ equipped with pointwise multiplication is also a topological ring which will be utilized in the sequel. The topological dual of $\mathscr{C}(X)$ as a topological vector space can be identified with the space $\mathscr{M}_c(X)$ of all compactly supported complex Borel measures with the pairing

$$\langle \mu, f\rangle = \int_X f\, d\mu.$$

In fact, $\mathscr{M}_c(X)$ itself is a commutative algebra equipped with the pointwise linear operations and with the convolution of measures defined by

$$\langle \mu * \nu, f\rangle = \int_X \int_X f(x*y)\,d\mu(x)\,d\nu(y)$$

for each μ, ν in $\mathscr{M}_c(X)$ and f in $\mathscr{C}(X)$. We shall call $\mathscr{M}_c(X)$ the *measure algebra* of X. The measure algebra bears its natural weak*-topology, and its topological dual can be identified with $\mathscr{C}(X)$: every weak*-continuous linear functional Λ on $\mathscr{M}_c(X)$ arises from a continuous function f in $\mathscr{C}(X)$ in the natural way

$$\Lambda(\mu) = \int_X f\,d\mu$$

whenever μ is in $\mathscr{M}_c(X)$ (see e.g. [4, Theorem 3.43]).

Besides the algebra structure $\mathscr{M}_c(X)$ can be considered as a module over the ring $\mathscr{C}(X)$ via the following action: for each μ in $\mathscr{M}_c(X)$ and φ, f in $\mathscr{C}(X)$ we define

$$\langle \varphi \cdot \mu, f\rangle = \int_X f\cdot\varphi\,d\mu.$$

It is easy to check that $\mathscr{M}_c(X)$ is a module over the ring $\mathscr{C}(X)$. We simple write $\varphi\mu$ for $\varphi\cdot\mu$. We shall call *module homomorphism* a mapping $F : \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ if it satisfies the equations

$$F(\mu+\nu) = F(\mu) + F(\nu), \qquad \text{and} \qquad F(\varphi\mu) = \varphi F(\mu)$$

for each μ, ν in $\mathscr{M}_c(X)$ and f in $\mathscr{C}(X)$. We note that every module homomorphism of $\mathscr{M}_c(X)$ is also a linear homomorphism, as multiplication by scalars is the same as multiplication by constant functions. Module homomorphisms of the measure algebra of a commutative hypergroup are described in the recent paper of the present authors [2].

In this paper we use multi-index notation. We recall that, besides the usual vector-notation for the basic operations we use the following notation: for every multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_r)$ in $\mathbb{N}^r$, we shall write $\alpha \le \beta$ whenever $\alpha_i \le \beta_i$ for $i = 1, 2, \dots r$, and the symbol $\alpha < \beta$ means that $\alpha \le \beta$ and $\alpha \ne \beta$. Further, we use the notations

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_r, \quad \alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_r!,$$
$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta! \cdot (\alpha-\beta)!}, \quad x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_r^{\alpha_r}.$$

If there is no misunderstanding, the zero of $\mathbb{N}^r$ will be denoted by 0 instead of $(0, 0, \dots, 0)$.

Let X be a commutative hypergroup and r a positive integer. The family of continuous functions $\varphi_\alpha : X \to \mathbb{C}$ is called a *moment function sequence of rank* r, if

$$\varphi_\alpha(x*y) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \varphi_\beta(x)\varphi_{\alpha-\beta}(y) \tag{1.1}$$

holds whenever x, y are in X and α is in $\mathbb{N}^r$. We may consider finite sequences as well, if we restrict $|\alpha| \leq N$ with some nonnegative integer N. Clearly, the function φ_0 is an exponential, and we shall say that the moment function sequence *corresponds to* the exponential φ_0.

We call a continuous function $\varphi : X \to \mathbb{C}$ a *moment function*, if there is a positive integer r, a moment sequence $(\varphi_\alpha)_{\alpha \in \mathbb{N}^r}$ of rank r, and a multi-index α in $\mathbb{N}^r$ such that $\varphi = \varphi_\alpha$. In this case α is called the *multi-degree* of φ.

Given the commutative hypergroup X and an element y in X, the operator τ_y defined by

$$\tau_y f = \delta_{\check{y}} * f$$

for each f in $\mathscr{C}(X)$ is called the *translation by* y. Given an exponential m the *modified difference corresponding to* m *with increment* y is the measure

$$\Delta_{m;y} = \delta_{\check{y}} - m(y)\delta_o.$$

The corresponding convolution operator can be written as

$$\Delta_{m;y} * f = (\tau_y - m(y)\,\mathrm{id})f.$$

A closed linear subspace in $\mathscr{C}(X)$ is called a *variety*, if it is invariant with respect all translation operators.

Products of modified differences corresponding to f will be written in the form

$$\Delta_{f;y_1,y_2,\ldots,y_{n+1}} = \Delta_{f;y_1} * \Delta_{f;y_2} * \cdots * \Delta_{f;y_{n+1}}$$

whenever $y_1, y_2, \ldots, y_{n+1}$ are in X. It is clear that the continuous function $m : X \to \mathbb{C}$ is an exponential if and only if $m(o) = 1$, and

$$\Delta_{m;y} * m = 0$$

for each y in X.

Given an exponential m on X the continuous function $\varphi : X \to \mathbb{C}$ is called a *generalized* m*-exponential monomial* if there exists a natural number n such that

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * \varphi = 0$$

whenever $y_1, y_2, \ldots, y_{n+1}$ are in X. If φ is nonzero, then the the exponential m and the smallest n with this property are uniquely determined, and n is called the *degree* of φ. Clearly, φ is of degree $n \geq 1$ if and only if $\Delta_{m;y} * \varphi$ is of degree $n-1$ for some y. For instance, the m-exponential monomials of degree 0 are exactly the constant multiples of m. The m-exponential monomials of degree 1, which vanish at o are called m*-sine functions*. A generalized m-exponential monomial is called an m*-exponential monomial* if its variety is finite dimensional.

An important property of moment functions is expressed in the following theorem.

Theorem 1.1. *Let X be a commutative hypergroup and m an exponential on X. Every moment function corresponding to m is an m-exponential monomial.*

Proof. We prove the statement by induction on $|\alpha|$, where α is the multi-degree of the moment function φ of rank r, which corresponds to the exponential m. If $|\alpha| = 0$, then α is the zero of $\mathbb{N}^r$ hence, by (1.1), we infer that $\varphi = m$ is an exponential. Assume that we have proved the statement for each α in $\mathbb{N}^r$ with $|\alpha| \leq N$, and let φ be a moment function of rank r corresponding to the exponential m, having multi-degree α with $|\alpha| = N + 1$. From (1.1) we infer

$$\Delta_{m;y}\varphi_\alpha(x) = \varphi_\alpha(x * y) - m(y)\varphi_\alpha(x) = \sum_{0\leq\beta<\alpha} \binom{\alpha}{\beta} \varphi_\beta(x)\varphi_{\alpha-\beta}(y)$$

for each x, y in X. On the right side we have a linear combination of the moment functions φ_β, each of them having multi-degree $\beta < \alpha$, hence $|\beta| \leq N$. By the induction hypothesis, we have that

$$\begin{aligned}\Delta_{m;y_1,y_2,\ldots,y_{N+1},y}\varphi_\alpha(x) &= \Delta_{m;y_1,y_2,\ldots,y_{N+1}} * \Delta_{m;y}\varphi_\alpha(x) \\ &= \sum_{0\leq\beta<\alpha} \binom{\alpha}{\beta} \Delta_{m;y_1,y_2,\ldots,y_{N+1}} * \varphi_\beta(x) \cdot \varphi_{\alpha-\beta}(y) = 0,\end{aligned}$$

hence φ_α is a generalized m-exponential monomial of degree at most $N + 1$. By (1.1), the variety of every moment function is finite dimensional, as it is spanned by the finitely many functions φ_β with $0 \leq \beta \leq \alpha$, every moment function is an exponential monomial. □

In fact, as an additional information we have proved that the multi-degree α of a moment function φ of rank r satisfies $\deg\varphi \leq |\alpha|$, where $\deg\varphi$ denotes its degree, as an exponential monomial. We note that, in general, we do not have equality here: for instance, the zero function may have any nonzero multi-degree, as a moment function.

Now we are going to give an example of a hypergroup, which is not a group, where generalized m-exponential monomials exist. Let us consider the hypergroup joins $C \vee D$. The construction and properties of a hypergroup joins can be found e.g. in [5]. We consider a particular case, where the compact part C is the two-point hypergroup $D(\theta) = \{0, i\}$ and the discrete part D is $\mathbb{R}$ endowed with the discrete topology. Using [5, Theorem 7] we know the form of generalized exponential monomials on $D(\theta) \vee \mathbb{R}$. More precisely: a continuous function f on $D(\theta) \cup \mathbb{R}$ is a generalized exponential monomial of degree at most n if and only if one of the following cases holds:

(1) $f|_{D(\theta)}$ is a generalized exponential monomial of degree at most n associated with an exponential $m|_{D(\theta)} \neq 1$ on $D(\theta)$ and $f|_{\mathbb{R}} = 0$,
(2) $f|_{D(\theta)}$ is constant and $f|_{\mathbb{R}}$ is a generalized exponential monomial on $\mathbb{R}$.

This means that there exists a generalized exponential monomials on $D(\theta) \vee \mathbb{R}$, which are not exponential monomials, as on $\mathbb{R}$ there exist nontrivial generalized exponential monomial. For more detailed discussion see [3] and references therein.

2. GENERALIZED DERIVATIONS

In our former paper [2] we investigated the relation between moment function sequences and higher order derivations on the measure algebra. We proved that there is a close connection between these two concepts, especially when we assume that the derivations are not just linear operators on the measure algebra, but they are also module homomorphisms if the measure algebra is considered as a module over the ring of continuous functions. As moment functions are special exponential monomials, it is reasonable to ask if it is possible to extend the concept of higher order derivations in a way such that this extension relates to generalized exponential monomials. We present a possible answer this question.

Let X be a commutative hypergroup and r a positive integer. The measure algebra $\mathscr{M}_c(X)$ will be considered as a module over the ring $\mathscr{C}(X)$ of continuous complex valued functions on X with the action

$$\langle \varphi \cdot \mu, f \rangle = \int_X f \cdot \varphi \, d\mu.$$

We shall write simply $\varphi\mu$ for $\varphi \cdot \mu$.

We recall (see [2]) that the family of module homomorphisms $(D_\alpha)_{\alpha \in \mathbb{N}^r}$ on $\mathscr{M}_c(X)$ is called a *higher order derivation of rank* r, or simply *higher order derivation*, if for each α in $\mathbb{N}^r$ we have

$$D_\alpha(\mu * \nu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D_\beta \mu * D_{\alpha-\beta} \nu \tag{2.1}$$

whenever μ, ν are in $\mathscr{M}_c(X)$. We underline that the module homomorphism property means that each D_α satisfies the two functional equations

$$D_\alpha(\mu + \nu) = D_\alpha \mu + D_\alpha \nu,$$
$$D_\alpha(\varphi\mu) = \varphi D_\alpha \mu$$

for each μ, ν in $\mathscr{M}_c(X)$ and φ in $\mathscr{C}(X)$. We say that this higher order derivation is *continuous*, if each operator D_α is continuous with respect to the weak*-topology on $\mathscr{C}(X)$. We note that D_0 is not just a module homomorphism, but it is also an algebra homomorphism, that is $D_0(\mu * \nu) = D_0\mu * D_0\nu$. We shall say that D_0 is a *multiplicative module homomorphism*, and that this higher order derivation *corresponds to* D_0.

We note that our terminology is somewhat different from the usual one. Namely, in general a linear operator $D : \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is called a *derivation*, if we have $D(\mu * \nu) = \mu * D\nu + D\mu * \nu$ for each μ, ν in $\mathscr{M}_c(X)$. So, in the above definition, for the case $|\alpha| = 1$ instead of homogeneity with respect to multiplication by complex numbers we require homogeneity with respect to multiplication by continuous functions, and instead of the identity operator id we use an arbitrary multiplicative module homomorphism D_0. This leads to the equation

$$D(\mu * \nu) = D_0\mu * D\nu + D\mu * D_0\nu.$$

Now we introduce a more general concept of derivations on $\mathscr{M}_c(X)$ recursively. Let r a positive integer. For each α with $|\alpha| = 1$ in $\mathbb{N}^r$, we say that the module homomorphism $D : \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is a *generalized derivation of order* α, if there exists a multiplicative module homomorphism D_0 of $\mathscr{M}_c(X)$ such that

$$D(\mu * \nu) = D_0\mu * D\nu + D\mu * D_0\nu$$

for each μ, ν in $\mathscr{M}_c(X)$. We say that D *corresponds to* D_0. Assume that α is in $\mathbb{N}^r$, and we have defined generalized derivations corresponding to the multiplicative module homomorphism D_0 of order β for each $\beta < \alpha$ in $\mathbb{N}^r$. Then the module homomorphism $D : \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is called a *generalized derivation of order* α corresponding to the multiplicative module homomorphism D_0, if the two variable function

$$(\mu, \nu) \mapsto D(\mu * \nu) - D_0\mu * D\nu - D\mu * D_0\nu$$

is a generalized derivation of order less than α corresponding to D_0 in both variables, when the other variable is fixed. We extend this terminology for the case $\alpha = 0$ by saying that any multiplicative module homomorphism is a generalized derivation of order zero – corresponding to itself. We note that we may call these generalized derivations of rank r, if we want to underline the role of r in the definition.

First we show that this concept is indeed a generalization of the concept of higher order derivations.

Theorem 2.1. *Let X be a commutative hypergroup, r a positive integer, and assume that the sequence of functions $(D_\alpha)_{\alpha\in\mathbb{N}^r}$ is a higher order derivation of rank r on $\mathscr{M}_c(X)$. Then D_α is a generalized derivation of order α, for each α in $\mathbb{N}^r$.*

Proof. We prove this statement by induction on $|\alpha|$. For $|\alpha| = 1$,we assume that $D_\alpha \colon \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is a derivation of order α corresponding to the multiplicative module homomorphism D_0. This means that we have

$$D_\alpha(\mu * \nu) = D_0\mu * D_\alpha\nu + D_\alpha\mu * D_0\nu$$

for each μ, ν in $\mathscr{M}_c(X)$. In other words,

$$D_\alpha(\mu * \nu) - D_0\mu * D_\alpha\nu - D_\alpha\mu * D_0\nu = 0$$

for each μ, ν in $\mathscr{M}_c(X)$, and this is obviously a generalized derivation of order zero in both variables.

Assume now that there exists an α in $\mathbb{N}^r$ with the property that the statement holds for all multi-indices less then α. Let further $D_\alpha \colon \mathscr{M}_c(X) \times \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ be a generalized derivation of order α. Then

$$D_\alpha(\mu * \nu) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D_\beta\mu * D_{\alpha-\beta}\nu$$

for each μ, ν in $\mathscr{M}_c(X)$. After some rearrangement, we get that

$$D_\alpha(\mu * \nu) - D_0\mu * D_\alpha\nu - D_\alpha\mu * D_0\nu = \sum_{0<\beta<\alpha} \binom{\alpha}{\beta} D_\beta\mu * D_{\alpha-\beta}\nu$$

for each μ, ν in $\mathscr{M}_c(X)$. Here the right side is a symmetric mapping and, due to the induction hypothesis, it is a generalized derivation of order at most $\beta < \alpha$ in both variables. This proves our statement. □

Theorem 2.2. *Let X be a commutative hypergroup, r a positive integer and α in $\mathbb{N}^r$. Given a continuous generalized derivation D of order α on $\mathscr{M}_c(X)$ corresponding to the continuous multiplicative module homomorphism D_0 we define*

$$m(x) = \langle D_0\delta_x, 1\rangle \quad \textit{and} \quad \varphi(x) = \langle D\delta_x, 1\rangle \tag{2.2}$$

for each x in X. Then m is an exponential, and φ is a generalized m-exponential monomial of degree $|\alpha|$, further we have

$$\langle D_0\mu, f\rangle = \int_X f \cdot m\, d\mu \quad \textit{and} \quad \langle D\mu, f\rangle = \int_X f \cdot \varphi\, d\mu \tag{2.3}$$

for each μ in $\mathscr{M}_c(X)$ and f in $\mathscr{C}(X)$.

Proof. The continuity of m and φ follows easily from the continuity of D_0 and D. For each x, y in X we have

$$\begin{aligned} m(x*y) = \langle D_0\delta_{x*y}, 1\rangle &= \int_X 1\, dD_0\delta_{x*y} = \int_X 1\, dD_0(\delta_x * \delta_y) \\ &= \int_X\int_X 1\, dD_0\delta_x\, dD_0\delta_y = \int_X 1\, dD_0\delta_x \cdot \int_X 1\, dD_0\delta_y = m(x)m(y). \end{aligned}$$

As D_0 is nonzero, we have $m(o) = 1$, hence m is an exponential.

The statement about φ will be proved by induction on $|\alpha|$, and it is true for $|\alpha| = 0$. Suppose that the statement holds for $|\alpha| \le N$, and now we prove it for an arbitrary multi-index α' with $|\alpha'| = N + 1$. Let $x, y, y_1, y_2, \dots, y_{N+1}$ be any elements in X, and we show that

$$\Delta_{m;y_1,y_2,\dots,y_{N+1},y} * \varphi(x) = \Delta_{m;y_1,y_2,\dots,y_{N+1}} * \Delta_{m;y} * \varphi(x) = 0.$$

We let $\psi(x) = \varphi(x*y) - m(y)\varphi(x)$, then it is enough to show that

$$\Delta_{m;y_1,y_2,\dots,y_{N+1}} * \psi(x) = 0.$$

We introduce the notation

$$B(\mu,\nu) = D(\mu*\nu) - D_0\mu * D\nu - D\mu * D_0\nu$$

whenever μ, ν are in $\mathscr{M}_c(X)$.

Then B is a generalized derivation of order at most α in both variables, and we can compute as follows:

$$\begin{aligned}\psi(x) &= \varphi(x*y) - m(y)\varphi(x)\\ &= \langle D\delta_{x*y} - D_0\delta_y * D\delta_x, 1\rangle\\ &= \langle D\delta_x * D\delta_y - D_0\delta_y * D\delta_x, 1\rangle\\ &= \langle D\delta_y * D_0\delta_x, 1\rangle + \langle B(\delta_x,\delta_y),1\rangle\\ &= \int_X 1\, d(D\delta_y * D_0\delta_x) + \langle B(\delta_x,\delta_y),1\rangle\\ &= \int_X\int_X 1\, dD\delta_y\, dD_0\delta_x + \langle B(\delta_x,\delta_y),1\rangle\\ &= \int_X 1\, dD\delta_y \int_X 1\, dD_0\delta_x + \langle B(\delta_x,\delta_y),1\rangle\\ &= \varphi(y)m(x) + \langle B(\delta_x,\delta_y),1\rangle.\end{aligned}$$

The function $x \longmapsto \langle B(\delta_x,\delta_y),1\rangle$ is a generalized exponential monomial of degree less than $|\alpha'|$, hence its degree is at most N. It follows that ψ is a generalized exponential monomial of degree less than N. We conclude that

$$\Delta_{m;y_1,y_2,\dots,y_{N+1},y} * \varphi(x) = \Delta_{m;y_1,y_2,\dots,y_{N+1}} * \psi(x) = 0,$$

which proves that φ a generalized exponential monomial of degree $N+1$.

To prove (2.3), observe that the mapping $\mu \mapsto \langle D_0\mu, 1\rangle$ is a weak*-continuous linear functional on $\mathscr{M}_c(X)$, hence it arises from a function φ_0 in $\mathscr{C}(X)$ in the following way:

$$\langle D_0\mu, 1\rangle = \int_X \varphi_0\, d\mu$$

for each μ in $\mathscr{M}_c(X)$. For $\mu = \delta_x$ this gives

$$\varphi_0(x) = \int_X \varphi\, d\delta_x = \langle D_0\delta_x, 1\rangle = m(x).$$

Similarly, we obtain the second part of (2.3), and the proof is complete. □

Proposition 2.3. *Let X be a commutative hypergroup and $F\colon \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ be a continuous module homomorphism such that $\mathscr{M}_c(X)\big/\ker(F)$ is finite dimensional and define the continuous function $f\colon X \to \mathbb{C}$ by*

$$f(x) = \langle F\delta_x, 1\rangle$$

for each $x \in X$. Then the variety $\tau(f)$ is finite dimensional.

Proof. Suppose that we are given a continuous module homomorphism $F\colon \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ such that $\mathscr{M}_c(X)\big/\ker(F)$ is finite dimensional. Due to the Fundamental Theorem of Homomorphisms, we immediately get that the range of F is of finite dimension, since we have $\operatorname{rng}(F) \simeq \mathscr{M}_c(X)\big/\ker(F)$.

Thus there exists a positive integer n and there exists continuous functions $F_i\colon \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ such that

$$F(\mu) = \sum_{i=1}^{n} F_i(\mu)$$

for all μ in $\mathscr{M}_c(X)$. Especially, for all $x \in X$ we have

$$F(\delta_x) = \sum_{i=1}^{n} F_i(\delta_x)$$

and also for all x, y in X

$$F(\delta_x * \delta_y) = \sum_{i=1}^{n} G_i(\delta_y) F_i(\delta_x).$$

Consider the function $f \in \mathscr{C}(X)$ defined by

$$f(x) = \langle F\delta_x, 1\rangle \quad (x \in X)\,.$$

Then for all x, y in X we have

$$\begin{aligned}
\tau_y * f(x) = f(x*y) = \langle F(\delta_x * \delta_y), 1\rangle &= \Big\langle \sum_{i=1}^{n} F_i(\delta_x) * G_i(\delta_y), 1 \Big\rangle \\
&= \sum_{i=1}^{n} \langle F_i(\delta_x) * G_i(\delta_y), 1\rangle = \sum_{i=1}^{n} \int_X 1 d\left(F_i(\delta_x) * G_i(\delta_y)\right) \\
&= \sum_{i=1}^{n} \int_X \int_X 1 dF_i(\delta_x) dG_i(\delta_y) = \sum_{i=1}^{n} \int_X 1 dF_i(\delta_x) \cdot \int_X 1 dG_i(\delta_y) \\
&= \sum_{i=1}^{n} f_i(x) g_i(y),
\end{aligned}$$

where the continuous functions $f_i, g_i\colon X \to \mathbb{C}$, $i = 1, \ldots, n$ are defined through

$$f_i(x) = \langle F_i\delta_x, 1\rangle = \int_X 1 dF_i(\delta_x) \quad \text{and} \quad g_i(x) = \langle G_i\delta_x, 1\rangle = \int_X 1 dG_i(\delta_x)$$

for all x in X. This means however that all the translates of the function f belong to a finite dimensional linear space. So the variety $\tau(f)$ is finite dimensional. $\square$

Theorem 2.4. *Let X be a commutative hypergroup, r a positive integer and α in $\mathbb{N}^r$. Given a continuous generalized derivation F of order α on $\mathscr{M}_c(X)$ corresponding to the continuous multiplicative module homomorphism D_0, the following statements are equivalent.*

(i) $\mathscr{M}_c(X)\,/\ker(F)$ *is finite dimensional.*
(ii) *The mapping F is a higher order derivation of rank r with degree α.*

Proof. Firstly we prove the direction (ii)⇒(i). Accordingly, let X be a commutative hypergroup, r a positive integer and α in $\mathbb{N}^r$. Assume further that F is a higher order derivation of rank r with degree α on $\mathscr{M}_c(X)$ corresponding to the continuous multiplicative module homomorphism D_0. Then for all $\beta \in \mathbb{N}^r$ there exists a continuous module homomorphism D_β on $\mathscr{M}_c(X)$ such that $F = D_\alpha$ and for all $\beta \in \mathbb{N}^r$ we have

$$D_\beta(\mu * \nu) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D_\gamma(\mu) * D_{\beta-\gamma}(\nu)$$

for all μ, ν in $\mathscr{M}_c(X)$. Especially, we have that for all $\mu \in \mathscr{M}_c(X)$

$$F(\mu) = D_\alpha(\mu) = D_\alpha(\mu * \delta_o) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D_\beta(\mu) * D_{\alpha-\beta}(\delta_o)$$

holds. In other words,

$$\operatorname{rng}(F) \subseteq \operatorname{lin}\{D_\beta \mid \beta \leq \alpha\}.$$

At the same time, due to the Fundamental Theorem of Homomorphisms

$$\operatorname{rng}(F) \simeq \mathscr{M}_c(F)\,/\ker(F).$$

Further the linear space $\operatorname{lin}\{D_\beta \mid \beta \leq \alpha\}$ is obviously of finite dimension.
Thus $\mathscr{M}_c(F)\,/\ker(F)$ is finite dimensional.

Finally, we consider the direction (i) ⇒ (ii). Suppose now that X is a commutative hypergroup, r a positive integer and α in $\mathbb{N}^r$. Let further F be a continuous generalized derivation of order α on $\mathscr{M}_c(X)$ corresponding to the continuous multiplicative module homomorphism D_0 so that $\mathscr{M}_c(X)\,/\ker(F)$ is finite dimensional. We prove the statement by induction on the height of the multi-index α. If $|\alpha| = 1$, then we have

$$F(\mu * \nu) = D_0(\mu) * F(\nu) + F(\mu) * D_0(\nu)$$

for all $\mu, \nu \in \mathscr{M}_c(X)$ showing that the statement holds trivially in case $|\alpha| = 1$. Assume now that there exists a multi-index α such that the statement holds for all multi-indices β that are (strictly) less then α. In this case the mapping D defined on $\mathscr{M}_c(X) \times \mathscr{M}_c(X)$ by

$$D(\mu, \nu) = F(\mu * \nu) - D_0(\mu) * F(\nu) - F(\mu) * D_0(\nu) \qquad (\mu, \nu \in \mathscr{M}_c(X))$$

is symmetric and it is a generalized derivation of degree (strictly) less than α in each of its variables.

On the other hand, due to the Fundamental Theorem of Homomorphisms, we have

$$\mathscr{M}_c(X) \big/ \ker(F) \simeq \operatorname{rng}(F)$$

and due to condition (i), $\mathscr{M}_c(X) \big/ \ker(F)$ or equivalently, $\operatorname{rng}(F)$ is finite dimensional. Thus for all fixed $\nu^* \in \mathscr{M}_c(X)$, the values of the mapping

$$\mathscr{M}_c(X) \ni \mu \longmapsto D(\mu, \nu^*)$$

belong to this finite dimensional algebra. Similarly, for all fixed $\mu^* \in \mathscr{M}_c(X)$, the values of the mapping

$$\mathscr{M}_c(X) \ni \nu \longmapsto D(\mu^*, \nu)$$

belong to this finite dimensional algebra. Therefore there exists a multi-index $\widetilde{\alpha} \in \mathbb{N}^r$ and there exist selfmappings F_β, G_β of $\mathscr{M}_c(X)$ for all multi-index $\beta \leq \widetilde{\alpha}$ such that

$$D(\mu, \nu) = \sum_{\beta \leq \widetilde{\alpha}} F_\beta(\mu) * G_\beta(\nu)$$

holds for all μ, ν in $\mathscr{M}_c(X)$. Due to the induction hypothesis, this mapping is also a symmetric function which is a generalized derivation of degree (strictly) less than α in each of its variables. However, this is only possible if

$$D(\mu, \nu) = \sum_{\beta < \alpha} \binom{\alpha}{\beta} D_\beta(\mu) * D_{\alpha - \beta}(\nu)$$

for all μ, ν in $\mathscr{M}_c(X)$, with an appropriate higher order derivation $(D_\alpha)_{\alpha \in \mathbb{N}^r}$. □

Theorem 2.5. *Let X be a commutative hypergroup, n – a natural number, m – an exponential, and φ – a generalized exponential monomial of degree n. Then there exists a continuous multiplicative module homomorphism D_0, and a continuous generalized derivation D of order n corresponding to D_0 such that* (2.2) *holds for each x in X.*

Proof. We define D_0 and D in the obvious way, as given in (2.3). Then it is a routine calculation to show that $\mu \longmapsto \langle D_0 \mu, 1 \rangle$ is a multiplicative linear functional of the measure algebra $\mathscr{M}_c(X)$. Hence D_0 is a continuous multiplicative module homomorphism of $\mathscr{M}_c(X)$. Then the first equation of (2.2) follows. We prove the second equation of (2.2) by induction on n. For $n = 0$ the statement is equivalent to the first equation in (2.2). Now we assume $n \geq 1$ and we suppose that the second equation of (2.2) defines a continuous generalized derivation of degree k for each generalized m-exponential monomial φ of degree k, whenever $k = 0, 1, \ldots, n$. Now let φ be a generalized m-exponential monomial φ of degree $n + 1$, and we define

$$\langle B(\mu, \nu), 1 \rangle = \int_X \int_X (\varphi(x * y) - m(y)\varphi(x) - m(x)\varphi(y))\, d\mu(x)\, d\nu(y)$$

whenever μ, ν are in $\mathscr{M}_c(X)$. Clearly, $B : \mathscr{M}_c(X) \times \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is a continuous module homomorphism in both variables – in fact, it is a continuous generalized derivation of order at most n, by the induction hypothesis, as the integrand is a generalized m-exponential monomial of degree at most n, in both variables. On the other hand, for each μ, ν in $\mathscr{M}_c(X)$ we have

$$\langle D(\mu * \nu) - D_0\mu * D\nu - D\mu * D_0\nu, 1\rangle = \langle B(\mu, \nu), 1\rangle.$$

Using the homogeneity with respect to multiplication by continuous functions of all these operators in their arguments we get

$$\langle D(\mu * \nu) - D_0\mu * D\nu - D\mu * D_0\nu, f\rangle = \langle B(\mu, \nu), f\rangle.$$

for each f in $\mathscr{C}(X)$. This proves that $D : \mathscr{M}_c(X) \to \mathscr{M}_c(X)$ is a continuous generalized derivation of order at $n + 1$, and our proof is complete. □

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