Opuscula Math. 43, no. 4 (2023), 507–546
https://doi.org/10.7494/OpMath.2023.43.4.507 Opuscula Mathematica

PERIODIC, NONPERIODIC, AND CHAOTIC SOLUTIONS FOR A CLASS OF DIFFERENCE EQUATIONS WITH NEGATIVE FEEDBACK

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Communicated by Petr Stehlík

Abstract. We study the scalar difference equation

$$x(k+1) = x(k) + \frac{f(x(k-N))}{N},$$

where f is nonincreasing with negative feedback. This equation is a discretization of the well-studied differential delay equation

$$x'(t) = f(x(t-1)).$$

We examine explicit families of such equations for which we can find, for infinitely many values of N and appropriate parameter values, various dynamical behaviors including periodic solutions with large numbers of sign changes per minimal period, solutions that do not converge to periodic solutions, and chaos. We contrast these behaviors with the dynamics of the limiting differential equation. Our primary tool is the analysis of return maps for the difference equations that are conjugate to continuous self-maps of the circle.

Keywords: difference equation, negative feedback, circle map.

Mathematics Subject Classification: 39A12, 39A23, 39A33.

1. INTRODUCTION

In this paper we study the difference equation

$$x(k+1) = x(k) + \frac{f(x(k-N))}{N}, \tag{E_N}$$

where f is bounded above, continuous, and satisfies the negative feedback condition $uf(u) < 0$ for all $u \neq 0$. Solutions of this equation can be viewed as the points of an Euler-method approximation for the differential delay equation

$$x'(t) = f(x(t-1)) \tag{E_∞}$$

with step size $1/N$. Equation (E_∞) has been intensively studied for many decades; see [3] and [6] for background. We record here some landmark results (all of which apply or extend to more general equations than (E_∞)). Assume that f is bounded above and continuous with negative feedback.

(a) If f is differentiable at 0 with $f'(0) < -\pi/2$, then the equilibrium solution of (E_∞) at 0 is unstable and there exists a nontrivial *slowly oscillating periodic solution*, that is, a periodic solution p whose successive zeros are more than 1 time unit apart [19].
(b) If f is smooth and strictly decreasing, then any non-constant periodic solution p of Equation (E_∞) "oscillates simply" in the sense that, if $z_0 < z_1 < z_2$ are three successive zeros of p, then $z_2 - z_0$ is the minimal period of p ([17] and [16], going back to ideas in [9]).
(c) If f is smooth and strictly decreasing and 0 is a hyperbolic equilibrium for Equation (E_∞), then the ω-limit set of any solution consists either of the equilibrium at 0 or of a single non-constant periodic orbit ([17] and [16], going back to ideas in [9]).

If f is not strictly decreasing, then it is well known that (b) and (c) need not hold: there can be periodic solutions with large numbers of zeros per minimal period, and "chaotic" solution semiflows are possible [20, 21].

In this paper, we shall be working mostly with cases where f is merely Lipschitz and non-increasing, rather than smooth and strictly decreasing. In this case, minimal modification of the necessary work in [17] shows that point (b) still holds.

Proposition 1.1. *Suppose that f is Lipschitz continuous, bounded above, nonincreasing, and satisfies the negative feedback condition. Then any non-constant periodic solution p of Equation (E_∞) "oscillates simply" in the sense that, if $z_0 < z_1 < z_2$ are three successive zeros of p, then $z_2 - z_0$ is the minimal period of p.*

Remark 1.2. Dynamically, the main effect of relaxing the strict monotonicity of f is to create the possibility of distinct solutions of (E_∞) "running together" and so coinciding after finite time. It seems plausible that point (c) can be extended to situations where f is merely non-increasing, though we have not done this.

Our main theorem for this paper is the following.

Theorem 1.3. *Assume that f is Lipschitz continuous, bounded above, and nonincreasing with negative feedback.*

(I) *For $N = 1$, f can be chosen so that 0 is an unstable equilibrium for Equation (E_N), but Equation (E_N) has no nontrivial periodic solution.*
(II) *For infinitely many $N \in \mathbb{N}$, f can be chosen so that Equation (E_N) has a nontrivial periodic solution with more than two sign changes per minimal period.*

(III) *For infinitely many $N \in \mathbb{N}$, f can be chosen so that Equation (E_N) has a solution that does not approach any periodic or equilibrium solution.*

(IV) *For infinitely many $N \in \mathbb{N}$, f can be chosen so that there is a subset of the phase space that contains infinitely many distinct periodic orbits for Equation (E_N), as well as a dense orbit for Equation (E_N).*

Point (I) of Theorem 1.3 is in contrast to point (a) above for Equation (E_∞). (It seems plausible that statement (I) might also hold when $N > 1$, but we have not proven this.) Point (II) is in contrast to point (b) and Proposition 1.1. Points (III) and (IV) are in contrast to point (c) (though we emphasize that, since the hypotheses on f are not exactly the same for all three of (c), (III), and (IV), the contrast here is somewhat less vivid). Point (IV) means that there is a nontrivial set on which Equation (E_N) is in some sense "chaotic".

Theorem 1.3 shows that Equation (E_N) can exhibit dynamics, even for large N, not present in the "limiting" Equation (E_∞). It is, of course, a commonplace to observe that discrete systems can be dynamically very distinct from their continuous counterparts (the one-dimensional logistic difference and differential equations are an especially famous pair of examples of this kind). There is, in addition, a more specific reason why Theorem 1.3 might not be too surprising. Consider a parameterized family of smooth maps on $\mathbb{R}^n$, all with a fixed point and Jacobian J at the origin. As a dominant pair of eigenvalues of J crosses the unit circle outward, a so-called Neimark–Sacker bifurcation can occur, whereby a stable invariant (topological) circle emerges near the origin (see, for example, Section 4.6 of [14] for an overview of this phenomenon). It seems plausible that a detailed analysis of this bifurcation for Equation (E_N) (which might be quite challenging) could allow us to identify cases where the dynamics on this invariant circle include periodic points of very long period, nonperiodic solutions, or chaotic solutions; in such situations, even if the invariant circle in some sense approximated a periodic orbit of a corresponding continuous-time system, the qualitative behaviors of *individual solutions* on the invariant circle might be quite distinct from that of the continuous orbit. (We emphasize that this is not the approach we take here; while our work also hinges on circle maps, the circles arise in a different way.)

Our main tool for studying Equation (E_N) will be the analysis of certain return maps: loosely speaking, maps that take initial conditions in some subset of the phase space and output the states of the corresponding solutions when they return to that subset. Such maps are prominent in the study of Equation (E_∞). The direct analogs of the most familiar return maps for Equation (E_∞), however, fail to be continuous when we are studying Equation (E_N); to remedy this, we need to redefine the return maps on certain topological quotients of their domains. In our examples, the topological differences between these domains and their quotients drives the distinction between Equations (E_∞) and (E_N). For example, we shall see that the global dynamics of Equation (E_N) are in some cases closely akin to an irrational rotation of the circle, and this can't really happen for Equation (E_∞).

The interest in the examples we give resides, in our view, in their relatively explicit nature: we will be able to clearly understand the sources of the dynamical behaviors we are interested in and the mechanism by which they "disappear" in the continuous limit.

(Roughly speaking, the part of the phase space where the dynamics of interest can occur becomes, in our examples, smaller as N gets larger; if solutions are only known with a certain resolution then the distinctive behaviors of Equation (E_N) become undetectable.) We also hope that our quotient-based approach to defining "discrete" return maps might prove useful in other settings.

Other authors have studied equations similar to or encompassing (E_N), often with an eye to comparison with Equation (E_∞). Several of these works focus on the effectiveness (or ineffectiveness) of various discretizations in capturing dynamical features of the continuous equation. For example, in [7] in 't Hout and Lubich show that stable hyperbolic periodic orbits of equations of the form $x'(t) = f(x(t), x(t-1))$ are well-approximated by attractive invariant curves for Runge–Kutta approximations of the equation. On the other hand, in [8] Ivanov gives examples of equations $\varepsilon[\dot{x}(t) + a\dot{x}(t-1)] + x(t) = f(x(t-1))$ that have Euler-method discretizations (of arbitrarily small step size) possessing stable periodic solutions whose continuous "counterparts" are not stable; see also the related [2].

In [15], Li, Zhao, and Zhang prove chaotic behavior for an Euler-method discretization of an equation of the form $x'(t) = -\mu x(t) + \gamma x(t-1)(1 - x^2(t-1))$.

In the pair of papers [12] and [13], Koto demonstrates that Neimark–Sacker bifurcation (mentioned above) occurs for Euler-method discretizations for a class of equations of the form (E_∞) (a similar type of result is obtained by Ding and Li in [4]). In [13] Koto also observes that single orbits of the discretized system sometimes seem to fill out the Neimark–Sacker invariant curves, and sometimes do not; this phenomenon is partially explained by a rigourous analysis of a family of periodic solutions of the discretization that appear to lie on the Neimark–Sacker invariant curve. In discussing the distinct dynamical possibilities for solutions of the discrete system on an invariant curve, [13] is similar in spirit to our work here.

In [5], Garab and Pötzsche develop a non-increasing "oscillation speed" for a class of difference equations of the form $x(k+1) = g(x(k), x(k-N))$, and use this oscillation speed to establish a Morse decomposition; we shall make rudimentary use of this oscillation speed below.

The paper is organized as follows. In Section 2 we discuss some basic theory for Equation (E_N). Much of this theory is both elementary and familiar, motivated as it is by comparison with Equation (E_∞); but we also discuss the less-familiar quotient-space return maps mentioned above. In Sections 3, 4, and 5 we introduce a particular family of equations, and use this family to prove Theorem 1.3. Section 6 is an Appendix where we establish some basics about the topological quotients that we shall study.

2. BASIC THEORY FOR EQUATION (E_N)

We will frame the initial value problem for Equation (E_N) as follows: we take as an initial condition the $N+1$ real numbers $x(0), x(1), \ldots, x(N-1), x(N)$. The corresponding solution of (E_N) is the (unique) sequence $x = (x(k))_{k=0}^{\infty}$ given by

$$x(k+1) = x(k) + \frac{f(x(k-N))}{N} \quad \text{for all } k \geq N.$$

Throughout this entire paper, we shall write $n = N + 1$. If $x = (x(k))_{k=0}^{\infty}$ is any solution of Equation (E_N), we write $\mathbf{x}_k$ for the vector

$$\mathbf{x}_k = \begin{pmatrix} x(k+N) \\ x(k+N-1) \\ \vdots \\ x(k+1) \\ x(k) \end{pmatrix} \in \mathbb{R}^n = \mathbb{R}^{N+1}$$

and call $\mathbf{x}_k$ the *segment of the solution x at k*. We write $F_N : \mathbb{R}^n \to \mathbb{R}^n$ for the discrete dynamical system

$$F_N \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 + \frac{f(v_n)}{N} \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}.$$

The difference equation (E_N) is equivalent to the map F_N in the sense that, for any $k, m \geq 0$, we have

$$F_N^m(\mathbf{x}_k) = \mathbf{x}_{k+m}.$$

We shall move freely between considering the difference equation (E_N) and the map F_N. (Observe that if f is Lipschitz (resp. smooth), then F_N is also.)

Solutions of (E_N) have a non-increasing "oscillation speed"; the analogous quantity of the same name has long been a valuable tool in the study of (E_∞). We give only a skeletal discussion here; a more thorough (and general) treatment is in [5] (see also [18]).

Definition 2.1. Let $x = (x(k))_{k=0}^{\infty}$ be any solution of (E_N). We say that x is *slowly oscillating at k* if there are no three indices $k \leq j_1 < j_2 < j_3 \leq k + N$ such that

$$x(j_1)x(j_2) < 0 \quad \text{and} \quad x(j_2)x(j_3) < 0.$$

Thus, if x is slowly oscillating at k, the vector $\mathbf{x}_k$ contains at most one (strict) sign change. The following proposition is a very special case of Theorem 3.3 in [5].

Proposition 2.2. *If x is slowly oscillating at k, then x is slowly oscillating at j for all $j \geq k$.*

If x is slowly oscillating for all $k \in \mathbb{Z}_+$, we say that x is *slowly oscillating*. Here is an elementary observation about slowly oscillating solutions that we shall use later.

Lemma 2.3. *Let $\theta < 0$ and $N \geq 2$. Suppose that x is a slowly oscillating solution of Equation (E_N). Suppose that $k \geq N$ and that*

$$x(k) \leq \theta \quad \textit{and} \quad x(k+J) \leq \theta$$

where $2 \leq J \leq N$. Then in fact

$$x(k+j) \leq \theta \textit{ for all } 0 \leq j \leq J.$$

Proof. Imagine that $x(k+j) > \theta$ for some $0 < j < J$. Then the negative feedback condition implies that

$$x(i_1) < 0 \quad \text{for some } k - N \leq i_1 \leq k + J - 2 - N$$

and

$$x(i_2) > 0 \quad \text{for some } i_1 < i_2 \leq k + J - 1 - N.$$

We have also assumed that $x(k) \leq \theta < 0$. Thus x has at least two sign changes on the discrete interval $[k-N, k]$; this is a contradiction. □

We now turn to a discussion of return maps for Equation (E_N). We begin by discussing, informally, a return map that is closely analogous to one commonly used in the study of Equation (E_∞); this discussion will motivate our overall approach. Given $0 < \nu < \mu$, let us define the following set:

$$S^n_{\nu,\mu} = \{\mathbf{v} \in \mathbb{R}^n : 0 \leq v_n \leq v_{n-1} \leq \ldots \leq v_2 \leq v_1 \text{ and } \nu \leq v_1 \leq \mu\}.$$

Observe that $S^n_{\nu,\mu}$ is compact and convex. Figure 1 illustrates $S^2_{\nu,\mu}$.

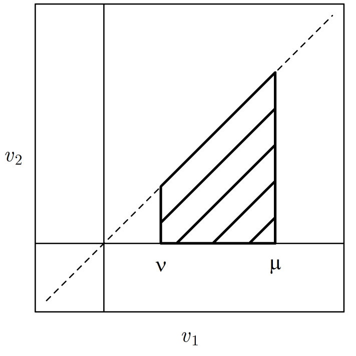

Fig. 1. The set $S^2_{\nu,\mu}$

Suppose now that $\mathbf{x}_0$ is in $S^n_{\nu,\mu}$; note that the solution x will be slowly oscillating. If $x(0)$ happens to be exactly zero, then

$$\mathbf{x}_1 = \begin{pmatrix} x(N) \\ x(N) \\ x(N-1) \\ \vdots \\ x(1) \end{pmatrix}$$

lies in $S^n_{\nu,\mu}$ also; but otherwise $x(N+1)$ is strictly less than $x(N)$ and so $\mathbf{x}_1$ is not in $S^n_{\nu,\mu}$. At any rate, x will be nonincreasing through the first N terms, and since $x(N)$ is strictly positive

$$x(2N+1) = x(2N) + \frac{f(x(N))}{N} < x(2N)$$

and so $\mathbf{x}_{N+1} = \mathbf{x}_n$ is *definitely* not in $S^n_{\nu,\mu}$. Let us now assume, though, that there is some minimal $m(\mathbf{x}_0) > N+1$ such that $\mathbf{x}_{m(\mathbf{x}_0)}$ lies in $S^n_{\nu,\mu}$. (This will be the case for appropriately chosen ν and μ provided, for example, that $f'(0)$ exists and is negative enough; for then x is guaranteed to have infinitely many sign changes and to grow away from the origin. We omit the details.)

Experience with Equation (E_∞) now suggests that we define a return map on $S^n_{\nu,\mu}$ by the formula

$$\mathbf{x}_0 \mapsto \mathbf{x}_{m(\mathbf{x}_0)}$$

but this map is *not* continuous. Figure 2 illustrates the problem. Two solutions x and y with "close" initial conditions in $S^4_{\nu,\mu}$ are shown. The segments $\mathbf{x}_0$ and $\mathbf{x}_{m(\mathbf{x}_0)}$ are shown in blue, and the segments $\mathbf{y}_0$ and $\mathbf{y}_{m(\mathbf{y}_0)}$ are shown in red. In this picture, $x(m(\mathbf{x}_0)) = 0$ but $y(m(\mathbf{x}_0)) < 0$, so we have $m(\mathbf{y}_0) = m(\mathbf{x}_0) + 1$, and $\mathbf{x}_{m(\mathbf{x}_0)}$ and $\mathbf{y}_{m(\mathbf{y}_0)}$ are not particularly close together.

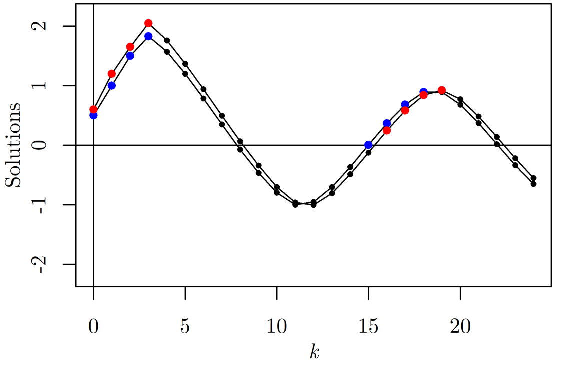

Fig. 2. The action of the map $\mathbf{u} \mapsto F_N^{m(\mathbf{u})}(\mathbf{u})$

Observe, though, that in this situation it will be the case that

(i) $\mathbf{x}_{m(\mathbf{y}_0)} = \mathbf{x}_{m(\mathbf{x}_0)+1}$ also lies in $S^n_{\nu,\mu}$, and
(ii) $\mathbf{x}_{m(\mathbf{y}_0)}$ and $\mathbf{y}_{m(\mathbf{y}_0)}$ *are* close together by the continuity of F_N.

The basic idea, therefore, is to replace $S^n_{\nu,\mu}$ with a quotient $\tilde{S}$ where the points $\mathbf{x}_{m(\mathbf{x})}$ and $\mathbf{x}_{m(\mathbf{x})+1}$ are identified. This will make the above-described return map continuous, at the price of altering the topology of its domain. We now describe a general framework that captures this idea.

Suppose that Y is a metric space with metric d, and that $F : Y \to Y$ is a Lipschitz continuous map. Let X be a subset of Y, and write $X_0 = F^{-1}(X) \cap X$. Let N be a positive integer. Assume that F, X and N together satisfy the following assumptions.

(H1) X is compact.
(H2) F is injective on X_0.

(H3) For every $x \in X$, there are integers $0 < \ell(x) \le N+1$ and $2N < m(x)$ with the features that $F^k(x) \in X$ for $0 \le k < \ell(x)$, $F^k(x) \notin X$ for $\ell(x) \le k < m(x)$, and $F^{m(x)}(x) \in X$.

(H4) The function $m : X \to \mathbb{N}$ is lower semicontinuous: given $x \in X$, there is an open neighborhood U about x such that $m(y) \ge m(x)$ for all $y \in U$. Moreover, U can be chosen so that $\max_{y\in U} m(y)$ exists and $F^j(x) \in X$ for all $m(x) \le j \le \max_{y\in U} m(y)$.

Observe that the continuity of F and the compactness of X imply that X_0 is also compact. Hypothesis (H3) says that points in X have forward orbits that leave X (within $N+1$ time units) and then return to X (after at least N time units away). Hypothesis (H4) guarantees, roughly speaking, that the return times of close-together points in X, though they may be different, can't be too different.

Hypothesis (H2) implies that each $x \in X$ has associated to it a unique set of points $[x]$ which is the maximal orbit segment of F (forwards and backwards) through x that lies entirely in X. $[x]$ might consist of x alone, or might consist of up to $N+1$ points. These sets $[x]$ constitute a partition of X. We define $\tilde{X}$ to be the corresponding set of equivalence classes

$$\tilde{X} = \big\{[x] : x \in X\big\}.$$

We define a return map $R : \tilde{X} \to \tilde{X}$ by the following formula:

$$R([u]) = \left[F^{m(u)}(u)\right].$$

We now describe how we topologize $\tilde{X}$ to make R continuous; this is a standard construction (see, for example, Section 3.1 of [1]). Let us write $x \sim y$ to mean that $x \in [y]$. We define $\mathcal{P}$ to be the set of finite ordered paths

$$\{w_1, z_1, w_2, z_2, \dots, w_k, z_k\} \subseteq X$$

where $w_i \sim z_i$. Given $u, v \in X$, we write

$$\rho(u,v) = \inf_{\mathcal{P}} \Big(d(u,w_1) + d(z_1,w_2) + \cdots + d(z_{k-1},w_k) + d(z_k,v)\Big).$$

(The intuition here is that $\rho(u,v)$ is the infimum length of all paths from u to v where we can make finitely many "jumps" from one equivalent point to another.)

We then define a metric $\tilde{\rho}$ on $\tilde{X}$ as follows:

$$\tilde{\rho}([u],[v]) = \rho(u,v).$$

We have the following general theorem; we defer the proof to Section 6.

Theorem 2.4. *With notation and hypotheses* (H1)–(H4) *as above, the map $\tilde{\rho}$ is well-defined and is in fact a metric. The map R is well-defined and is continuous.*

It turns out that, for appropriate f, ν, and μ, the return map on $S^n_{\nu,\mu}$ described earlier fits into the framework of Theorem 2.4; since we will not be working with this particular return map any more in this paper, we omit the proof.

This concludes our rather abstract general discussion of Equation (E_N). In the following sections we study a simple family of equations in greater depth, and supply examples that establish Theorem 1.3. Our main tool will be a version of the above-discussed return map R that turns out to be conjugate to a continuous self-map of the circle; our results will then follow from standard theorems about circle maps (which we review at the end of Section 3).

3. A FAMILY OF EQUATIONS WITH "CIRCLE-LIKE" RETURN MAPS

This section and the next two constitute the heart of the paper. Here is the family of feedback functions that we shall study: where $s \geq 1$,

$$f(x) = f_s(x) = \begin{cases} 1, & x \leq -\frac{1}{s}, \\ -sx, & x \geq -\frac{1}{s}. \end{cases}$$

Observe that f_s is Lipschitz and nonincreasing. Figure 3 illustrates the function f_2.

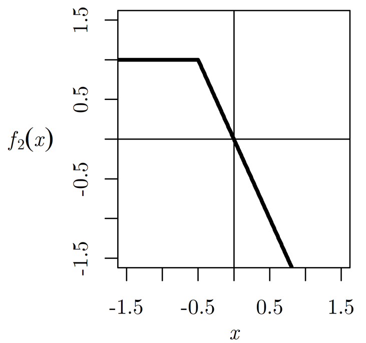

Fig. 3. The function f_2

For all of Sections 3, 4, and 5, we shall adhere to this notation: by f we shall always mean the function f_s just described. We shall continue to write $n = N + 1$; and we shall write $F = F_N$ for the corresponding map on $\mathbb{R}^n$ that advances (segments of) solutions of Equation (E_N) by one step.

We now define the set on which we will be studying return maps for Equation (E_N). It is the following line segment in $\mathbb{R}^n$:

$$L = \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \ : \ v_n \in \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N} \right), \right.$$
$$\left. v_k = v_{k+1} + \frac{1}{N} \text{ for all } 1 \leq k \leq (n-1) \right\}.$$

The closure of L is

$$\overline{L} = \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \ : \ v_n \in \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N} \right], \right.$$

$$\left. v_k = v_{k+1} + \frac{1}{N} \text{ for all } 1 \le k \le (n-1) \right\}.$$

We shall also, for all of Sections 3, 4, and 5, write I for the half-open interval

$$I = \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N} \right)$$

and $\overline{I}$ for its closure:

$$\overline{I} = \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N} \right].$$

The following lemma is immediate from the formula for f and illustrates why L is of interest: it "absorbs" all solutions that are below $-1/s$ for long enough.

Lemma 3.1. *Suppose that x is a solution of Equation (E_N) with feedback function f. If there is a point k such that the numbers*

$$x(k+1), \ \ldots, \ x(k+N-1), \ x(k+N)$$

are all less than or equal to $-1/s$, then $\mathbf{x}_j \in L$ for some $j \ge k+N$.

We now show that we can apply the return-map apparatus of Theorem 2.4 to our current setting.

Proposition 3.2. *Let f, F and L be as above. Write x for the solution of Equation (E_N) with initial condition $\mathbf{x}_0$. Assume that, for any $\mathbf{x}_0 \in \overline{L}$, there is some minimal $m(\mathbf{x}_0) \ge N$ such that $\mathbf{x}_{m(\mathbf{x}_0)} \in \overline{L}$, and such that*

$$x(m(\mathbf{x}_0) - N) < -\frac{1}{s} \quad \text{and} \quad x(m(\mathbf{x_0}) - 1) < -\frac{1}{s}.$$

Then all the hypotheses of Theorem 2.4 hold with $\overline{L}$ in the role of X.

Proof. $\overline{L}$ is compact; this is (H1).

Suppose that x is a solution of Equation (E_N) with $\mathbf{x}_0 \in \overline{L}$. In particular we have

$$x(0) \in \overline{I}, \quad x(1) = x(0) + \frac{1}{N}, \quad x(2) = x(0) + \frac{2}{N}, \quad \ldots, \quad x(N) = x(0) + 1.$$

Observe that

$$x(N+1) = x(N) - \frac{s}{N} x(0) \quad \text{and} \quad x(N+2) = x(N+1) - \frac{s}{N} x(1).$$

It follows that $\mathbf{x}_1 \in \overline{L}$ if and only if $x(0) = -1/s$; in this case $\mathbf{x}_0$ is the "lower" endpoint of $\overline{L}$ and $\mathbf{x}_1$ is the "upper" endpoint of $\overline{L}$. No matter what, $\mathbf{x}_2$ is definitely not in $\overline{L}$. Thus, in the notation of Theorem 2.4, the role of X_0 is played by the one-point set

$$L_0 = \left\{ \begin{pmatrix} -\frac{1}{s} + 1 \\ \vdots \\ -\frac{1}{s} + \frac{1}{N} \\ -\frac{1}{s} \end{pmatrix} \right\}.$$

We have verified (H2).

Now, we are assuming that $\mathbf{x}_{m(\mathbf{x}_0)} \in \overline{L}$. This means that

$$\begin{aligned} x(m(\mathbf{x}_0)) \in \overline{I} &= \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N} \right], \\ x(m(\mathbf{x}_0) + 1) &= x(m(\mathbf{x}_0)) + \frac{1}{N}, \\ x(m(\mathbf{x}_0) + 2) &= x(m(\mathbf{x}_0)) + \frac{2}{N}, \\ &\vdots \\ x(m(\mathbf{x}_0) + N) &= x(m(\mathbf{x}_0)) + 1. \end{aligned}$$

It follows that

$$x(m(\mathbf{x}_0) - N),\ x(m(\mathbf{x}_0) - (N-1)),\ \ldots,\ x(m(\mathbf{x}_0) - 1)$$

are all less than or equal to $-1/s$.

Now, if we imagine that $x(m(\mathbf{x}_0)) = -1/s + 1/N$, then since x can increase by no more than $1/N$ in one time step we must have $x(m(\mathbf{x}_0) - 1) = -1/s$; but in this case $\mathbf{x}_{m(\mathbf{x}_0)-1} \in \overline{L}$, contradicting the minimality of $m(\mathbf{x_0})$. Thus we see that $x(m(\mathbf{x}_0))$ actually lies in I, and that $\mathbf{x}_{m(\mathbf{x}_0)}$ actually lies in L.

Since $x(m(\mathbf{x}_0) - N) \leq -1/s$ and $x(j) > -1/s$ for all $1 \leq j \leq N$, we must have $m(\mathbf{x_0}) - N > N$ and hence that $m(\mathbf{x_0}) > 2N$. We have established point (H3).

Since $\overline{L}$ is compact and none of $\mathbf{x}_2, \ldots, \mathbf{x}_{m(\mathbf{x}_0)-1}$ lies in $\overline{L}$, by the continuity of F there is some $\varepsilon > 0$ such that $\|\mathbf{y}_0 - \mathbf{x}_0\| < \varepsilon$ implies that none of $\mathbf{y}_2, \ldots, \mathbf{y}_{m(\mathbf{x}_0)-1}$ lies in $\overline{L}$ either. Thus m is lower semicontinuous.

We can also choose ε small enough that

$$y(m(\mathbf{x}_0) - N) < -\frac{1}{s} \quad \text{and} \quad y(m(\mathbf{x}_0) - 1) < -\frac{1}{s} \quad \text{whenever } \|\mathbf{y}_0 - \mathbf{x}_0\| < \varepsilon.$$

Now, choose $\mathbf{y}_0 \in \overline{L}$ with $\|\mathbf{y}_0 - \mathbf{x}_0\| < \varepsilon$. Since

$$y(m(\mathbf{x}_0) - N) < -\frac{1}{s} \quad \text{and} \quad y(m(\mathbf{x_0}) - 1) < -\frac{1}{s},$$

Lemma 2.3 tells us that

$$y(m(\mathbf{x}_0) - N),\ y(m(\mathbf{x}_0) - (N-1)),\ \ldots,\ y(m(\mathbf{x}_0) - 2),\ y(m(\mathbf{x}_0) - 1)$$

are all less than $-1/s$. Accordingly,

$$y(m(\mathbf{x}_0)+j) = y(m(\mathbf{x}_0)) + \frac{j}{N} \quad \text{for all } 1 \le j \le N.$$

We now proceed in two cases. If $x(m(\mathbf{x}_0)) > -1/s$, shrinking ε if necessary we may assume that $y(m(\mathbf{x}_0)) > -1/s$ also, and so $\mathbf{y}_{m(\mathbf{x}_0)} \in \overline{L}$ and $m(\mathbf{y}_0) = m(\mathbf{x}_0)$.

If $x(m(\mathbf{x}_0)) = -1/s$, then $\mathbf{x}_{m(\mathbf{x}_0)}$ and $\mathbf{x}_{m(\mathbf{x}_0)+1}$ both lie in $\overline{L}$. Shrinking ε if necessary, we may assume that

$$|y(m(\mathbf{x}_0)) - x(m(\mathbf{x}_0))| < \frac{1}{N}.$$

If $y(m(\mathbf{x}_0)) \ge -1/s$ then in fact $y(m(\mathbf{x}_0)) \in I$ and $m(\mathbf{y}_0) = m(\mathbf{x}_0)$. If $y(m(\mathbf{x}_0)) < -1/s$, then $y(m(\mathbf{x}_0) + N + 1)) = y(m(\mathbf{x}_0) + N) + 1/N$ and, since $|y(m(\mathbf{x}_0)) - x(m(\mathbf{x}_0))| < 1/N$, we have $y(m(\mathbf{x}_0) + 1) \in I$. In this situation, $m(\mathbf{y}_0) = m(\mathbf{x}_0) + 1$. We have verified (H4). $\square$

If Proposition 3.2 applies, the return map we're interested in is given by the formula

$$G(\mathbf{x}) = F^{m(\mathbf{x})}(\mathbf{x}).$$

G is not continuous viewed as a map on $\overline{L}$, but is continuous viewed as a map on the quotient $\tilde{L}$ described in Theorem 2.4. In this situation, the quotient structure is particularly simple: the only two points of the line segment $\overline{L}$ that get identified in the quotient are its endpoints, and $\tilde{L}$ is topologically a circle.

As a final step in setting up our concrete calculations, we describe a simpler map that is conjugate to G. Any point of $\overline{L}$ is completely determined by its last coordinate, which must lie in $\overline{I}$. Let us identify the endpoints of I to obtain the circle

$$\tilde{I} \simeq S^1.$$

We equip $\tilde{I}$ with the circle metric

$$\lambda(x, y) = \min\left\{|x-y|, \frac{1}{N} - |x-y|\right\}.$$

Now consider the homeomorphism $h : \tilde{L} \to \tilde{I}$ whose output is the last coordinate of the input point:

$$h(\mathbf{x}) = x_n.$$

Define the map $g : \tilde{I} \to \tilde{I}$ by the formula

$$g(t) = h(G(h^{-1}(t))).$$

g is conjugate to G under the conjugacy h, and so we have

$$g^j(t) = h(G^j(h^{-1}(t))) \ \text{ for all } j \in \mathbb{Z}_+.$$

This map g is an example of a *circle map* — a continuous function from a circle to itself. The map g captures the dynamics of Equation (E_N), at least for those solutions that pass through L. It is these maps g that we shall actually compute.

Remark 3.3. The ideas we have just presented can be extended to slightly more general functions f: in particular, we can define $f(u)$ however we want for $u \geq -1/s$, provided that solutions beginning in $\overline{L}$ re-enter $\overline{L}$. Numerical experiments suggest that many such equations (E_N) obtained this way are interesting, but the calculations are of course more cumbersome.

We conclude this section by recalling some of the facts about circle maps that we shall need. These facts are standard; see, for example, [10].

Suppose that $\tilde{J} = [c, d)$ is a half-open interval of length $\ell = d - c$, equipped with the *circle metric*

$$\lambda(s,t) = \min\{|s-t|, \ell - |s-t|\}.$$

Let us write $q : \mathbb{R} \to \tilde{J}$ for the covering projection of $\mathbb{R}$ onto $\tilde{J}$:

$$q(x) = x - \left\lfloor \frac{x-c}{d-c} \right\rfloor (d-c).$$

Now suppose that $g : \tilde{J} \to \tilde{J}$ is a continuous map. Then the following hold:

- There is a continuous map $\gamma : \mathbb{R} \to \mathbb{R}$ with the feature that $q\gamma = gq$. This map γ is called a *lift* of g, and is unique up to the addition of an integer multiple of ℓ.
- If γ is any lift of g, the number $(\gamma(d) - \gamma(c))/\ell$ is integer-valued and independent of the choice of lift, and is called the *degree* of g.
- If g is a homeomorphism of degree one and γ is a lift of g, then the quantity

$$r(\gamma, x) = \lim_{k\to\infty} \frac{\gamma^k(x) - x}{k}$$

  exists. This quantity is independent of the choice of x, and independent modulo $\ell\mathbb{Z}$ of choice of γ. Accordingly, the quantity $\rho(g) := q(r(\gamma, x) + c) - c$ is well-defined and is called the *rotation number* of g.
- If g has a lift of the form $\gamma(x) = x + \alpha$, where $\alpha \in [0, \ell)$, then $\rho(g) = \alpha$.
- If g is a homeomorphism of degree one, then g has periodic points if and only if $\beta = \rho(g)/\ell$ is rational. Moreover, if $\beta = j/k \in \mathbb{Q}$ with $\gcd(j,k) = 1$, then any periodic point of g has minimal period k.
- $\rho(g)$ is continuous with respect to g (where the set of degree 1 homeomorphisms of $\tilde{J}$ has the sup norm).

Here is a description of a set of conditions under which $g : \tilde{J} \to \tilde{J}$ is in some sense “chaotic”. Our discussion is both informal and quite narrow, but the ideas are standard. Suppose the following hold for a circle map $g : \tilde{J} \to \tilde{J}$.

(C1) $\mu > 1$, and $c < c_0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < d_0 < d$. Write $A = [\alpha_1, \alpha_2]$ and $B = [\beta_1, \beta_2]$.
(C2) g maps both A and B homeomorphically onto $[c_0, d_0]$.
(C3) g has expansion constant μ on A and on B: that is, if x and y are both in A or both in B, then $|g(x) - g(y)| \geq \mu|x - y|$ (note that we are using the standard absolute value metric here, not the circle metric).

Figure 4 shows the graph of such a circle map.

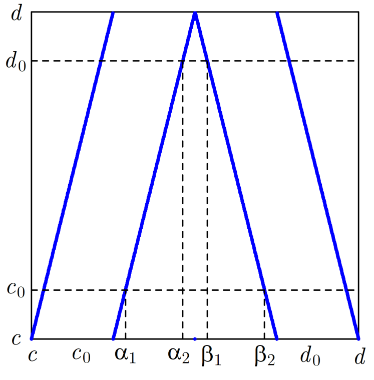

Fig. 4. A "chaotic" circle map g

Now consider any continuous interval map $\phi : [c_0, d_0] \to [c_0, d_0]$ that agrees with g on $A \cup B$. Such a map is pictured in Figure 5.

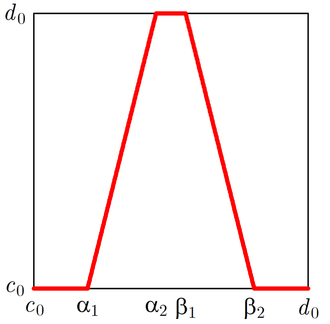

Fig. 5. The map ϕ

Let

$$K = \{x \in A \cup B \ : \ \phi^k(x) \in A \cup B \text{ for all } k \in \mathbb{N} \ \}.$$

Standard results in one-dimensional dynamics (see, for example, Chapter 9 of [11]) show that K is compact and nonempty, and that $\phi : K \to K$ is "chaotic" in the following sense: K contains infinitely many distinct periodic orbits for ϕ, as well as a dense orbit for ϕ. (In fact, in this situation, $\phi : K \to K$ is chaotic in the sense of Devaney.) Since $\phi = g$ on K and the standard metric and the circle metric induce the same topology on $A \cup B$, we can say that the restriction $g : K \to K$ has the same features. We summarize.

Proposition 3.4. *Suppose that $\tilde{J} = [c, d)$, equipped with the circle metric, and suppose that $g : \tilde{J} \to \tilde{J}$ is a continuous circle map satisfying conditions* (C1)–(C3) *above. Then there is a nonempty compact subset $K \subset \tilde{I}$ such that the restriction $g : K \to K$ is chaotic in the following sense: $g : K \to K$ has infinitely many distinct periodic orbits, as well as a dense orbit.*

Remark 3.5. The theory of circle maps is usually presented for maps of the standard circle $\mathbb{R}/\mathbb{Z} = [0, 1)$ (equipped with the circle metric). Since it will be more convenient for us to work with the more general intervals $J = [c, d)$, we are extending the standard theory to this slightly more general setting via the conjugating maps

$$H(x) = \ell x + c \quad \text{and} \quad H^{-1} = \frac{x - c}{\ell}.$$

$H : [0, 1) \to [c, d)$ is a homeomorphism. With this notation, if g is any circle map on J, then $\psi := H^{-1} g H$ is a circle map on $[0, 1)$, conjugate to g.

4. PROOF OF PART (I) OF THEOREM 1.3

In this section we use the ideas described in Section 3 to prove Part (I) of Theorem 1.3 (and to illustrate the rest of Theorem 1.3 in the $N = 1$ case). The work here will serve as a preview for the more involved calculations in Section 5.

For this section, we fix $N = 1$ and $f = f_s$, with $s \geq 1$. Write

$$t \in I = \left[-\frac{1}{s}, -\frac{1}{s} + 1\right).$$

We write $x = (x(0), x(1), \ldots)$ for the solution of (E_N) with initial condition

$$x(0) = t, \quad x(1) = t + 1.$$

In the notation of Section 3, we have $\mathbf{x}_0 \in L$.

Let us also write $B(k)$ for the solution of the linear equation

$$B(k + 1) = B(k) - sB(k - 1)$$

with $B(0) = t$ and $B(1) = t + 1$. Here are the first several terms of this solution:

$$\begin{aligned}
B(0) &= t,\\
B(1) &= 1 + t,\\
B(2) &= 1 + (1-s)t,\\
B(3) &= 1 - s + (1-2s)t,\\
B(4) &= 1 - 2s + (1 - 3s + s^2)t,\\
B(5) &= 1 - 3s + s^2 + (1 - 4s + 3s^2)t,\\
B(6) &= 1 - 4s + 3s^2 + (1 - 5s + 6s^2 - s^3)t.
\end{aligned}$$

Let $\chi = \chi(s,t)$ be the minimal value of $k \geq 1$ (assuming it exists) for which $x(k) < -1/s$; observe that $x(k) = B(k)$ for all $0 \leq k \leq \chi + 1$ and that

$$x(\chi+1) \leq x(\chi) + 1 \quad \text{and} \quad x(\chi+2) = x(\chi+1) + 1.$$

After time $k = \chi + 2$, the solution x will then continue to climb by 1 each time step until it re-enters L; that is, there will be some minimal $m \geq \chi + 1$ such that

$$\mathbf{x}_m = \begin{pmatrix} x(m+1) \\ x(m) \end{pmatrix} \in L.$$

Our map $g : \tilde{I} \to \tilde{I}$ is given by the formula $g(t) = x(m)$.

We now explicitly calculate the maps g (or lifts of them) for a range of values of s. In particular, we shall focus on $s \in [1, 4/3]$.

If $s = 1$, straightforward calculations show that $-1 < x(k)$ for all $1 \leq k \leq 4$, that $x(5) = -1$, that $x(6) = t$, and that $x(7) = t + 1$. Thus $\mathbf{x}_6 \in L$ and $m(\mathbf{x}_0) = 6$; furthermore, $\mathbf{x}_6 = \mathbf{x}_0$ and g is the identity map. Figure 6 shows the first few terms of such a solution.

If $s = 4/3$, then for all $t \in I$ we have

$$x(k) > -\frac{1}{s} \quad \text{for} \quad 1 \leq k \leq 3 \quad \text{and} \quad x(4) \leq -\frac{1}{s}.$$

Thus $x(k) = B(k)$ for $1 \leq k \leq 5$; and then $x(5+j) = x(5) + j$ until the solution re-enters L. (Moreover, for all $t \in I$ either $x(4)$ or $x(5)$ will be strictly less than $-1/s$, and so Proposition 3.2 applies.) Figure 7 shows the first few terms of two different solutions. For the solution in red, $m = 6$; and for the solution in blue, $m = 7$.

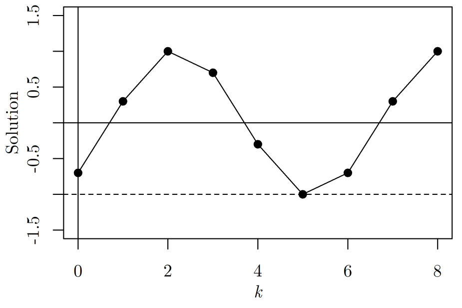

Fig. 6. The first few terms of a solution x when $s = 1$. The dashed line is at height $-1/s = -1$

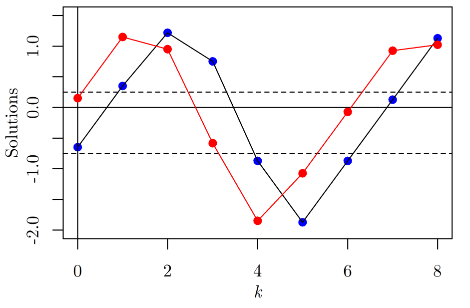

Fig. 7. The first few terms of two solutions x when $s = 4/3$. The dashed lines are at height $-1/s$ and $-1/s + 1$

Therefore, for each $t \in I$, $g(t)$ will be of the form

$$g(t) = x(5) + \text{some non-negative integer depending on } t.$$

If we add various integers to the various values of $g(t)$ to create a continuous real-valued function γ of t, that function γ is a lift of g. Therefore we see that, when $s = 4/3$,

$$\gamma(t) = x(5) = t - \frac{11}{9}$$

is a lift of g. In this case, g is a rotation by $7/9$. Figure 8 shows a plot of γ (in red) and g (in blue) over the interval I. (The thin dashed line shows the diagonal $t = t$. The thick dashed lines are at heights $-1/s$ and $-1/s + 1$.)

Now let us suppose that s is strictly between 1 and $4/3$. In this case, for all $t \in I$ we have

$$x(k) > -\frac{1}{s} \quad \text{for} \quad 1 \le k \le 3.$$

Write

$$t_* = \frac{-1/s - 1 + 2s}{1 - 3s + s^2} \in I.$$

Tedious but straightforward calculations (along with reasoning similar to what we did above) show the following. Given any $t \in I$, either $x(4)$ or $x(5)$ is strictly less than $-1/s$. In addition,

(A) if $t \ge t_*$, then $x(4) \le -1/s$, and so for such t we have

$$g(t) = x(5) + \text{some non-negative integer depending on } t;$$

(B) if $t < t_*$, then $x(4) > -1/s$ and $x(5) \le -1/s$, and so for such t we have

$$g(t) = x(6) + \text{some non-negative integer depending on } t.$$

Figure 9 shows the first few terms of three solutions when $s = 1.2$. The solution in blue has $t < t_*$ and $g(t) = x(6) + 1$; the solution green has $t = t_*$ and $g(t) = x(5) + 1$, and the solution in red has $t > t_*$ and $g(t) = x(5) + 1$.

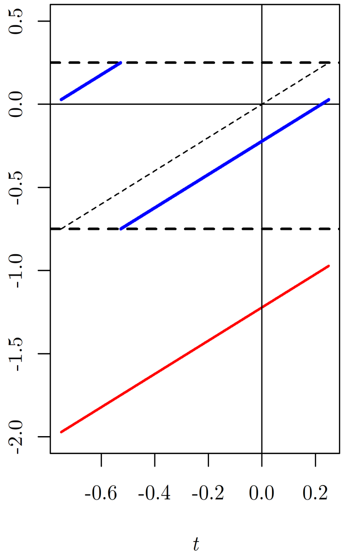

Fig. 8. The function g (in blue) and a lift γ (in red) when $s = 4/3$

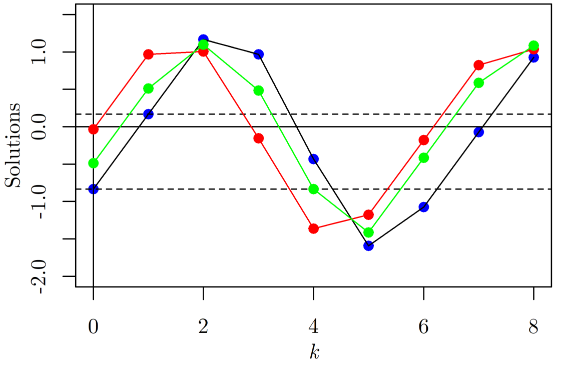

Fig. 9. The first few terms of three solutions x when $s = 1.2$. The dashed lines are at heights $-1/s$ and $-1/s + 1$

As before, we can add multiples of 1 to various values of g to try to obtain a lift γ of g; in particular, we want to achieve continuity at t_*. Observing that $x(6) = x(5) + 1$ when $t = t_*$, we accordingly obtain the following formula for a lift γ of g:

$$\gamma(t) = \begin{cases} x(6) = B(6), & t \in [-\frac{1}{s}, t_*], \\ x(5) + 1 = B(5) + 1, & t \in [t_*, -\frac{1}{s} + 1]. \end{cases}$$

Since $B(5)$ and $B(6)$ are affine in t, the graph of γ consists of two linear pieces. Figure 10 shows this function γ and the corresponding circle map g when $s = 1.2$.

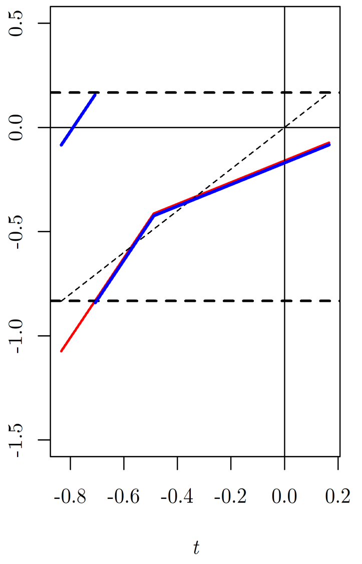

Fig. 10. The function g (in blue) and a lift γ (in red) when $s = 1.2$

Further elementary calculations reveal the following: first, that

$$\gamma\left(-\frac{1}{s}+1\right)-\gamma\left(-\frac{1}{s}\right)=1 \quad \text{for all } s \in \left(1, \frac{4}{3}\right);$$

second, that

$$0 < \frac{\partial}{\partial t}B(5) \quad \text{and} \quad 0 < \frac{\partial}{\partial t}B(6) \quad \text{for all } s \in \left(1, \frac{4}{3}\right).$$

The first observation shows that g has degree one; the second shows that g is injective. It follows that g is in fact a degree one homeomorphism of $\tilde{I}$ for each value of $s \in [1, 4/3]$, and so $\rho(g)$ is defined. Since g varies continuously with s, $\rho(g)$ does as well. (Strictly speaking, the domains $\tilde{I}$ of g vary with respect to s also; if we like, we can transfer all the maps g to circle maps on – say – $[0, 1)$ by a homeomorphic change of variable.) Since g is a rotation when $s = 1$ and when $s = 4/3$, we can calculate the rotation number of g in these cases: we have that $\rho(g) = 0$ when $s = 1$ and $\rho(g) = 7/9$ when $s = 4/3$. Therefore there are some values of $s \in (1, 4/3)$ for which $\rho(g)$ is irrational; as explained at the end of Section 3, for these values of s, g *has no periodic points*. It follows that the corresponding map $G : L \to L$ has no periodic points, and that Equation (E_N) has no periodic solutions passing through L.

Proof of part (I) *of Theorem 1.3.* Suppose that $s \in (1, 4/3)$ is such that Equation (E_N) has no periodic points passing through L; we have just shown that there are such values of s. Since $s > 1$ and $N = 1$, it is easy to show that the eigenvalues of the linearization of F at 0 are outside the unit circle. Since f is linear on $[-1/s, \infty)$, it follows that any nonzero solution of Equation (E_N) must, at some point, attain a value less than or equal to $-1/s$. As explained above, though, such a solution will, as it rises again, inevitably enter L and then return to L regularly. So when $s > 1$, any nonzero periodic solution of Equation (E_N) would have to have segments in L, but there are no periodic solutions with segments in L. This completes the proof of part (I) of Theorem 1.3. □

Since $\overline{L}$ is compact, we also see that if $g : \tilde{I} \to \tilde{I}$ has no periodic points, then no solution passing through $\overline{L}$ can approach a periodic point. Thus we have also illustrated point (III) of Theorem 1.3 in the $N = 1$ case.

We also see that, in the same family of examples we just gave, there will be maps g with rational rotation numbers that (when reduced) have large denominators; again as explained at the end of Section 3, g has periodic points with large minimal period in this case. These points will correspond to solutions of Equation (E_N) with large numbers of sign changes per minimal period. Thus we have also illustrated (in the $N = 1$ case) point (II) of Theorem 1.3.

Example 4.1 (A chaotic map g in the $N = 1$ case.)**.** If we take $s = 5$, then the same sort of computations that we have done already show that g has a lift γ given by the following formula:

$$\gamma(t) = \begin{cases} B(4) = -9 + 11t, & t \in \left[-\frac{1}{5}, \frac{3}{10}\right], \\ B(3) + 1 = -3 - 9t, & t \in \left[\frac{3}{10}, \frac{4}{5}\right]. \end{cases}$$

Figure 11 shows a graph of the circle map g when $s = 5$. It is evident that Proposition 3.4 applies, and so the map g is chaotic in the sense of Proposition 3.4.

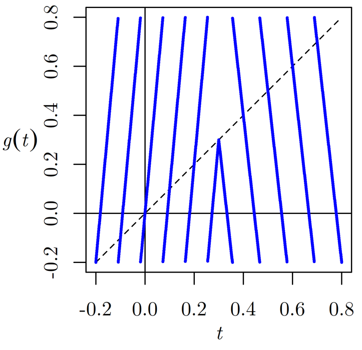

Fig. 11. The function g when $s = 5$

We close this section with numerical plots in the plane of the first several points of solutions with initial condition

$$\mathbf{x}_0 = \begin{pmatrix} -\frac{1}{s}+1 \\ -\frac{1}{s} \end{pmatrix} \in L,$$

for various values of s. For all of these plots, the set L is illustrated with a thick red line and the first several points of the orbit of $\mathbf{x}_0$ under F are shown as black dots.

In Figure 12, $s = 4/3$. We know that g is a rotation by $7/9$ in this case, and so $t = -1/s$ is a 9-periodic point of g. The corresponding solution x is periodic and passes through L in 9 distinct places.

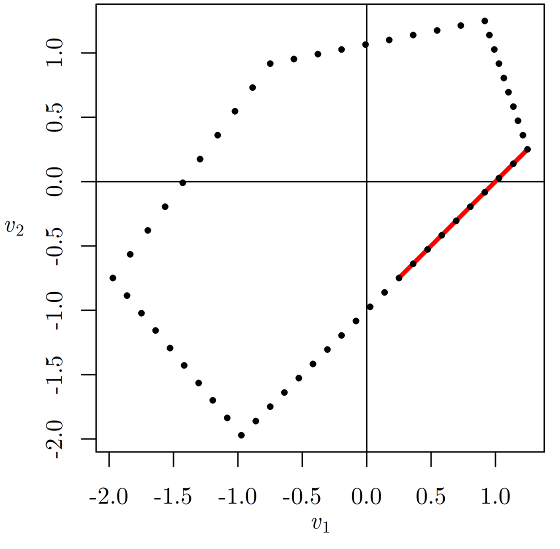

Fig. 12. First 1000 points of the orbit of $\mathbf{x}_0$ under F when $s = 4/3$

In Figure 13, $s = 1.2$. The orbit of $t = -1/s$ under g seems to be converging to a fixed point of g, and the corresponding solution of x appears to be converging to a 6-periodic solution.

In Figure 14, $s = 5$. The orbit of of $t = -1/s$ under g appears plausibly dense in I, and the corresponding solution of x appears to be densely filling out a subset of the plane. Solutions with behaviors analogous to those illustrated in Figures 13 and 14 are also contrasted in [13].

5. SOME EXPLICIT SOLUTIONS

In this section we extend the ideas in Sections 3 and 4 to complete the proof of Theorem 1.3.

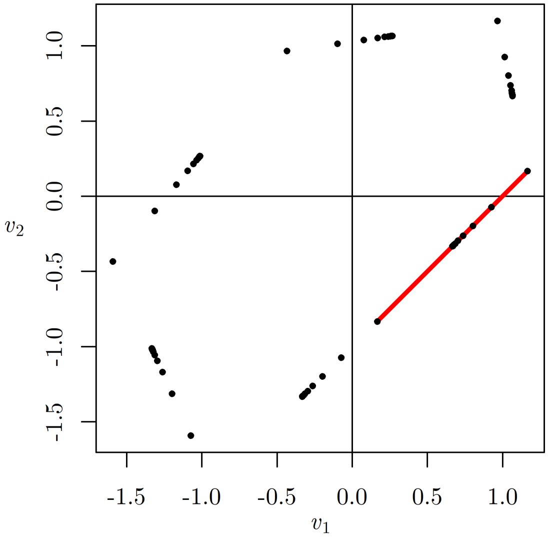

Fig. 13. First 1000 points of the orbit of $\mathbf{x}_0$ under F when $s = 1.2$

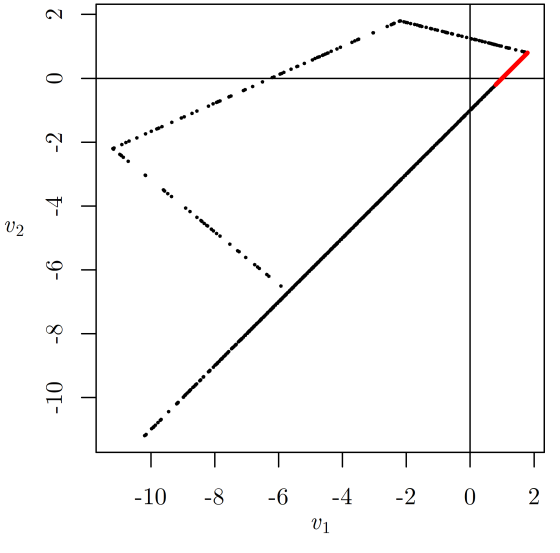

Fig. 14. First 1000 points of the orbit of $\mathbf{x}_0$ under F when $s = 5$

Let $N \in \mathbb{N}$ and $f = f_s$ with $s \geq 1$, and suppose that x is a solution of Equation

(E_N) with $\mathbf{x}_0 \in \overline{L}$. Suppose that χ is the first index (if it exists) for which $x(\chi) < -1/s$. Then, for all $k \leq \chi + N - 1$, x will satisfy the linear equation

$$x(k+1) = x(k) - \frac{s}{N} x(k-N). \tag{EL_N^s}$$

We begin this section by developing formulas for some solutions of Equation (EL_N^s). We will then apply these formulas to calculate, in certain cases, formulas for the maps $G : L \to L$ and $g : \tilde{I} \to \tilde{I}$ described in Section 3. The work is very similar to that in Section 4.

We begin with a reminder of a standard combinatorial fact, which is clear either from consulting Pascal's triangle or from explicit calculation.

Lemma 5.1. *Suppose that c and d are any two positive integers with $c \leq d$. Then*

$$\binom{d+1}{c} = \binom{d}{c} + \binom{d}{c-1}.$$

We now give a solution for a particular solution of Equation (EL_N^s).

Lemma 5.2. *The solution of Equation (EL_N^s) with initial condition*

$$x(0) = x(1) = \cdots = x(N-1) = x(N) = 1$$

is given by the following formula:

$$x(k) = Q_s^N(k) = \sum_{j \geq 0} (-1)^j \binom{k - Nj}{j} \left(\frac{s}{N}\right)^j, \quad \textit{where } \binom{d}{c} = 0 \ \ \textit{if} \ \ d < c.$$

Proof. We proceed by induction. Since $k - Nj < j$ for all $k \leq N$ and all $j \geq 1$, we have $Q_s^N(k) = 1$ for all $k \in \{0, \dots, N\}$, so the formula yields the correct initial condition. To finish the proof we need to show that

$$Q_s^N(k+1) = Q_s^N(k) - \frac{s}{N} Q_s^N(k-N)$$

for all $k \geq N$. The coefficient of the $(s/N)^j$ term in the left-hand expression above is

$$(-1)^j \binom{k+1-Nj}{j},$$

while (using Lemma 5.1) the coefficient of the $(s/N)^j$ term in the right-hand expression is

$$\begin{aligned}
&(-1)^j \binom{k - Nj}{j} - (-1)^{j-1} \binom{k - N - N(j-1)}{j-1} \\
&= (-1)^j \left[\binom{k - Nj}{j} + \binom{k - N - N(j-1)}{j-1} \right] = (-1)^j \binom{k+1-Nj}{j},
\end{aligned}$$

as desired. □

Let us write $Q_s^N(k) = 0$ for all $k < 0$. Linearity now yields the following additional special formula.

Lemma 5.3. *Let $t, z \in \mathbb{R}$. The solution of Equation (EL_N^s) with initial condition*

$$x(0) = t,\ x(1) = t + z,\ x(2) = t + 2z,\ \dots, x(N) = t + Nz$$

is given by the following formula:

$$x(k) = tQ_s^N(k) + z\left(Q_s^N(k-1) + Q_s^N(k-2) + \cdots + Q_s^N(k-N)\right).$$

We introduce the following notation:

$$B_s^N(t,k) = tQ_s^N(k) + \frac{1}{N}\Big[Q_s^N(k-1) + Q_s^N(k-2) + \cdots + Q_s^N(k-N)\Big]. \qquad (B)$$

These are the terms of the solution of Equation (EL_N^s) with initial condition

$$x(0) = t,\ x(1) = t + \frac{1}{N},\ x(2) = t + \frac{2}{N},\ \dots,\ x(N) = t + 1.$$

Viewed as a point $\mathbf{x}_0 \in \mathbb{R}^n = \mathbb{R}^{N+1}$, this initial condition lies in L provided that

$$t \in I = \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N}\right).$$

We shall need a few more results detailing the relationships among these various solutions.

Lemma 5.4. *With notation as above, we have the following formulas for $k \geq N$.*

(i)
$$B_s^N(t, k-N) = \Big(\frac{Nt}{s} + \frac{N}{s^2} - \frac{1}{s}\Big)Q_s^N(k) - \Big(\frac{N}{s^2} + \frac{Nt}{s}\Big)Q_s^N(k+1).$$

(ii)
$$B_s^N\left(-\frac{1}{s}, k-N\right) = \frac{-Q_s^N(k)}{s}.$$

(iii)
$$B_s^N\Big(-\frac{1}{s}, k-N\Big) = -\frac{1}{s} \quad \textit{if and only if} \quad Q_s^N(k) = 1.$$

(iv)
$$B_s^N(t, k-N) = \frac{-Q_s^N(k)}{s} + \left(t + \frac{1}{s}\right)Q_s^N(k-N).$$

Proof. Combining the facts that

$$B_s^N(t,k+1) = B_s^N(t,k) - \frac{s}{N}B_s^N(k-N) \ \text{ and } \ Q_s^N(k+1) = Q_s^N(k) - \frac{s}{N}Q_s^N(k-N)$$

with the formula for $B_s^N(t,k)$ given in (B), we have

$$\begin{aligned}
B_s^N(t,k-N) &= \frac{N}{s}\Big[B_s^N(t,k) - B_s^N(t,k+1)\Big] \\
&= \frac{N}{s}\Big[tQ_s^N(k) + \frac{1}{N}Q_s^N(k-N) - tQ_s^N(k+1) - \frac{1}{N}Q_s^N(k)\Big] \quad \text{(use } (B))\\
&= \frac{N}{s}\Big[tQ_s^N(k) + \frac{1}{N}\Big(\frac{N}{s}\left[Q_s^N(k) - Q_s^N(k+1)\right]\Big) - tQ_s^N(k+1) - \frac{1}{N}Q_s^N(k)\Big] \\
&= \Big(\frac{Nt}{s} + \frac{N}{s^2} - \frac{1}{s}\Big)Q_s^N(k) - \Big(\frac{N}{s^2} + \frac{Nt}{s}\Big)Q_s^N(k+1).
\end{aligned}$$

This proves (i). (ii) is just (i) with $t = -1/s$, and (iii) follows immediately from (ii). To prove (iv), we first note from (B) that

$$\frac{\partial}{\partial t}B_s^N(t,k-N) = Q_s^N(k-N).$$

Combining this fact and (ii) we get

$$\begin{aligned}
B_s^N(t,k-N) &= B_s^N\left(-\frac{1}{s}, k-N\right) + \left(t - \left(-\frac{1}{s}\right)\right)Q_s^N(k-N) \\
&= \frac{-Q_s^N(k)}{s} + \left(t + \frac{1}{s}\right)Q_s^N(k-N),
\end{aligned}$$

as desired. □

There is one more formula for $B_s^N(t,k)$ that we shall find useful. Observe that

$$B_s^N\left(-\frac{1}{s} + \frac{1}{N}, k\right) = B_s^N\left(-\frac{1}{s}, k+1\right) \quad \text{for all} \quad k \geq 0.$$

Thus, if $k \geq 0$ and

$$t = (1-\tau)\left(-\frac{1}{s}\right) + \tau\left(-\frac{1}{s} + \frac{1}{N}\right) = -\frac{1}{s} + \frac{\tau}{N}, \ 0 \leq \tau < 1,$$

then

$$\begin{aligned}
B_s^N(t,k) &= tQ_s^N(k) + \frac{1}{N}\Big[Q_s^N(k-1) + \cdots + Q_s^N(k-N)\Big] \\
&= \left[(1-\tau)\left(-\frac{1}{s}\right) + \tau\left(-\frac{1}{s} + \frac{1}{N}\right)\right]Q_s^N(k) \\
&\quad + \frac{1}{N}\Big[Q_s^N(k-1) + \cdots + Q_s^N(k-N)\Big] \\
&= (1-\tau)B_s^N\left(-\frac{1}{s}, k\right) + \tau B_s^N\left(-\frac{1}{s} + \frac{1}{N}, k\right) \\
&= (1-\tau)B_s^N\left(-\frac{1}{s}, k\right) + \tau B_s^N\left(-\frac{1}{s}, k+1\right).
\end{aligned}$$

We record this fact, as well as a version of it we get by applying part (ii) of Lemma 5.4.

Lemma 5.5. *If*

$$t = -\frac{1}{s} + \frac{\tau}{N}, \quad 0 \le \tau < 1,$$

then for all $k \ge 0$ we have

$$\begin{aligned} B_s^N(t,k) &= (1-\tau) B_s^N\left(-\frac{1}{s}, k\right) + \tau B_s^N\left(-\frac{1}{s}, k+1\right) \\ &= \frac{\tau - 1}{s} Q_s^N(k+N) - \frac{\tau}{s} Q_s^N(k+1+N). \end{aligned}$$

Here is a reminder of what we're doing. If every solution with initial condition $\mathbf{x}_0 \in L$ returns to L, then for every such $\mathbf{x}_0$ there is a minimal $m = m(\mathbf{x}_0) > N$ such that

$$\mathbf{x}_m = \begin{pmatrix} x(m+N) \\ x(m+N-1) \\ \vdots \\ x(m) \end{pmatrix} \in L.$$

Since $\mathbf{x}_0$ is completely determined by $t = x(0)$, as already explained this return to L is completely characterized by the map $g : \tilde{I} \to \tilde{I}$ given by the formula $g(x(0)) = x(m)$. This map g is continuous under appropriate conditions (for example, if the conditions of Proposition 3.2 apply).

We now describe a family of situations under which this map g exists, is continuous, and has a formula that we can compute.

Proposition 5.6. *Suppose that N is given and fixed. Let us choose a natural number $K > N$. Assume that there are numbers $s(N, K+1)$ and $s(N, K)$ with the following features.*

(a) $1 < s(N, K+1) < s(N, K)$.

(b) *If $s = s(N, K+1)$, then $K+1$ is the minimal value of $k \ge 1$ for which*

$$B_s^N\left(-\frac{1}{s}, k\right) = -\frac{1}{s}.$$

(c) *If $s = s(N, K)$, then K is the minimal value of $k \ge 1$ for which*

$$B_s^N\left(-\frac{1}{s}, k\right) = -\frac{1}{s}.$$

(d) *For any $s \in [s(N, K+1), s(N, K)]$, any $t \in I$, and any $1 \le k \le K-1$ we have*

$$B_s^N(t,k) > -\frac{1}{s}.$$

(e) *For any $s \in [s(N, K+1), s(N, K)]$ we have*

$$Q_s^N(K) < 0 \text{ and } Q_s^N(K+1) < 0.$$

(f) *For any $s \in (s(N,K+1), s(N,K))$ we have*

$$\frac{\partial}{\partial s} B_s^N\left(-\frac{1}{s}, K\right) < 0 \quad \textit{and} \quad \frac{\partial}{\partial s} B_s^N\left(-\frac{1}{s}, K+1\right) < 0.$$

(g) *For any $s \in [s(N,K+1), s(N,K)]$ and any $t \in \overline{I}$, we have*

$$B_s^N(t, K+N) < -\frac{1}{s}.$$

Then, for any $s \in [s(N,K+1), s(N,K)]$, the map $g : \tilde{I} \to \tilde{I}$ is defined and continuous. Moreover, g has a lift γ with the following formulas on $\overline{I}$.

– *If $s = s(N,K+1)$, then*

$$\gamma(t) = B_s^N\left(-\frac{1}{s}, K+N+1\right) + \left(t + \frac{1}{s}\right), \quad t \in \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N}\right]$$

(and so g is a rotation).

– *If $s = s(N,K)$, then*

$$\gamma(t) = B_s^N\left(-\frac{1}{s}, K+N\right) + \left(t + \frac{1}{s}\right), \quad t \in \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N}\right]$$

(and so g is a rotation).

– *If $s \in (s(N,K+1), s(N,K))$, then*

$$\gamma(t) = \begin{cases} B_s^N(t_*, K+N) - Q_s^N(K+1+N)(t_* - t), & t \in \left[-\frac{1}{s}, t_*\right], \\ B_s^N(t_*, K+N) + Q_s^N(K+N)(t - t_*), & t \in \left[t_*, -\frac{1}{s} + \frac{1}{N}\right] \end{cases}$$

where t_* is the number strictly between $-\frac{1}{s}$ and $-\frac{1}{s} + \frac{1}{N}$ given by the formula

$$t_* = t_*(s) = \frac{1 - Q_s^N(K+N)}{NQ_s^N(K+1+N) - NQ_s^N(K+N)} - \frac{1}{s}$$

(and so g is consists of two linear pieces).

Finally, in any of the three cases above,

$$\gamma\left(-\frac{1}{s} + \frac{1}{N}\right) - \gamma\left(-\frac{1}{s}\right) = \frac{1}{N}$$

and so g is of degree 1.

The last of the above formulas for γ admittedly looks complicated, but the main point is that γ is piecewise linear and continuous, with slope $Q_s^N(K+1+N)$ on the first subinterval and slope $Q_s^N(K+N)$ on the second subinterval.

Proof. Assume that conditions (a)–(g) hold. As above, we write x for the solution of Equation (E_N) with initial condition

$$x(0) = t,\ x(1) = t + \frac{1}{N},\ x(2) = t + \frac{2}{N},\ \cdots,\ x(N) = t + 1$$

where

$$t \in \overline{I} = \left[-\frac{1}{s}, -\frac{1}{s} + \frac{1}{N}\right].$$

We begin by proving that g is defined and continuous for any $s \in [s(N, K+1), s(N, K)]$. The idea, roughly speaking, is that our hypotheses guarantee that solutions starting in $\overline{L}$ will all be *strictly below* $-1/s$ for at least N time steps before returning to L, and so Proposition 3.2 applies.

If $s = s(N, K+1)$ we know that $B_s^N(-1/s, K+1) = -1/s$. By the second part of (f) we conclude that $B_s^N(-1/s, K+1) < -1/s$ for all $s \in (s(N, K+1), s(N, K)]$ (since $B_s^N(-1/s, K+1)$ decreases and $-1/s$ increases as s increases). Similarly, since

$$\frac{\partial}{\partial t} B_s^N(t, k) = Q_s^N(k),$$

the second part of (e) tells us that

$$B_s^N(t, K+1) \leq -1/s \text{ for all } s \in [s(N, K+1), s(N, K)] \text{ and all } t \in \overline{I}.$$

In fact, the above equality is *strict except in the case that* $s = s(N, K+1)$ *and* $t = -1/s$. In the special case that $t = -1/s$ and $s = s(N, K+1)$, though, we do have

$$B_s^N\left(-\frac{1}{s}, K+2\right) = B_s^N\left(-\frac{1}{s} + \frac{1}{N}, K+1\right) < -\frac{1}{s}.$$

We have shown that, given any $s \in [s(N, K+1), s(N, K)]$ and any $t \in \overline{I}$, there is a first value of k for which $B_s^N(t, k)$ is *strictly less* than $-1/s$. Point (d) combined with the work above shows us that this first value of k is either K or $K+1$ in every case except when $s = s(N, K+1)$ and $t = -1/s$, in which case this first value of k is $K+2$ (the first part of (f) rules out the value being K in this particular case). Note that we are therefore guaranteed that

- $x(k) = B_s^N(t, k)$ for all $0 \leq k \leq K+N$; and
- if $B_s^N(t, K) \geq -1/s$, then $x(K+1+N) = B_s^N(t, K+1+N)$ as well.

We now proceed in two cases.

Case I. either $t > -1/s$ or $s > s(N, K+1)$. We have already shown that $x(K+1) < -1/s$ in this case. Moreover, (g) guarantees that $x(K+N) = B_s^N(t, K+N) < -1/s$. Therefore, by Lemma 2.3 (or trivially, in the $N = 1$ case) we know that the N successive terms

$$x(K+1), \ldots, x(K+N)$$

are all strictly less than $-1/s$. Therefore, by Lemma 3.1, $m(\mathbf{x}_0)$ is defined; moreover, we see that $m(\mathbf{x}_0) \geq K+N+1$ and that $x(k)$ is strictly below $-1/s$ for at least N terms before re-entering I. In particular we have

$$x(m(\mathbf{x}_0)-N) < -\frac{1}{s} \text{ and } x(m(\mathbf{x}_0)-1) < -\frac{1}{s},$$

and the hypotheses of Proposition 3.2 hold.

Case II. both $t = -1/s$ and $s = s(N, K+1)$. We have already shown that $x(K+2) < -1/s$. Moreover, (g) and the same sort of calculation we did before shows that

$$B_s^N\left(-\frac{1}{s}, K+1+N\right) = B_s^N\left(-\frac{1}{s}+\frac{1}{N}, K+N\right) < -\frac{1}{s}.$$

Thus Lemma 2.3 implies that the N successive terms

$$x(K+2), \ldots, x(K+1+N)$$

are all strictly less than $-1/s$. Just as before, we conclude that

$$x(m(\mathbf{x}_0)-N) < -\frac{1}{s} \quad \text{and} \quad x(m(\mathbf{x}_0)-1) < -\frac{1}{s},$$

and the hypotheses of Proposition 3.2 hold.

We have shown that g is always defined and continuous. We now turn to computing it.

Case 1. $s = s(N, K+1)$. Then

$$B_s^N\left(-\frac{1}{s}+\frac{1}{N}, K\right) = B_s^N\left(-\frac{1}{s}, K+1\right) = -\frac{1}{s}.$$

By assumption (e) we have

$$\frac{\partial}{\partial t} B_s^N(t, K) = Q_s^N(K) < 0;$$

we conclude that $B_s^N(t, K) > -1/s$ for all $t \in I$ (note that we really do mean I here, not $\overline{I}$). Therefore, for all $t \in I$, $K+1$ is the minimal value of $k \geq 1$ for which $x(k) \leq -1/s$. We have already shown in this case that $x(K+1+N) = B_s^N(t, K+1+N)$. After time $K+1+N$, x will climb by $1/N$ each time period until the solution segment has returned to L: more particularly, we will have

$$x(K+1+N+j) = B_s^N(t, K+1+N) + \frac{j}{N} \quad \text{for all } 1 \leq j \leq m(\mathbf{x}_0) - (K+1).$$

Therefore we have

$$g(t) = x(m(\mathbf{x}_0)) = B_s^N(t, K+1+N) + \frac{\text{some nonnegative integer depending on } t}{N}.$$

Since lifts of g are determined only up to integer multiples of $1/N$, if we add or subtract multiples of $1/N$ from $g(t)$ at various points to create a continuous real-valued function γ, this γ is a lift of g. Therefore the following is a valid formula for γ:

$$\gamma(t) = B_s^N(t, K+1+N).$$

Now, (B) tells us that

$$\frac{\partial}{\partial t} B_s^N(t, K+1+N) = Q_s^N(K+1+N);$$

but since $B_s^N(-1/s, K+1) = -1/s$, point (iii) of Lemma 5.4 yields that

$$\frac{\partial}{\partial t} B_s^N(t, K+1+N) = 1 \quad \text{for all } t \in I.$$

Thus we obtain the formula for γ given in the first conclusion of the proposition.

Case 2. $s = s(N, K)$. This case is very similar to Case 1. Combining (c), (d) and (e) shows that K is, for all $t \in I$, the minimal value of $k \geq 1$ for which $B_s^N(t, k) \leq -1/s$. Then, just as above, we get

$$g(t) = B_s^N(t, K+N) + \frac{\text{some nonnegative integer depending on } t}{N}.$$

Since $Q_s^N(K) = 1$ in this case, we can choose our lift γ to have the formula given in the proposition.

Case 3. $s \in (S(N, K+1), S(N, K)$. By (b),(c), and (f) we have

$$B_s^N\left(-\frac{1}{s} + \frac{1}{N}, K\right) = B_s^N\left(-\frac{1}{s}, K+1\right) < -\frac{1}{s} \text{ and } B_s^N\left(-\frac{1}{s}, K\right) > -\frac{1}{s}.$$

Applying Lemma 5.5, we get that

$$\begin{aligned} B_s^N\left(-\frac{1}{s} + \frac{\tau}{N}, K\right) &= (1-\tau) B_s^N\left(-\frac{1}{s}, K\right) + \tau B_s^N\left(-\frac{1}{s}, K+1\right) \\ &= \frac{\tau - 1}{s} Q_s^N(K+N) - \frac{\tau}{s} Q_s^k(K+1+N). \end{aligned}$$

The unique value of $\tau \in [0, 1]$ that makes the above expression equal to $-1/s$ is

$$\tau_* = \frac{1 - Q_s^N(K+N)}{Q_s^N(K+1+N) - Q_s^N(K+N)}.$$

Write $t_* = -1/s + \tau_*/N$. By (e), we see that $B_s^N(t, K) > -1/s$ for $t \in [-1/s, t_*)$ and that $B_s^N(t, K) \leq -1/s$ for $t \in [t_*, -1/s + 1/N)$. The same sort of reasoning as in the first two steps now shows that, for $t \in [t_*, -1/s + 1/N)$, we have

$$g(t) = B_s^N(t, K+N) + \frac{\text{some nonnegative integer depending on } t}{N},$$

and so a valid formula for a lift *on the subinterval* $[t_*, -1/s + 1/N)$ is

$$t \mapsto B_s^N(t_*, K+N) + (t - t_*)Q_s^N(K+N).$$

Similarly, if $t \in [-1/s, t_*)$, we have

$$g(t) = B_s^N(t, K+1+N) + \frac{\text{some nonnegative integer depending on } t}{N},$$

and so a valid formula for a lift *on the subinterval* $[-1/s, t_*)$ is

$$t \mapsto B_s^N\,(t_*, K+1+N) - (t_* - t)Q_s^N(K+1+N).$$

Now, since $B_s^N(t_*, K)$ is equal to exactly $-1/s$, the difference between $B_s^N(t_*, K+N)$ and $B_s^N(t_*, K+N+1)$ is governed by Equation (EL_N^s), and so

$$B_s^N(t_*, K+1+N) = B_s^N(t_*, K+N) - \frac{s}{N}B_s^N(t_*, K) = B_s^N(t_*, K+N) + \frac{1}{N}.$$

Thus we can subtract $1/N$ from the formula for γ on $[-1/s, t_*)$ to get the continuous formula for γ on all of $\overline{I}$ given in the Proposition.

It remains to show that g has degree 1. In Cases 1 and 2, since γ has constant slope 1 it is clear that

$$\gamma\left(-\frac{1}{s} + \frac{1}{N}\right) - \gamma\left(-\frac{1}{s}\right) = \frac{1}{N}.$$

To obtain the same conclusion for Case 3, we recall that $t_* = -1/s + \tau_*/N$ (and the defining equation for τ_*) and compute

$$\begin{aligned}
&\gamma\left(-\frac{1}{s} + \frac{1}{N}\right) - \gamma\left(-\frac{1}{s}\right) \\
&= \left(t_* + \frac{1}{s}\right) Q_s^N(K+1+N) + \left(-\frac{1}{s} + \frac{1}{N} - t_*\right) Q_s^N(K+N) \\
&= \frac{\tau_*}{N}Q_s^N(K+1+N) + \frac{(1-\tau_*)}{N}Q_s^N(K+N) = \frac{1}{N}.
\end{aligned}$$

This completes the proof. $\square$

In seeking to apply Proposition 5.6, we are guided by the fact that the formula for Q_s^N makes our calculations relatively easy for values of K that are close to (small) integer multiples of N; and indeed, we will find what we want for such K. For the convenience of the reader, we collect several formulas we need to finish our work. These are all consequences of the definitions of Q and B and various parts of Lemma 5.4.

$$
\begin{aligned}
Q_s^N(N+1) &= 1 - \frac{1}{N}s, \\
Q_s^N(N+2) &= 1 - \frac{2}{N}s, \\
Q_s^N(2N-1) &= 1 - \frac{N-1}{N}s, \\
Q_s^N(2N) &= 1 - s, \\
Q_s^N(2N+1) &= 1 - \frac{N+1}{N}s, \\
Q_s^N(2N+2) &= 1 - \frac{N+2}{N}s + \frac{1}{N^2}s^2, \\
Q_s^N(3N) &= 1 - 2s + \frac{N-1}{2N}s^2, \\
Q_s^N(3N+1) &= 1 - \frac{2N+1}{N}s + \frac{N+1}{2N}s^2, \\
Q_s^N(3N+2) &= 1 - \frac{2N+2}{N}s + \frac{N^2+3N+2}{2N^2}s^2, \\
Q_s^N(4N) &= 1 - 3s + \frac{2N-1}{N}s^2 - \frac{(N-1)(N-2)}{6N^2}s^3, \\
Q_s^N(4N+1) &= 1 - \frac{3N+1}{N}s + \frac{2N+1}{N}s^2 - \frac{(N+1)(N-1)}{6N^2}s^3.
\end{aligned}
$$

$$B_s^N(t, N+1) = t\left(1 - \frac{1}{N}s\right) + 1,$$

$$B_s^N(t, N+2) = t\left(1 - \frac{2}{N}s\right) + 1 - \frac{1}{N^2}s,$$

$$B_s^N(t, 2N) = t(1-s) + 1 - \frac{N-1}{2N}s,$$

$$B_s^N(t, 2N+1) = t\left(1 - \frac{N+1}{N}s\right) + 1 - \frac{N+1}{2N}s,$$

$$B_s^N(t, 2N+2) = t\left(1 - \frac{N+2}{N}s + \frac{1}{N^2}s^2\right) + 1 - \frac{N+3}{2N}s,$$

$$B_s^N(t, 3N) = t\left(1 - 2s + \frac{N-1}{2N}s^2\right) + 1 - \frac{3N-1}{2N}s + \frac{(N-1)(N-2)}{6N^2}s^2,$$

$$B_s^N(t, 3N+1) = t\left(1 - \frac{2N+1}{N}s + \frac{N+1}{2N}s^2\right) + 1 - \frac{3N+1}{2N}s + \frac{(N+1)(N-1)}{6N^2}s^2.$$

5.1. EXISTENCE OF SOLUTIONS OF LONG PERIOD, AND SOLUTIONS THAT DO NOT APPROACH PERIODIC SOLUTIONS

In this subsection, we consider values of s between $s(N, 2N+1)$ and $s(N, 2N)$. We omit the lengthy but straightforward proof of the following lemma.

Lemma 5.7. *Let $N > 1$. With notation as above, we have*

$$s(N, 2N+1) = \frac{4N+2}{N+1} \quad \textit{and} \quad s(N, 2N) = \frac{4N}{N-1}.$$

Furthermore, there is some $N_* \in \mathbb{N}$ such that, whenever $N \geq N_*$, all of the hypotheses of Proposition 5.6 are satisfied, with $2N$ in the role of K.

Lemma 5.8. *There is a natural number N_{**} such that the following hold.*

- *For all $N \geq N_{**}$ and all $s \in [s(N, 2N+1), s(N, 2N)]$, both $Q_s^N(3N)$ and $Q_s(3N+1)$ are strictly between 0 and 2.*

– *There are infinitely many values of $N \geq N_{**}$ for which the two numbers*

$$-\frac{1}{s(N,2N)} - B^N_{s(N,2N)}\left(-\frac{1}{s(N,2N)}, 3N\right)$$

and

$$-\frac{1}{s(N,2N+1)} - B^N_{s(N,2N+1)}\left(-\frac{1}{s(N,2N+1)}, 3N+1\right)$$

do not differ by a multiple of $1/N$.

Before proving this lemma, let's explain why it establishes points (II) and (III) of Theorem 1.3; the ideas are basically the same as in Section 4.

Let N be one of the (infinitely many) natural numbers for which all points of Lemmas 5.7 and 5.8 hold. Then, since $N > N_*$, the circle map g has the form described in Proposition 5.6. In particular, for $s = s(N, 2N+1)$ and $s = s(N, 2N)$, the maps g are rotations; but by the second point of Lemma 5.8 they are *two different rotations*. For all s between $s(N, 2N+1)$ and $s(N, 2N)$, the first point of Lemma 5.8 shows that g is a circle homeomorphism of degree 1 (since it has a strictly increasing lift γ whose slope is always less than 2). Thus, as s ranges from $s(N, 2N+1)$ and $s(N, 2N)$, the rotation number of g is always defined and varies continuously from one number to another (distinct) number.

Thus there are some values of s for which the rotation number of g is irrational, and so g has no periodic points. In this case there are no periodic solutions of Equation (E_N) passing through $\overline{L}$. Since $\overline{L}$ is compact, solutions of Equation (E_N) passing through $\overline{L}$ do not approach any periodic solution. There are other values of s for which the rotation number is rational but has large denominator, and so g has periodic points of long period. These periodic points correspond to periodic solutions of Equation (E_N) with long minimal period and many sign changes per minimal period.

Proof of Lemma 5.8. When N is large, $s \in [s(N, 2N+1), s(N, 2N)]$, and $t \in \overline{I}$, we have $s \approx 4$ and $t \approx -1/4$ and so

$$Q^N_s(3N) \approx Q^N_s(3N+1) \approx 1.$$

The establishes the first point of the lemma.

For the second part of the lemma, plugging in our various formulas yields (for $N \geq 3$) that

$$-\frac{1}{s(N,2N)} - B^N_{s(N,2N)}\left(-\frac{1}{s(N,2N)}, 3N\right) = \frac{7N+13}{3N-3}$$

and

$$\frac{1}{s(N,2N+1)} - B^N_{s(N,2N+1)}\left(-\frac{1}{s(N,2N+1)}, 3N+1\right) = \frac{7N^3+12N^2+9N+2}{3N^3+3N^2}.$$

The difference of these two quantities is

$$\frac{1}{N} \times \frac{15N^3+16N^2+7N+2}{3N^3-3N}.$$

Since the factor on the right is never equal to 5 but approaches 5 as $N \to \infty$, for all N sufficiently large it is not an integer. This completes the proof. $\square$

As discussed above, we have established points (II) and (III) of Theorem 1.3.

5.2. CHAOTIC SOLUTIONS

The approach here is very similar to that already demonstrated. We leave most of the algebraic and arithmetic verifications to the reader. Given N, for the remainder of this section let us fix

$$s_* = 2N^2 + 3N \quad \text{and} \quad t_* = -\frac{1}{s_*} + \frac{1}{2N} = \frac{2N+1}{2N(2N+3)}.$$

Observe that t_* is the midpoint of the interval I.

Plugging $s = s_*$ into the formulas for $B_s^N(t,k)$ listed above, we get

$$B_{s_*}^N(t, N+1) = -t(2N+2) + 1,$$

$$B_{s_*}^N(t, N+2) = -t(4N+5) - 1 - \frac{3}{N},$$

$$B_{s_*}^N(t, 2N+1) = -t(2N^2+5N+2) - \frac{2N^2+5N+1}{2},$$

$$B_{s_*}^N(t, 2N+2) = t(2N^2+5N+4) - \frac{2N^2+9N+7}{2}.$$

The following facts are readily verified.

(1) $B_{s_*}^N(t, N+1) \le -1/s$ if and only if $t \ge t_*$.
(2) $B_{s_*}^N(t, N+2) \le -1/s$ for all $t \in I$.
(3) $B_{s_*}^N(t, 2N+1) \le -1/s$ for all $t \in I$.
(4) $B_{s_*}^N(t, 2N+2) \le -1/s$ for all $t < t_*$.
(5) $B_{s_*}^N(t_*, 2N+2) - B_{s_*}^N(t_*, 2N+1) = 1/N$.

The same sort of reasoning we have used multiple times now shows that the following is a valid lift γ of the return map for g:

$$\gamma(t) = \begin{cases} B_{s_*}^N(t, 2N+2) + \frac{1}{N}, & t \in \left[-\frac{1}{s}, t_*\right], \\ B_{s_*}^N(t, 2N+1), & t \in \left[t_*, -\frac{1}{s} + \frac{1}{N}\right]. \end{cases}$$

In other words, on I, γ is piecewise linear, with slope

$$2N^2 + 5N + 4 \text{ on } \left[-\frac{1}{s}, t_*\right]$$

and slope

$$-(2N^2+5N+2) \text{ on } \left[t_*, -\frac{1}{s}+\frac{1}{N}\right].$$

Since each of these two subintervals is $1/(2N)$ units long, the image of each of these subintervals under γ is at least $N+2$ units long. In other words, g "wraps each subinterval around the circle" several times, and the graph of g looks something like Figure 4. In particular, the hypotheses of Proposition 3.4 are satisfied. This establishes point (IV) of Theorem 1.3.

6. APPENDIX – PROOF OF THEOREM 2.4

We retain all the notation from Section 2. We repeat that the function ρ is a general and standard way to define a semimetric on the quotient of a metric space by an equivalence relation (see Section 3.1 of [1]). In particular, the following properties are straightforward to verify.

(i) $\rho(u,v) \geq 0$ for all $u, v \in X$.
(ii) If $u \sim v$, then $\rho(u,v) = 0$.
(iii) $\rho(u,v) = \rho(v,u)$.
(iv) The triangle inequality: given any $u, v, y \in X$, we have

$$\rho(u,v) \leq \rho(u,y) + \rho(y,v).$$

(v) If $u \sim v$, then for any $y \in X$ we have

$$\rho(u,y) = \rho(v,y) \text{ and } \rho(y,u) = \rho(y,v).$$

Property (v) ensures that the map $\tilde{\rho}([u],[v]) = \rho(u,v)$ is well-defined, and properties (i)–(iv) ensure that $\tilde{\rho}$ is a semimetric on $\tilde{X}$. In general, however, it is possible for $\rho(u,v)$ to be zero even if $u \not\sim v$. That this is not the case in our situation is the substance of the following proposition.

Proposition 6.1. *If $\rho(u,v) = 0$, then $u \sim v$ (and so $\tilde{\rho}$ is in fact a metric).*

Proof. Let us choose $u, v \in X$ with $u \not\sim v$. By point (v) above, we can choose u to be the "most advanced" member of $[u]$ — in other words, we can insist that $u \in X \setminus X_0$. Since X_0 is closed in X, if $w \in X$ and $d(u,w)$ is sufficiently small, then $w \in X \setminus X_0$ also. Also observe that, by (H3), u is disjoint from the compact set

$$F^{N+1}(X) \cup F^{N+2}(X) \cup \cdots \cup F^{2N}(X).$$

Finally observe that, if $0 \leq j \leq N$, then $F^j(v)$ either belongs to $[v]$ or (by (H3)) is not in X at all; since $u \not\sim v$ the sets $[u]$ and $[v]$ are disjoint, and accordingly we see that u is disjoint from the finite set

$$\{v, F(v), \ldots, F^N(v)\}.$$

Putting these observations together yields the following lemma.

There is some $D > 0$ such that the following hold:

(a) *If $d(u, w) < D$ and $w \in X$, then $w \in X \setminus X_0$;*
(b) *If $d(u, w) < D$ and $w = F^j(z)$ for some $z \in X$ and $0 \le j \le 2N$, then in fact $0 \le j \le N$;*
(c) *$d(u, F^j(v)) \ge D$ for all $0 \le j \le N$.*

Now let $L > 1$ be a number with the feature that $Lip(F^j) \le L$ for all $0 \le j \le N$, and choose $\varepsilon < D/L$.

Imagine that $\rho(u, v) = 0$. Then we can choose

$$w_i \sim z_i \in X, \quad 1 \le i \le k$$

such that

$$d(u, w_1) < \varepsilon_0, \; d(z_1, w_2) < \varepsilon_1, \quad \ldots, \quad d(z_k, v) < \varepsilon_k$$

and

$$\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_k < \varepsilon.$$

Now, since $d(u, w_1) < D$, we know that w_1 is the "most advanced" member of its orbit (that is, $w_1 \in X \setminus X_0$); thus there is some $0 \le j_1 \le N$ such that

$$F^{j_1}(z_1) = w_1.$$

Therefore

$$\begin{aligned} d(u, F^{j_1}(w_2)) &\le d(u, w_1) + d(w_1, F^{j_1}(w_2)) \\ &= d(u, w_1) + d(F^{j_1}(z_1), F^{j_1}(w_2)) \\ &\le d(u, w_1) + Ld(z_1, w_2) \\ &< \varepsilon_0 + L\varepsilon_1 < D. \end{aligned}$$

Now, if $F^{j_1}(w_2)$ actually lies in X, by point (a) above it is the "most advanced" member of $[w_2] = [z_2]$; if not, it is the result of applying F no more than $j_1 \le N$ times to the "most advanced" member of $[w_2] = [z_2]$. Either way (remember that $[z_2]$ contains at most $N + 1$ elements by (H3)) there is some $0 \le j_2 \le 2N$ such that $F^{j_2}(z_2) = F^{j_1}(w_2)$, and so

$$d(u, F^{j_2}(z_2)) < D.$$

By point (b) above we conclude that we actually have $0 \le j_2 \le N$.

We now have

$$\begin{aligned} d(u, F^{j_2}(w_3)) &\le d(u, F^{j_1}(w_2)) + d(F^{j_1}(w_2), F^{j_2}(w_3)) \\ &= d(u, F^{j_1}(w_2)) + d(F^{j_2}(z_2), F^{j_2}(w_3)) \\ &\le d(u, F^{j_1}(w_2)) + Ld(z_2, w_3) \\ &< \varepsilon_0 + L\varepsilon_1 + L\varepsilon_2 < D. \end{aligned}$$

Imitating the same sort of argument as before yields that there is some $0 \le j_3 \le N$ such that

$$F^{j_2}(w_3) = F^{j_3}(z_3),$$

and so

$$\begin{aligned} d(u, F^{j_3}(w_4)) &\leq d(u, F^{j_2}(w_3)) + d(F^{j_2}(w_3), F^{j_3}(w_4)) \\ &= d(u, F^{j_2}(w_3)) + d(F^{j_3}(z_3), F^{j_3}(w_4)) \\ &\leq d(u, F^{j_2}(w_3)) + Ld(z_3, w_4) \\ &< \varepsilon_0 + L\varepsilon_1 + L\varepsilon_2 + L\varepsilon_3 < D. \end{aligned}$$

Continuing in this way for finitely many steps yields that there is some $0 \leq j_k \leq N$ such that

$$d(u, F^{j_k} v) < L\varepsilon < D,$$

Contradicting point (c). We therefore cannot have $\rho(u, v) = 0$. □

Let us define the map $\Pi : X \to \tilde{X}$ by $\Pi(u) = [u]$.

Lemma 6.2. *The map $\Pi : X \to \tilde{X}$ is continuous, and $\tilde{X}$ is compact. In fact, Π is strongly continuous: that is, the set $U \subseteq \tilde{X}$ is open if and only if $\Pi^{-1}(U)$ is open in X.*

In the following proof we write $B_r^\tau(q)$ to mean the open ball of radius r centered at q with respect to the metric τ.

Proof. Suppose that $U \subseteq \tilde{X}$ is open. Choose $u \in \Pi^{-1}(U)$. Since U is open there is some $\varepsilon > 0$ such that $B_\varepsilon^{\tilde{\rho}}([u]) \subseteq U$.

Suppose that $d(u, v) < \varepsilon$. Then $\rho(u, v) < \varepsilon$ and so $[v] \in B_\varepsilon^{\tilde{\rho}}([u]) \subseteq U$. This shows that $B_\varepsilon^d(u) \subseteq \Pi^{-1}(U)$, and so $\Pi^{-1}(U)$ is open. Therefore $\Pi : X \to \tilde{X}$ is continuous; since X is compact, it follows that $\tilde{X}$ is compact.

Suppose now that $U \subseteq \tilde{X}$ and that $\Pi^{-1}(U)$ is open. Write W for the complement of $\Pi^{-1}(U)$ in X; observe that $\Pi(W)$ is the complement of U in $\tilde{X}$. Since W is a closed subset of a compact set, it is compact; thus $\Pi(W)$ is compact and so U is open. □

Remark 6.3. The compactness of X is crucial in the proof of Lemma 6.2. If X is any metric space equipped with an equivalence relation and we topologize $\tilde{X}$ with the semimetric $\tilde{\rho}$, the map $x \mapsto [x]$ will always be continuous but need not be strongly continuous.

Recalling the notation in Hypothesis (H3), given $v \in X$ let us write $m(x)$ for the first integer greater than $2N$ satisfying $F^{m(x)}(x) \in X$. We now define the return map $R : \tilde{X} \to \tilde{X}$ by the formula

$$R([v]) = \left[F^{m(v)}(v)\right].$$

The following proposition will complete the proof of Theorem 2.4.

Proposition 6.4. $R : \tilde{X} \to \tilde{X}$ *is well-defined and continuous.*

Proof. Since all the elements of $[v]$ have the same forward orbit under F (up to a shift), the map R is well-defined. It remains to show that R is continuous.

Let us choose and fix v, and let $\varepsilon > 0$ be given. Now,

$$R^{-1} B_\epsilon^{\tilde{\rho}}([v]) = \{[u] : \tilde{\rho}(R([u]), [v]) < \varepsilon\}.$$

Define the set

$$V = \Pi^{-1} R^{-1} B_\epsilon^{\tilde{\rho}}([v]) = \{u : \rho(F^{m(u)}(u), v) < \varepsilon\}.$$

We wish to show that V is open in X. Then the strong continuity of Π will imply that $R^{-1} B_\epsilon^{\tilde{\rho}}([v])$ is open in $\tilde{X}$, and hence that R is continuous.

Now, choose and fix $u \in V$, and suppose that $\rho(F^{m(u)}(u), v) = \varepsilon_0 < \varepsilon$. Take $r < \varepsilon - \varepsilon_0$. Choose an open set U in X about u such that $w \in U$ implies that $m(u) \leq m(w)$ and that $F^{m(w)}(u) \in X$, with $[F^{m(w)}(u)] = [F^{m(u)}(u)]$ (we are using hypothesis (H4) here). Since u is fixed and $m(w) \leq m(u) + N$, shrinking U if necessary we may assume that $d(F^{m(w)}(w), F^{m(w)}(u)) < r$. In this case we have

$$\rho(F^{m(w)}(w), F^{m(u)}(u)) = \rho(F^{m(w)}(w), F^{m(w)}(u)) \leq d(F^{m(w)}(w), F^{m(w)}(u)) < r,$$

and so by the triangle inequality for ρ we have

$$\rho(F^{m(w)}(w), v) < \varepsilon.$$

Thus $U \subseteq V$ and V is open. $\square$

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Received: January 13, 2023.
Revised: April 5, 2023.
Accepted: April 14, 2023.