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ON THE EXISTENCE OF OPTIMAL SOLUTIONS TO THE LAGRANGE PROBLEM GOVERNED BY A NONLINEAR GOURSAT–DARBOUX PROBLEM OF FRACTIONAL ORDER

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Abstract. In the paper, the Lagrange problem given by a fractional boundary problem with partial derivatives is considered. The main result is the existence of optimal solutions based on the convexity assumption of a certain set. The proof is based on the lower closure theorem and the appropriate implicit measurable function theorem.

Keywords: fractional partial derivative, fractional boundary problem, existence of optimal solutions, Lagrange problem, lower closure theorem.

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1. INTRODUCTION

In this paper we investigate the following optimal control problem

$$J(z,u)=\iint\limits_P F\left(x,y,z(x,y),D_x^{\alpha}z(x,y),D_y^{\beta}z(x,y),u(x,y)\right)\longrightarrow \min, \tag{1.1}$$

where z is the solution to a fractional partial equation of the form

$$\begin{aligned}&D_{x,y}^{\alpha,\beta}z(x,y)=f(x,y,z(x,y),D_x^{\alpha}z(x,y),D_y^{\beta}z(x,y),u(x,y))\\&\text{for } (x,y)\in P:=[0,a]\times[0,b] \text{ a.e.,}\end{aligned} \tag{1.2}$$

corresponding to a control u and satisfying

$$\begin{aligned}I_{x,y}^{1-\alpha,1-\beta}z(x,0)&=\gamma(x)\quad \text{for } x\in[0,a],\\ I_{x,y}^{1-\alpha,1-\beta}z(0,y)&=\delta(y)\quad \text{for } y\in[0,b].\end{aligned} \tag{1.3}$$

where F, $f:P\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$. The symbols $I_{x,y}^{1-\alpha,1-\beta}$ and D_x^{α}, D_y^{β}, $D_{x,y}^{\alpha,\beta}$ stand for partial integral and derivatives resp. of fractional orders ($\alpha,\beta\in(0,1]$).

Differential problem (1.2)–(1.3) can be treated as a fractional counterpart to the Goursat–Darboux problem with control u of the form

$$\frac{\partial^2 z}{\partial x \partial y}(x,y) = f\Big(x, y, z\,(x,y)\,, \frac{\partial z}{\partial x}(x,y), \frac{\partial z}{\partial y}(x,y), u(x,y)\Big) \tag{1.4}$$

$$\begin{aligned} z(x,0) &= \gamma(x) \text{ for } x \in [0,a]\,, \\ z(0,y) &= \delta(x) \text{ for } y \in [0,b]\,. \end{aligned} \tag{1.5}$$

Classical problem (1.4)–(1.5) has been quite thoroughly studied both in terms of existence of solutions, continuous dependence on a parameter and existence of optimal solutions (see [1, 6, 7, 9, 10, 13]).

It is worth emphasising that this work is part of a series of papers on (1.2)–(1.3). In the first paper [5] in this series the existence, uniqueness and continuous dependence of solutions on control has been investigated. The space of solutions for the classical problem (1.4)–(1.5) is the space AC of absolutely continuous functions of two variables. For problem (1.2)–(1.3), an analogon of this space (denoted $AC^{\alpha,\beta}_{x,y}$) is introduced. This space has many interesting properties investigated in the next paper [4] of the aforementioned series . Among other things, we find there a characterisation of weak convergence in $AC^{\alpha,\beta}_{x,y}$ very crucial for our approach.

The main result of this paper is the existence of optimal solutions to the Lagrange problem governed by (1.1)–(1.3). The result is obtained assuming the convexity of some subset of finite-dimensional space (see (A6)) and applying Measurable Selection Theorem 3.4. Importantly, it does not require the assumption on the convexity of the functional J or the linearity of the equation (1.2). The approach is adopted from the book of Lamberto Cesari [2] and is based on the use of the so-called Lower Closure Theorem 3.3. There are many articles in the literature applying this idea for both ordinary derivative problems and equations with fractional operators (see e.g. [1, 8]). To the best knowledge of the author, however, this approach has not been applied to problems with fractional derivatives of fractional order.

The discrete counterpart of the problem (1.4)–(1.5) is known in the technical literature as the Fornasini-Marchesini problem and has wide applications in electronics and signal processing, among others (see [3]). It therefore seems reasonable to ask whether also the discrete equivalent of the problem (1.2)–(1.3) can be used to model physical phenomena. At this stage of the research, the author of the paper are unable to answer this question. Modelling phenomena based on equations with fractional derivatives is often based on replacing classical derivatives with fractional-order derivatives. This approach, although it seems naive, gives very good numerical results. Perhaps this is due to the non-local nature of the concept of fractional derivatives, which in a sense averages out the errors associated with the imperfection of models based on integer-order derivatives. Undoubtedly, it is worth undertaking further research in this area starting by proposing a discretisation of fractional order derivatives.

2. SPACE $AC^{\alpha,\beta}_{x,y}$

As mentioned in the previous section, we look for solutions to the differential problem (1.2)–(1.3) in the space $AC^{\alpha,\beta}_{x,y}$, which is the equivalent of the space of absolutely continuous functions of two variables AC introduced in [13]. In this section we will define the concepts of integrals and derivatives of fractional order necessary to introduce the space $AC^{\alpha,\beta}_{x,y}$. We will limit ourselves here to the most important concepts and facts necessary for our further considerations. A thorough discussion of the properties of these concepts along with their connections with classical equivalents can be found in [5].

We start with a definition of the fractional integral. For a function $\varphi \in L^1(P,\mathbb{R}^n)$ and $\alpha > 0$, we define the functions $I^\alpha_x\varphi : P \to \mathbb{R}^n$ and $I^\alpha_y\varphi : P \to \mathbb{R}^n$ by

$$I^\alpha_x\varphi(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(s,y)}{(x-s)^{1-\alpha}}ds,$$
$$I^\alpha_y\varphi(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^y \frac{\varphi(x,t)}{(y-t)^{1-\alpha}}dt.$$

Of course by Fubini's theorem $I^\alpha_x\varphi, I^\alpha_y\varphi \in L^1(P,\mathbb{R}^n)$.

Next, for $\alpha,\beta > 0$ and $z \in L^1(P,\mathbb{R}^n)$ we define

$$I^{\alpha,\beta}_{x,y}z(x,y) := \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\iint_{(0,x)\times(0,y)} \frac{z(s,t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}}d(s,t), \quad (x,y) \in P \text{ a.e.}$$

Using Fubini's theorem, one can show that $I^{\alpha,\beta}_{x,y}z \in L^1(P,\mathbb{R}^n)$ and

$$I^{\alpha,\beta}_{x,y}z(x,y) = I^\alpha_x I^\beta_y z(x,y) = I^\beta_y I^\alpha_x z(x,y)$$

for $(x,y) \in P$ a.e.

To define the notion of fractional partial derivative we need to provide the notation of the set AC_x (resp. AC_y) of functions of two variables which are absolutely continuous in x (in y resp.) Namely, we say that $\varphi \in AC_x$ if $\varphi \in L^1(P,\mathbb{R}^n)$, $\varphi(\cdot,y)$ is absolutely continuous on $[0,a]$ for $y \in [0,b]$ a.e., $\varphi(0,\cdot) \in L^1([0,b],\mathbb{R}^n)$ and $\frac{\partial\varphi}{\partial x} \in L^1(P,\mathbb{R}^n)$. In an analogous way we define the set AC_y.

We say that a function $z \in L^1(P,\mathbb{R}^n)$ has the left fractional partial derivative $D^\alpha_x z$ (resp. $D^\alpha_y z$) of the order $\alpha \in (0,1)$ in the Riemann–Liouville sense, with respect to x (resp. y) on the interval P, if $I^{1-\alpha}_x z \in AC_x$ (resp. $I^{1-\alpha}_y z \in AC_y$). We put in such a case

$$D^\alpha_x z(x,y) = \frac{\partial}{\partial x}(I^{1-\alpha}_x z)(x,y),\ (x,y) \in P \text{ a.e.}$$
$$(D^\alpha_y z(x,y) = \frac{\partial}{\partial y}(I^{1-\alpha}_y z)(x,y),\ (x,y) \in P \text{ a.e., resp.)}$$

The set of all functions z possessing the partial derivatives $D^\alpha_x z$ (resp. $D^\alpha_y z$) will be denoted by AC^α_x (resp. AC^α_y).

Finally, we say that a function $z \in L^1(P, \mathbb{R}^n)$ possesses a mixed fractional derivative $D^{\alpha,\beta}_{x,y} z$ of the order $(\alpha, \beta) \in (0,1) \times (0,1)$ in the Riemann–Liouville sense, on the interval P, if $I^{1-\alpha,1-\beta}_{x,y} z \in AC$. In such a case we put

$$D^{\alpha,\beta}_{x,y} z(x,y) = \frac{\partial^2}{\partial x \partial y} \left(I^{1-\alpha,1-\beta}_{x,y} z \right)(x,y), \quad (x,y) \in P \text{ a.e.}$$

The set of all functions possessing the mixed fractional derivative of the order (α, β) will be denoted by $AC^{\alpha,\beta}_{x,y}$.

A very crucial observation is the following theorem.

Theorem 2.1 ([5])**.** *If $z \in L^1(P, \mathbb{R}^n)$, then z has the mixed fractional derivative $D^{\alpha,\beta}_{x,y} z$ if and only if there exist functions $\varphi \in L^1(P, \mathbb{R}^n)$, $\mu \in L^1([0,a], \mathbb{R}^n)$, $\nu \in L^1([0,b], \mathbb{R}^n)$ and a constant $e \in \mathbb{R}^n$ such that*

$$\begin{aligned} z(x,y) = I^{\alpha,\beta}_{x,y} \varphi(x,y) &+ \frac{1}{\Gamma(\alpha)} \frac{1}{x^{1-\alpha}} I^{\beta}_{0+} \nu(y) + \frac{1}{\Gamma(\beta)} \frac{1}{y^{1-\beta}} I^{\alpha}_{0+} \mu(x) \\ &+ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \frac{e}{x^{1-\alpha} y^{1-\beta}} \end{aligned} \tag{2.1}$$

for $(x,y) \in P$ a.e. In such a case

$$\begin{aligned} D^{\alpha,\beta}_{x,y} z(x,y) &= \varphi(x,y), \quad (x,y) \in P \ \text{a.e.}, \\ D^{\alpha}_{x} (I^{1-\beta}_{y} z)(x,0) &= \mu(x), \quad x \in [0,a] \ \text{a.e.}, \\ D^{\beta}_{y} (I^{1-\alpha}_{x} z)(0,y) &= \nu(y), \quad y \in [0,b] \ \text{a.e.} \\ I^{1-\alpha,1-\beta}_{x,y} z(0,0) &= e. \end{aligned}$$

Furthermore, using the representation (2.1), it can be shown (see [5, Remark 17]) that

$$\begin{aligned} D^{\alpha}_{x} z(x,y) &= I^{\beta}_{y} \varphi(x,y) + \frac{1}{\Gamma(\beta)} \frac{1}{y^{1-\beta}} \mu(x), \\ D^{\beta}_{y} z(x,y) &= I^{\alpha}_{x} \varphi(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{x^{1-\alpha}} \nu(y) \end{aligned} \tag{2.2}$$

for $(x,y) \in P$ a.e.

Note that the space $AC^{\alpha,\beta}_{x,y}$ is a Banach space with the following norm

$$\begin{aligned} \|z\|_{AC^{\alpha,\beta}_{x,y}} := \left\| D^{\alpha,\beta}_{x,y} z \right\|_{L^1(P,\mathbb{R}^n)} &+ \left\| D^{\alpha}_{x} (I^{1-\beta}_{y} z)(\cdot, 0) \right\|_{L^1([0,a],\mathbb{R}^n)} \\ &+ \left\| D^{\beta}_{y} (I^{1-\alpha}_{x} z)(0, \cdot) \right\|_{L^1([0,b],\mathbb{R}^n)} + \left| I^{1-\alpha,1-\beta}_{x,y} z(0,0) \right|_{\mathbb{R}^n}, \end{aligned}$$

which, given representation (2.1), takes the form

$$\|z\|_{AC^{\alpha,\beta}_{x,y}} = \|\varphi\|_{L^1(P,\mathbb{R}^n)} + \|\mu\|_{L^1([0,a],\mathbb{R}^n)} + \|\nu\|_{L^1([0,b],\mathbb{R}^n)} + |e|_{\mathbb{R}^n}.$$

The weak convergence in the space $AC^{\alpha,\beta}_{x,y}$ can be easily characterised by the following proposition.

Proposition 2.2 ([4, Corollary 1]). *A sequence $(z^k) \subset AC^{\alpha,\beta}_{x,y}$ is weakly convergent to $z^0 \in AC^{\alpha,\beta}_{x,y}$ if and only if*

(i) $D^{\alpha,\beta}_{x,y} z^k \rightharpoonup D^{\alpha,\beta}_{x,y} z^0$ *weakly in* $L^1(P,\mathbb{R}^n)$,
(ii) $\frac{\partial}{\partial x}(I^{1-\alpha,1-\beta}_{x,y} z^k)(\cdot,0) \rightharpoonup \frac{\partial}{\partial x}(I^{1-\alpha,1-\beta}_{x,y} z^0)(\cdot,0)$ *weakly in* $L^1([0,a],\mathbb{R}^n)$,
(iii) $\frac{\partial}{\partial y}(I^{1-\alpha,1-\beta}_{x,y} z^k)(0,\cdot) \rightharpoonup \frac{\partial}{\partial y}(I^{1-\alpha,1-\beta}_{x,y} z^0)(0,\cdot)$ *weakly in* $L^1([0,b],\mathbb{R}^n)$,
(iv) $I^{1-\alpha,1-\beta}_{x,y} z^k(0,0) \to I^{1-\alpha,1-\beta}_{x,y} z^0(0,0)$ *in* $\mathbb{R}^n$.

Remark 2.3. Using (2.1), (2.2) and applying Fubini's theorem, it can be easily shown that

$$\begin{aligned}\|D^\alpha_x z\|_{L^1(P,\mathbb{R}^n)} &\le \int_0^a\int_0^b \left|I^\beta_y \varphi(x,y)\right| dxdy + \frac{1}{\Gamma(\beta)}\int_0^a\int_0^b \left|\frac{1}{y^{1-\beta}}\mu(x)\right| dxdy \\ &\le c_1 \|\varphi\|_{L^1(P,\mathbb{R}^n)} + c_2 \|\mu\|_{L^1([0,a],\mathbb{R}^n)} \le c_3 \|z\|_{AC^{\alpha,\beta}_{x,y}}\end{aligned}$$

and similarly

$$\left\|D^\beta_y z\right\|_{L^1(P,\mathbb{R}^n)} \le c_4 \|z\|_{AC^{\alpha,\beta}_{x,y}},$$

which implies that the operator $D^\alpha_x : AC^{\alpha,\beta}_{x,y} \to L^1(P,\mathbb{R}^n)$ is bounded (and of course linear). Hence, weak convergence of the sequence $z^k \rightharpoonup z^0$ in $AC^{\alpha,\beta}_{x,y}$ implies weak convergence of the derivatives $D^\alpha_x z^k \rightharpoonup D^\alpha_x z^0$, $D^\beta_y z^k \rightharpoonup D^\beta_y z^0$ in L^1.

3. MAIN RESULT

In this section we will investigate the existence of optimal solutions to (1.1)–(1.3).

Let $U \subset \mathbb{R}^m$ be a fixed compact set. We study (1.1)–(1.3) in the space $AC^{\alpha,\beta}_{x,y}$ of solutions z and the set

$$\mathcal{U} := \left\{u \in L^1(P,\mathbb{R}^m) : u(x,y) \in U \text{ for } (x,y) \in P \text{ a.e.}\right\}$$

of controls u.

Let us first address the problem of the existence of solutions to the differential problem (1.2)–(1.3) with a fixed parameter u. We will use here the result of [5].

Theorem 3.1. *Assume that:*

(A1) $f(\cdot,z,z_1,z_2,u)$ *is measurable for all* $(z,z_1,z_2,u) \in^n \times^n \times^n \times^m$, $f(x,y,z,z_1,z_2,\cdot)$ *is continuous for* $(x,y) \in P$ *a.e. and every* $(z,z_1,z_2) \in^n \times^n \times^n$ *and satisfies the Lipschitz condition with respect to* $(z,z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, *i.e. there exists a constant* $L > 0$ *such that*

$$\left|f(x,y,z,z^1,z^2,u) - f(x,y,w,w^1,w^2,u)\right| \le L(|z-w| + \left|z^1-w^1\right| + \left|z^2-w^2\right|)$$

for $(x,y) \in P$ *a.e. and any* $z,z^1,z^2,w,w^1,w^2 \in \mathbb{R}^n, u \in \mathbb{R}^m$,

(A2) $f(\cdot,0,0,0,u(\cdot)) \in L^1(P,\mathbb{R}^n)$ *for any control* $u \in L^1(P,\mathbb{R}^m)$.

Then, for any control $u \in L^1(P,\mathbb{R}^m)$, *there exists a unique solution* z_u *to* (1.2)–(1.3), *corresponding to* u.

Assume (A1), (A2) and

(A3) $F(\cdot, z, z_1, z_2, u)$ is measurable for any (z, z_1, z_2, u) and $F(x, y, \cdot)$ is continuous for (x, y) a.e.

We say that a pair $(z, u) \in AC_{x,y}^{\alpha,\beta} \times \mathcal{U}$ is admissible if z is a solution to (1.2)–(1.3) corresponding to u. The set of all admissible pairs will be denoted by the symbol Ω. We say that a pair $(z^*, u^*) \in \Omega$ is optimal if

$$J(z^*, u^*) \leq J(z, u) \text{ for } (z, u) \in \Omega.$$

In our approach, the main result on the existence of optimal solution for problem (1.1)–(1.3) requires a weak compactness of the set of solutions to (1.2)–(1.3). This property can be obtained, for example, using the following

Theorem 3.2 ([4, Theorem 4.2]). *Let $d \in L^1(P, \mathbb{R})$ be a fixed function and let*

$$\mathcal{U}_d := \{u \in L^1(P, \mathbb{R}^m) : |f(x, y, 0, 0, 0, u(x, y))| \leq d(x, y) \text{ for } (x, y) \in P \text{ a.e.}\}. \quad (3.1)$$

Then the set of solutions to (1.2)–(1.3), *corresponding to controls $u \in \mathcal{U}_d$ is relatively weakly compact in the space $AC_{x,y}^{\alpha,\beta}$.*

In our case, however, it can be shown that the assumption of compactness of the set U together with assumption (A2) guarantee the existence of a function $d \in L^1(P, \mathbb{R})$ such that for all $u \in \mathcal{U}$

$$|f(x, y, 0, 0, 0, u(x, y))| \leq d(x, y) \text{ for } (x, y) \in P \text{ a.e.},$$

and consequently the set of solutions to (1.2)–(1.3), corresponding to controls $u \in \mathcal{U}$ is relatively weakly compact in the space $AC_{x,y}^{\alpha,\beta}$.

Indeed, thanks to (A1) and the fact that U is compact we can define the function $g : P \to \mathbb{R}$ by the formula

$$g(x, y) = \inf_{u \in U} \left(- |f(x, y, 0, 0, 0, u)|\right).$$

Applying [12, Theorem 2K] we get that g and the following function

$$\bar{u}(x, y) = \operatorname{argmin}_{u \in U} \left(- |f(x, y, 0, 0, 0, u)|\right)$$

are measurable. Of course defining $d(x, y) = -g(x, y)$ for $(x, y) \in P$ a.e. we get that for any $u \in \mathcal{U}$

$$|f(x, y, 0, 0, 0, u(x, y))| \leq d(x, y) \quad \text{for } (x, y) \in P \text{ a.e.}$$

The fact that $d \in L^1$ follows from the fact that $g(x, y) = f(x, y, 0, 0, 0, \bar{u}(x, y))$ and from (A2).

Let us further assume that

(A4) there exists a constant $\gamma \in \mathbb{R}$ such that

$$\Omega_\gamma := \{(z, u) \in \Omega : J(z, u) \leq \gamma\} \neq \emptyset,$$

(A5) there exists a function $\lambda \in L^1(P, \mathbb{R})$ such that

$$F\left(x, y, z(x, y), D_x^\alpha z(x, y), D_y^\beta z(x, y), u(x, y)\right) \geq \lambda(x, y)$$

for $(x, y) \in P$ a.e. and all $(z, u) \in \Omega_\gamma$,

(A6) the set

$$\begin{aligned} Q((x, y), z) := \{(\eta, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{R} \times (\mathbb{R}^n)^3 : &\text{ there is } u \in U \text{ such that} \\ &\eta \geq F(x, y, z, \zeta_2, \zeta_3, u) \wedge \zeta_1 = f(x, y, z, \zeta_2, \zeta_3, u)\} \end{aligned}$$

is convex for $(x, y) \in P$ a.e. and every $z \in \mathbb{R}^n$.

Of course, by defining set Q we are in fact defining, for a given $z \in \mathbb{R}^n$ a multivalued mapping $Q(\cdot, z) : P \multimap \mathbb{R} \times (\mathbb{R}^n)^3$.

For the reader's convenience, we will now recall some important facts about multifunctions.

Let $\Phi : \mathbb{R}^s \supset G \multimap \mathbb{R}^r$ be a closed-valued multifunction.

We say that Φ is measurable iff for any closed set $C \subset \mathbb{R}^r$ the set

$$\Phi^{-1}(C) := \{t \in G : \Phi(t) \cap C \neq \emptyset\}$$

is measurable.

Let $\operatorname{Dom}\Phi$ be the so-called effective domain of Φ that is the set of all $t \in \mathbb{R}^s$ for which $\Phi(s) \neq \emptyset$. A function $\varphi : \operatorname{Dom}\Phi \to \mathbb{R}^r$ is called a selection of Φ if $\varphi(t) \in \Phi(t)$ for $t \in G$.

We say that Φ has property (K) at the point $t_0 \in \mathbb{R}^s$ iff

$$\Phi(t_0) = \bigcap_{\delta > 0} \overline{\bigcup \{\Phi(t) : |t - t_0| < \delta\}}.$$

Moreover, we say that Φ has property (K) if it has property (K) at any $t \in \mathbb{R}^s$.

In the proof of theorem on the existence of optimal solutions, we will use the following result.

Theorem 3.3 ([2, Lower Closure Theorem for Orientor Fields, 10.7.i]). *Let $G \subset \mathbb{R}^s$ be measurable and of finite measure and for any $(t,z) \in G \times \mathbb{R}^n$ $\tilde{Q}(t,z)$ be a given closed and convex subset of $\mathbb{R} \times \mathbb{R}^v$ that has property* (K) *with respect to z for $t \in G$. Let $\eta^k, \zeta, \zeta^k, z, z^k, \lambda, \lambda^k$, $k = 1,2,\ldots$ be L^1 functions defined on G such that $z^k \to z$ in L^1, $\zeta^k \rightharpoonup \zeta$ and $\lambda^k \rightharpoonup \lambda$ weakly in L^1 as $k \to \infty$ and*

$$\left(\eta^k, \zeta^k\right) \in \tilde{Q}(t, z^k) \quad \textit{for } t \in G\, \textit{a.e. and } k = 1,2,\ldots,$$

$$-\infty < j := \liminf_{k\to\infty} \int\limits_G \eta^k(t)dt < \infty,$$

$$\eta^k(t) \geq \lambda^k(t) \quad \textit{for } t \in G \textit{ a.e. and } k = 1,2,\ldots$$

Then there is a function $\eta \in L^1$ such that

$$(\eta(t), \zeta(t)) \in \tilde{Q}\left(t, z(t)\right) \quad \textit{for } t \in G \textit{ a.e.} \quad \textit{and} \quad \int\limits_G \eta(t)dt \leq j.$$

We shall also apply the following theorem on implicit measurable functions, which has been, for the convenience of the reader, adapted to our case.

Theorem 3.4 ([12, Theorem 2J]). *Let $\Phi : G \multimap \mathbb{R}^r$ be a multifunction of the form*

$$\Phi(t) = \left\{\mu \in M : F\left(t,\mu\right) = \alpha\left(t\right) \textit{ and } f\left(t,\mu\right) \leq \eta(t)\right\},$$

where $G \subset \mathbb{R}^s$, $M \subset \mathbb{R}^r$ is a compact set, $F : G \times \mathbb{R}^r \to \mathbb{R}^l$ and $f : G \times \mathbb{R}^r \to \mathbb{R}$ are Carathéodory mapping (measurable with respect to the first and continuous with respect to the second variable) and α, η measurable functions. Then Φ is measurable and has a measurable selection, i.e. there is a measurable function $\varphi : \operatorname{Dom}\Phi \to M$ such that $F\left(t,\varphi(t)\right) = \xi(t)$ and $f\left(t,\varphi(t)\right) \leq \eta(t)$, $t \in G$.

We are now ready to formulate the main result of the paper.

Theorem 3.5. *Let $U \subset \mathbb{R}^m$ be a compact set. Then, under assumptions* (A1)–(A6), *problem* (1.1)–(1.3) *has an optimal solution $\left(z^0, u^0\right) \in \Omega$.*

Proof. Let $\left(z^k, u^k\right) \subset \Omega$ be a minimizing sequence, i.e.

$$j := \lim_{k\to\infty} J\left(z^k, u^k\right) = \inf_{(z,u)\in\Omega} J(z,u) = \inf_{(z,u)\in\Omega_\gamma} J(z,u).$$

By (A4) and (A5), j is a finite number.

We will apply Theorem 3.3. In view of Theorem 3.2 we can assume that $z^k \rightharpoonup z^0$ weakly in $AC^{\alpha,\beta}_{x,y}$ for some $z^0 \in AC^{\alpha,\beta}_{x,y}$. Let $\zeta_1^k = D^{\alpha,\beta}_{x,y} z^k$, $\zeta_2^k = D^{\alpha}_{x} z^k$, $\zeta_3^k = D^{\beta}_{y} z^k$ and

$$\eta^k(x,y) = F\left(x, y, z^k(x,y), \zeta_2^k(x,y), \zeta_3^k(x,y), u^k\left(x,y\right)\right), \quad \text{for } (x,y) \in P \text{ a.e.},$$

for $k = 0,1,\ldots$ Consequently,

$$\left(\eta^k(x,y), \zeta_1^k(x,y), \zeta_2^k(x,y), \zeta_3^k(x,y)\right) \in Q\left((x,y), z^k(x,y)\right), \quad (x,y) \in P \text{ a.e.},$$

for $k = 1,2,\ldots$

Using now Proposition 2.2 we get that $\zeta_1^k \rightharpoonup \zeta_1^0$ weakly in L^1 and in view of Remark 2.3 we have that also $\zeta_2^k \rightharpoonup \zeta_2^0$ and $\zeta_3^k \rightharpoonup \zeta_3^0$ weakly in L^1 as $k \to \infty$.

Let us then note that the operators $I_{x,y}^{\alpha,\beta}, I_{0+}^{\beta}\ I_{0+}^{\alpha}$ are completely continuous (see [11, Lemma 1.1]). Thus, in view of Proposition 2.2, passing possibly to a subsequence we can assume that

$$\begin{gathered} I_{x,y}^{\alpha,\beta} D_{x,y}^{\alpha,\beta} z^k \to I_{x,y}^{\alpha,\beta} D_{x,y}^{\alpha,\beta} z^0, \\ I_{0+}^{\beta} D_y^{\beta} (I_x^{1-\alpha} z^k)(0,\cdot) \to I_{0+}^{\beta} D_y^{\beta} (I_x^{1-\alpha} z^0)(0,\cdot), \\ I_{0+}^{\alpha} D_x^{\alpha} (I_y^{1-\beta} z^k)(\cdot,0) \to I_{0+}^{\alpha} D_x^{\alpha} (I_y^{1-\beta} z^0)(\cdot,0), \end{gathered}$$

in L^1, which, in view of the representation (2.1), means that $z^k \to z^0$ in L^1.

Defining $\lambda^k(x,y) = \lambda(x,y)$ (see (A5)) we get that $\eta^k(x,y) \geq \lambda(x,y)$ for $(x,y) \in P$ a.e. and $k = 1,2,\ldots$ Moreover,

$$\infty > j = \liminf_{k\to\infty} J\left(z^k, u^k\right) = \liminf_{k\to\infty} \int_G \eta^k(t)dt > -\infty.$$

To apply Theorem 3.3, it is now sufficient to show that $Q\left((x,y),z\right)$ has property (K) with respect to z for any (x,y). From [2, Theorem 8.5.iii] we know that $Q\left((x,y),z\right)$ has property (K) with respect to z if and only if the graph

$$\operatorname{Gr} Q((x,y),\cdot) := \{(z,(\eta,\zeta_1,\zeta_2,\zeta_3)) : z \in \mathbb{R}^n, (\eta,\zeta_1,\zeta_2,\zeta_3) \in Q((x,y),z)\}$$

is closed subset of $\mathbb{R}^2 \times (\mathbb{R}^n)^3$ for any $z \in \mathbb{R}^n$. To show it assume that

$$\left(\bar{z}^k, \left(\bar{\eta}^k, \bar{\zeta}_1^k, \bar{\zeta}_2^k, \bar{\zeta}_3^k\right)\right) \to \left(\bar{z}, \left(\bar{\eta}, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3\right)\right) \quad \text{as} \quad k \to \infty,$$

where $\left(\bar{z}^k, \left(\bar{\eta}^k, \bar{\zeta}_1^k, \bar{\zeta}_2^k, \bar{\zeta}_3^k\right)\right) \in \operatorname{Gr} Q\left((x,y),\cdot\right)$. This means that for each $k = 1,2,\ldots,$ there exists $\bar{u}^k \in U$ such that

$$\bar{\eta}^k \geq F\left(x,y,\bar{z}^k,\bar{\zeta}_2^k,\bar{\zeta}_3^k,\bar{u}^k\right), \quad \bar{\zeta}_1^k = f\left(x,y,\bar{z}^k,\bar{\zeta}_2^k,\bar{\zeta}_3^k,\bar{u}^k\right) \quad \text{for } k = 1,2,\ldots \tag{3.2}$$

Since the set U is compact, we may assume that $\bar{u}^k \to \bar{u}$ for some $\bar{u} \in U$. Passing now with $k \to \infty$ in (3.2) we have, thanks to the appropriate continuity of functions f and F (see assumptions (A1) and (A3)) that $\left(\bar{z}, \left(\bar{\eta}, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3\right)\right) \in \operatorname{Gr} Q((x,y),\cdot)$, consequently the graph $\operatorname{Gr} Q((x,y),\cdot)$ is closed and $Q((x,y),z)$ has property (K) with respect to (x,y). It also follows from this fact that $Q((x,y),z)$ is closed.

Applying Theorem 3.3 we assert that there exists a function $\eta^0 \in L^1(P,\mathbb{R})$ such that

$$(\eta^0(x,y), \zeta_1^0(x,y), \zeta_2^0(x,y), \zeta_3^0(x,y)) \in Q\left((x,y), z^0(x,y)\right), \quad (x,y) \in P \text{ a.e.}, \tag{3.3}$$

$$\int_P \eta^0(x,y)dt \leq j. \tag{3.4}$$

Let us now define multifunction $\Phi : P \multimap \mathbb{R}^m$ by the formula

$$\Phi(x,y) = \Big\{ u \in M : \eta^0(x,y) \geq F\left(x,y,z^0(x,y),\zeta_2^0(x,y),\zeta_3^0(x,y),u\right), \\ \zeta_1^0(x,y) = f\left(x,y,z^0\left(x,y\right),\zeta_2^0(x,y),\zeta_3^0\left(x,y\right),u\right) \Big\}.$$

It is easy to check that for multifunction Φ all assumptions of Theorem 3.4 are satisfied. As a result of applying this theorem we get that there is a function $u^0 \in \mathcal{U}$ such that

$$D_{x,y}^{\alpha,\beta} z^0(x,y) = f(x,y,z^0\left(x,y\right), D_x^{\alpha} z^0(x,y), D_y^{\beta} z^0\left(x,y\right), u^0(x,y)) \text{ for } (x,y) \in P \text{ a.e.},$$

and by (3.4)–(3.3)

$$\inf_{(z,u)\in\Omega_\gamma} J(z,u) = j \geq \int_P \eta^0(x,y)dtdt \\ \geq \int_P F\left(x,y,z^0(x,y), D_x^{\alpha} z^0\left(x,y\right), D_y^{\beta} z^0(x,y), u^0\left(x,y\right)\right) dxdy$$

This means that $\left(z^0,u^0\right)$ is an optimal solution to (1.1)–(1.3). □

It is worth noting that, using our approach, we do not require the assumption on the convexity of the cost functional J. Such assumption is quite often made in the formulation of theorems on the existence of optimal solutions for Lagrange problems. It provides a weak semicontinuity of the cost functional, while the weak compactness of the minimizing sequence is relatively easy to obtain. However, the convexity of the set Q made in (A6) does not give convexity of the cost function, nor, still less, it is a sufficient condition for the existence of optimal solutions. Let us consider two examples.

Example 3.6. Let $P = [0,1] \times [0,1]$, $M = [0,1]$ and

$$f(x,y,z,\zeta_2,\zeta_3,u) = z\sqrt{|u|}, \\ F(x,y,z,\zeta_2,\zeta_3,u) = \sqrt{|u|}.$$

Then

$$Q(x,y,z) = \left\{ (\eta,\zeta_1,\zeta_2,\zeta_3) \in \mathbb{R}^4 : \text{there exist } u \text{ such that } \eta \geq \sqrt{|u|} \text{ and } \zeta_1 = z\sqrt{|u|} \right\}$$

is for fixed (x,y,z) convex, despite the fact that none of f, F is convex with respect to u.

Example 3.7. Consider

$$\begin{cases} D_{x,y}^{\alpha,\beta} z = u^4(x,y), \\ I_{x,y}^{1-\alpha,1-\beta} z(x,0) = 0, \quad x \in [0,a], \\ I_{x,y}^{1-\alpha,1-\beta} z(0,y) = 0, \quad x \in [0,b], \end{cases}$$

$$J(z,u) = \int_P u^2(x,y)dxdy,$$

where $P = [0,1] \times [0,1]$, $M = [0,1]$. Then the set

$$Q(x,y,z) = \left\{(\eta, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^4 : \text{there exists } u \text{ such that } \eta \geq u^2 \text{ and } \zeta^1 = u^4\right\}$$

is not compact, while obviously the optimal optimal pair is $(u^*, z^*) = (0,0)$.

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