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# THE FIRST EIGENCURVE FOR A NEUMANN BOUNDARY PROBLEM INVOLVING $p$-LAPLACIAN WITH ESSENTIALLY BOUNDED WEIGHTS

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**Abstract.** This article is intended to prove the existence and uniqueness of the first eigencurve, for a homogeneous Neumann problem with singular weights associated with the equation

$$-\Delta_p u = \alpha m_1|u|^{p-2}u + \beta m_2|u|^{p-2}u$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. We then establish many properties of this eigencurve, particularly the continuity, variational characterization, asymptotic behavior, concavity and the differentiability.

**Keywords:** $p$-Laplacian, first eigencurve, singular weight, Neumann boundary conditions.

**Mathematics Subject Classification:** 35J30, 35J60, 35J66.

## 1. INTRODUCTION

Let $N$ be an integer $\geq 1$, $\Omega$ be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, we denote by $\nu = \nu(x)$ is the unit outer normal at $x$, defined for all $x \in \partial\Omega$, $\alpha$ and $\beta$ are two real parameters. We study in the present work the following Neumann two-parameter eigenvalue problem for the $p$-Laplacian operator:

$$\begin{cases} -\Delta_p u = \alpha m_1(x)|u|^{p-2}u + \beta m_2(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the well known $p$-Laplacian operator, $1 < p < +\infty$, $\nabla u = (\partial_1 u, \ldots, \partial_N u)^T$ is the gradient of $u$, $\frac{\partial u}{\partial \nu}$ denotes the derivative of $u$ in the direction of the exterior unit normal to the boundary $\nu$, $m_1$ and $m_2$ are two possibly singular weight functions belonging to $L^\infty(\Omega)$ with $m_1$ changes sign in $\Omega$ and $m_2 \gneqq 0$ in $\Omega$.

 

It is well known that this type of differential equations involving the $p$-Laplacian operator widely appears in several physical and natural phenomena, such as the non-Newtonian fluids, nonlinear elasticity, glaciology, and population dynamics, etc. [4, 10, 11]. Our purposes in this paper are the following.

The first one consists in proving the existence and uniqueness of principal eigencurve the $p$-Laplacian with indefinite weight for an elliptic Neumann problem. we first recall the definition of the principal eigencurve. Let be $\alpha \in \mathbb{R}$, we define $\mathcal{C}_1$ the principal eigencurve as the graph of map $\beta_1 : \alpha \to \beta_1(\alpha) = \beta$ with $\beta$ is the unique real verifying $\lambda_1(\alpha\, m_1 + \beta\, m_2) = 1$ where $\lambda_1(m)$ is the first eigenvalue of $p$-Laplacian with weight $m$ and Neumann boundary conditions. This definition was first introduced by A. Dakkak and M. Hadda in [6]. More precisely, we will show that for a fixed $\alpha \in \mathbb{R}$, there exists a unique real $\beta$ such that the problem (1.1) admits a unique solution in the weak sense.

The second purpose is to study some properties of the first eigencurve of $p$-Laplacian with weight. We establish the continuity, concavity and the differentiability. We also give a variational characterization and obtain the asymptotic behavior of $\beta_1(\cdot)$.

Throughout this paper, we always assume that the following conditions hold. The weight functions $m_1$ and $m_2$ in problem (1.1) belong to $M^+(\Omega)$ and satisfy the following conditions:

(A1) $m_1$ changes sign in $\Omega$ and $\int_\Omega m_1 < 0$,
(A2) $m_2 \gneqq 0$ in $\Omega$ and $\Omega^*_{m_1} \subset \Omega^*_{m_2}$,

where

$$M^+(\Omega) = \big\{m \in L^\infty(\Omega) : \text{meas}(\{x \in \Omega : m(x) > 0\}) \neq 0\big\}$$

and

$$\Omega^*_m = \big\{x \in \Omega :\ m(x) \neq 0\big\}$$

for a given $m \in L^\infty(\Omega)$.

Some fundamental results about the eigencurves of the $p$-Laplacian with weight on domains subject to various boundary conditions (Dirichlet, Neumann, Sturm–Liouville, etc.) have been established, such as the existence, uniqueness, continuity, variational characterization, differentiability, asymptotic behavior, and so on. For example, we refer the readers to [3–7, 12, 13] and the references therein.

In [5] the authors investigated, for a Dirichlet problem, various properties such as concavity, differentiability, and asymptotic behavior.

The existence and uniqueness of the $n$-th eigencurve for a Dirichlet problem with $\operatorname{ess\,inf}_\Omega m_2 > 0$, have been studied in [6]. There, it was also proved a variational formulation for the eigencurves and their asymptotic behavior was studied, while in [7] under the assumption $m_2 \in M^+(\Omega)$ and $m_2 \geq 0$ the authors carried out the same study but only for the second eigencurve of the $p$-Laplacian with an indefinite weight.

In [12], under the assumption $\operatorname{ess\,inf}_\Omega m_2 > 0$, the authors investigated the existence of the first eigencurve for a Neumann problem. Also, in [13], it was studied the existence, variational characterization, differentiability and asymptotic behavior of the $n$-th eigencurve for the one-dimensional $p$-Laplacian with indefinite weight.

The rest of the paper is structured as follows. In Section 2, we present the functional framework of our problem and recall some basic results concerning the spectrum of the $p$-Laplacian with an indefinite weight which play an important role in the proof of our results. In Section 3, we will state our main results. At last, Section 4 contains their proofs.

## 2. PRELIMINARIES

Throughout this paper, $\Omega$ is a smooth bonded domain of $\mathbb{R}^N$. We denote by $W^{1,p}(\Omega)$ the usual Sobolev space endowed with its natural norm

$$\|u\|_{1,p} = (\|u\|_p^p + \|\nabla u\|_p^p)^{\frac{1}{p}},$$

where $\|\cdot\|_p$ is the Lebesgue norm of $L^p(\Omega)$ (see [1] for more details).

Next, let us recall some basic properties of the spectrum of $p$-Laplacian operator. For this, we consider the nonlinear Neumann eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda\, m|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in L^\infty(\Omega)$ and $\lambda$ is a real parameter. We are interested in the solutions of (2.1) in weak sense, i.e. functions $u \in W^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u\, \nabla v = \lambda \int_\Omega m|u|^{p-2}uv, \quad \forall v \in W^{1,p}(\Omega).$$

A real number $\lambda$ is said to be a Neumann eigenvalue of the $p$-Laplacian with weight $m$, if there exists $u \in W^{1,p}(\Omega) \setminus \{0\}$, called eigenfunction associated to $\lambda$, such that is a solution of problem (2.1). If $m \in M^+(\Omega)$, the set of positive eigenvalues, noted $\sigma_p^+(-\Delta_p, m, \Omega)$, constitutes the spectrum of $p$-Laplacian with weight $m$. It is well-known that the spectrum $\sigma_p^+(-\Delta_p, m, \Omega)$ contains an increasing sequence of non-negative eigenvalues obtained through the Ljusternik–Schnirelman theory (see [14]):

$$0 \leq \lambda_1(m) < \lambda_2(m) \leq \dots \leq \lambda_n(m) \leq \ldots \to +\infty.$$

The sequence of eigenvalues associated to the problem (2.1) is given for all $n \geq 2$ by

$$\frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \frac{\displaystyle\int_\Omega m|u|^p}{\displaystyle\int_\Omega |\nabla u|^p}, \tag{2.2}$$

where

$$\Gamma_n = \big\{K \subset S : K \text{ is symmetric, compact and } \gamma(K) \geq n\big\},$$

$S$ is the unit sphere of $W^{1,p}(\Omega)$ and $\gamma(K)$ denotes the Krasnoselskii genus of $K$, which is defined by

$$\gamma(K) = \min\big\{j \in \mathbb{N} : \text{ there exists } f : K \to \mathbb{R}^j \setminus \{0\} \text{ continuous and odd}\big\}.$$

Next, if $m$ changes sign in $\Omega$ and $\int_\Omega m\,dx < 0$, then we can characterize the first eigenvalue $\lambda_1(m)$ as follows:

$$\lambda_1(m) = \inf_{u \in \mathcal{A}(m)} \int_\Omega |\nabla u|^p, \tag{2.3}$$

where

$$\mathcal{A}(m) = \Big\{u \in W^{1,p}(\Omega) : \int_\Omega m\,|u|^p dx = 1\Big\}$$

(see [2, 9]).

We can also define the negative spectrum when $-m \in M^+(\Omega)$ by

$$\sigma_p^-(-\Delta_p, m, \Omega) = -\sigma_p^+(-\Delta_p, -m, \Omega)$$

which contains a decreasing sequence $(\lambda_{-n}(m))_{n\geq 1}$ of eigenvalues such that $\lambda_{-n}(m) \to -\infty$ as $n \to +\infty$, and

$$\lambda_{-n}(m) = -\lambda_n(-m).$$

In order to establish the existence and the uniqueness of the first eigencurve of the $p$-Laplacian with weight, it is necessary to recall the main properties of $\lambda_1(m)$.

**Proposition 2.1** (see [2, 8])**.** *Let $m \in M^+(\Omega)$, then the following assertions hold.*

- (i) *$\lambda_1(m) > 0$ and $\lambda_1(m)$ is the unique nonzero principal eigenvalue if and only if $m$ changes sign in $\Omega$ and $\int_\Omega m\,dx < 0$.*
- (ii) *$\lambda_1(m)$ is simple and the corresponding eigenfunction $u$ can be chosen such that $u(x) > 0$ in $\Omega$. Moreover, $\lambda_1(m)$ is isolated, i.e. there exists $\lambda > \lambda_1(m)$ such that $\sigma_p^+(-\Delta_p, m, \Omega) \cap ]0, \lambda[ = \{\lambda_1\}$.*
- (iii) *If $\int_\Omega m\,dx > 0$, then $\lambda_1(m) = 0$ and $0$ is the unique nonnegative principal eigenvalue.*
- (iv) *If $\int_\Omega m\,dx = 0$, then $\lambda_1(m) = 0$ and $0$ is the unique principal eigenvalue.*

**Proposition 2.2** (see [2])**.** *Let $m, m' \in M^+(\Omega)$. Then the following assertions hold.*

- (i) *If $m \leq m'$, then $\lambda_1(m) \geq \lambda_1(m')$. Furthermore, if*

$$\mathit{meas}(\{x \in \Omega : m < m'\}) \neq 0,$$

  *then $\lambda_n(m) > \lambda_n(m')$.*
- (ii) *The mapping $\lambda_1 : m \to \lambda_1(m)$ is continuous in $M^+(\Omega)$ for the distance associated with the infinity norm $\|\cdot\|_\infty$.*

**Proposition 2.3.** *Let $(m_k)_k$ be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$. Then*

$$\lim_{k\to\infty} \lambda_1(m_k) = +\infty \text{ if and only if } m \le 0 \text{ almost everywhere in } \Omega.$$

*Proof.* Let $(m_k)_k$ be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$. Assume first that $\lim_{k\to\infty} \lambda_1(m_k) = +\infty$. we prove that $m \le 0$ almost everywhere in $\Omega$. Indeed, assume by contradiction that $\mathrm{meas}(\{x \in \Omega : m(x) > 0\}) \neq 0$. Using the continuity of the $\lambda_1$ (cf. Proposition 2.2), we have $\lim_{k\to\infty} \lambda_1(m_k) = \lambda_n(m)$, and it is a finite quantity, which gives a contradiction.

Conversely, if $m \le 0$ almost everywhere in $\Omega$, suppose by contradiction that there exist $\lambda > 0$ and a subsequence of $(m_k)_k$, still denoted by $(m_k)_k$, such that

$$\lambda_1(m_k) \le \lambda.$$

We put $r = \frac{2\lambda}{\lambda_1(2)}$. Since $m_k \to m$ in $L^\infty(\Omega)$, then there exists $n_0 \in \mathbb{N}$ such that for all $k \ge n_0$, we have

$$\|m_k - m\|_\infty \le \frac{2}{r}.$$

Hence

$$m_k \le m + \frac{2}{r} \quad \text{a.e. } x \in \Omega.$$

So, using the fact that $m \le 0$ a.e. $x \in \Omega$, we conclude that

$$m_k \le \frac{2}{r} \quad \text{a.e. } x \in \Omega.$$

According to the first point of Proposition 2.2, we have

$$\lambda_1(m_k) \ge \lambda_1\left(\frac{2}{r}\right) = r\lambda_1(2) = 2\lambda,$$

which yields a contradiction. Consequently, $\lim_{k\to\infty} \lambda_1(m_k) = +\infty$. □

## 3. STATEMENTS OF MAIN RESULTS

For any $m \in L^\infty(\Omega)$, we introduce the following notations

$$\Omega_m^+ = \{x \in \Omega : \ m(x) > 0\} \quad \text{and} \quad \Omega_m^- = \{x \in \Omega : \ m(x) < 0\}.$$

We present in this section the main results of this work. Let us start with the existence and uniqueness result which reads as follows:

**Theorem 3.1.** *Let $m_1, m_2 \in M^+(\Omega)$. Assume assumptions* (A1) *and* (A2) *hold, and in addition* $\mathrm{meas}(\Omega_{m_1}^-) > 0$. *Then, for every $\alpha \in \mathbb{R}$, there exists a unique real number $\beta_1 = \beta_1(\alpha)$ such that*

$$\lambda_1(\alpha\, m_1 + \beta_1\, m_2) = 1. \tag{3.1}$$

Next, the following theorem tells us that the first eigencurve $\beta_1(\cdot)$ is continuous on $\mathbb{R}$.

**Theorem 3.2.** *Assume that the assumptions* (A1) *and* (A2) *are satisfied. Then:*

(i) $\lim_{\alpha\to 0}\beta_1(\alpha)=0$,
(ii) *for* $\alpha=0$, *it is appropriate to set* $\beta_1(0)=0$ *then, the function* $\alpha\to\beta_1(\alpha)$ *is continuous on* $\mathbb{R}$.

Furthermore, in the present theorem we use the min-max arguments, to give a variational characterization of $\beta_1(\cdot)$.

**Theorem 3.3.** *Under the assumptions of Theorem 3.1, for every* $\alpha\in\mathbb{R}$ *the unique real number* $\beta_1(\alpha)$ *such that* $\lambda_1(\alpha\, m_1+\beta_1(\alpha)\, m_2)=1$ *is characterized by the following relation*

$$\beta_1(\alpha)=\inf_{u\in W^*}\frac{\int\limits_\Omega|\nabla u|^p-\alpha\int\limits_\Omega m_1|u|^p}{\int\limits_\Omega m_2|u|^p},\tag{3.2}$$

*where*

$$W^*=\Big\{u\in W^{1,p}(\Omega):\int\limits_\Omega m_2|u|^p\neq 0\Big\}.$$

Concerning the asymptotic behavior of $\beta_1(\cdot)$, we state the following result.

**Theorem 3.4.** *Assume that the assumptions* (A1) *and* (A2) *hold. Then we have:*

(i) $\lim_{\alpha\to+\infty}\frac{\beta_1(\alpha)}{\alpha}=-\operatorname{ess\,sup}_{\Omega^*_{m_2}}\frac{m_1}{m_2}$,
(ii) *if* $\operatorname{meas}(\Omega^-_{m_1})>0$, *then* $\lim_{\alpha\to-\infty}\frac{\beta_1(\alpha)}{\alpha}=-\operatorname{ess\,inf}_{\Omega^*_{m_2}}\frac{m_1}{m_2}$.

Finally, we will obtain the concavity and differentiability of $\beta_1(\cdot)$. The result reads as follows.

**Theorem 3.5.** *Assume that the assumptions* (A1) *and* (A2) *hold. Then:*

(i) *the function* $\alpha\to\beta_1(\alpha)$ *is concave,*
(ii) *the function* $\alpha\to\beta_1(\alpha)$ *is differentiable. Moreover, for every* $\alpha_0\in\mathbb{R}$ *we have*

$$\beta_1'(\alpha_0)=-\frac{\int\limits_\Omega m_1|\varphi_{\alpha_0}|^p dx}{\int\limits_\Omega m_2|\varphi_{\alpha_0}|^p dx},$$

*where* $\varphi_{\alpha_0}$ *is an eigenfunction associated to* $\lambda_1(\alpha_0\, m_1+\beta_1(\alpha_0)\, m_2)=1$.

## 4. PROOFS OF THE MAIN RESULTS

*Proof of Theorem 3.1.* Let $\alpha\in\mathbb{R}$. We consider the real function defined by $f_\alpha: t\to\lambda_1(\alpha\, m_1+t\, m_2)$. Using the first point of Proposition 2.2 we can show that $f_\alpha$ is continuous. Moreover, $f_\alpha$ is strictly decreasing. Indeed, let $t<t'$.

Then for $\alpha \in \mathbb{R}$ (fixed) we have

$$m = \alpha\, m_1 + t\, m_2 \leq m' = \alpha\, m_1 + t'\, m_2 \;\text{ in a.e. }\; \Omega.$$

Since $m_2 > 0$ a.e. in $\Omega^+_{m_2}$, then $t\, m_2 < t'\, m_2$. It follows that $m' > m$ a.e. in $\Omega^+_{m_2}$, so according to the second point of Proposition 2.2 we have $\lambda_1(m) > \lambda_1(m')$, i.e. $\lambda_1(\alpha\, m_1 + t\, m_2) > \lambda_1(\alpha\, m_1 + t'\, m_2)$. Hence $f_\alpha(t) > f_\alpha(t')$. This is equivalent to saying that $f_\alpha$ is strictly decreasing. Consequently, $f_\alpha$ is injective.

In order to complete the proof of this theorem, we will distinguish the following three cases.

*Case 1.* $0 \leq \alpha \leq \lambda_1(m_1)$.

If $\alpha = 0$, we agree to put $\beta_1(0) = 0$ (later we will show that this convention makes sense) and if $\alpha = \lambda_1(m_1)$, it is obvious to take $\beta_1(\alpha) = 0$.

For $0 < \alpha < \lambda_1(m_1)$, we have

$$f_\alpha(0) = \lambda_1(\alpha\, m_1) = \frac{\lambda_1(m_1)}{\alpha} > 1 \tag{4.1}$$

and

$$\lim_{t \to +\infty} f_\alpha(t) = \lim_{t \to +\infty} \frac{\lambda_1(\frac{\alpha}{t} m_1 + m_2)}{t} = 0. \tag{4.2}$$

Combining (4.1) with (4.2) and taking into account that $f_\alpha$ is injective, it follows that there exists a unique real number $\beta_1(\alpha) \in ]0, +\infty[$ such that $f_\alpha(\beta_1(\alpha)) = 1$, i.e. $\lambda_1(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$.

*Case 2.* $\alpha > \lambda_1(m_1)$.

In this case, we note that

$$0 < f_\alpha(0) = \lambda_1(\alpha m_1) = \frac{\lambda_1(m_1)}{\alpha} < 1. \tag{4.3}$$

Next, we consider the following set

$$A_\alpha = \big\{ t \leq 0 :\; \alpha\, m_1 + t\, m_2 \leq 0 \;\text{ a.e. in }\; \Omega \big\}.$$

We denote $\tau_\alpha = \sup A_\alpha$ and since $\Omega^*_{m_1} \subset \Omega^*_{m_2}$, so we can define

$$\delta_\alpha = \frac{-\alpha \|m_1\|_\infty}{\operatorname{ess\,inf}_{\Omega^+_{m_1}} m_2}.$$

We easily see that $\delta_\alpha \in A_\alpha$. Then $A_\alpha \neq \emptyset$. Now we show that $\tau_\alpha \in A_\alpha$. Indeed, firstly we verify that $\tau_\alpha < 0$. Since $f_\alpha(0) > 0$ and $f_\alpha$ is a continuous function, then there exists $\eta < 0$ such that $f_\alpha(t) > 0$ for all $t \in [\eta, 0]$, i.e. $\lambda_1(\alpha m_1 + t m_2) > 0$ for all $t \in [\eta, 0]$. We conclude that $\alpha\, m_1 + t\, m_2$ changes sign in $\Omega$ for all $t \in [\eta, 0]$, so in particular $\alpha\, m_1 + \eta\, m_2$ changes sign in $\Omega$, hence $\tau_\alpha \leq \eta < 0$. Moreover, according to the definition of $\tau_\alpha$, for all $n \in \mathbb{N}^*$, there exists $t_n \in A_\alpha$ such that $\tau_\alpha - \frac{1}{n} < t_n$. It follows that

$$\alpha\, m_1 + \tau_\alpha\, m_2 \leq \alpha\, m_1 + t_n\, m_2 + \frac{1}{n}\, m_2 \leq \frac{1}{n} \|m_2\|_\infty \quad \text{a.e. in } \Omega.$$

Therefore, letting $n$ tend to $+\infty$ in the above inequality, we obtain

$$\alpha\, m_1 + \tau_\alpha\, m_2 \le 0 \quad \text{a.e. in } \Omega.$$

Thus $\tau_\alpha \in A_\alpha$. From Proposition 2.2 we get

$$\lim_{t \to \tau_\alpha^+} f_\alpha(t) = +\infty. \tag{4.4}$$

Hence, it follows from (4.3) and (4.4) that there exists a unique real $\beta_1(\alpha) \in ]\tau_\alpha, 0[$ which verifies $f_\alpha(\beta_1(\alpha)) = 1$, i.e. $\lambda_1(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$.

*Case 3.* $\alpha < 0$.

In this case we have $f_\alpha(0) = \lambda_1(\alpha m_1) = 0$, because $\int_\Omega \alpha m_1 dx > 0$.

On the other hand, similarly as in the case where $\alpha > \lambda_1(m_1)$, we will seek a real $\eta_\alpha$ such that $\lim_{t\to\eta_\alpha} f_\alpha(t) = +\infty$. To this end, we consider the set

$$B_\alpha = \big\{ t \le 0 :\ \alpha\, m_1 + t\, m_2 \le 0 \ \text{ a.e. in } \Omega \big\}.$$

Clearly

$$\mu_\alpha = \frac{\alpha \|m_1\|_\infty}{\operatorname{ess\,inf}_{\Omega^+_{m_1}} m_2} \in B_\alpha.$$

Then $B_\alpha \neq \emptyset$. We denote $\eta_\alpha = \sup B_\alpha$.

The rest of the proof can be done in a similar way to that of the case where $\alpha > \lambda_1(m_1)$. □

**Remark 4.1.** Let $\alpha \in \mathbb{R}^*$.

(i) If $\alpha > \lambda_1(m_1)$ or $\alpha < 0$, then we have $\beta(\alpha) < 0$.
(ii) If $0 < \alpha \le \lambda_1(m_1)$, then we have $\beta(\alpha) \ge 0$.

*Proof of Theorem 3.2.* (i) We put $L = \limsup_{\alpha\to 0^+} \beta_1(\alpha)$ and we will show that $L = 0$. Let $\alpha > 0$ be small enough, then $\beta(\alpha) > 0$ (because if $0 < \alpha < \lambda_1(m_1)$) we get that $L \ge 0$. Next we verify that $L$ is finite. Assuming by absurd that $L = +\infty$, there exists a sequence $(\alpha_k)$ such that $\alpha_k > 0$, $\lim_{k\to+\infty} \alpha_k = 0$ and $\lim_{k\to+\infty} \beta_1(\alpha_k) = +\infty$. Since $\lambda_1(\alpha_k m_1 + \beta_1(\alpha_k) m_2) = 1$, then by homogeneity of $\lambda_1$ with respect to the weight we obtain

$$\lambda_1\Big(\frac{\alpha_k}{\beta_1(\alpha_k)} m_1 + m_2\Big) = \beta_1(\alpha_k) \tag{4.5}$$

as $\lim_{k\to+\infty} \frac{\alpha_k}{\beta_1(\alpha_k)} = 0$ and according to the continuity of $\lambda_1$ with respect to the weight we have

$$\lambda_1\Big(\frac{\alpha_k}{\beta_1(\alpha_k)} m_1 + m_2\Big) \to \lambda_1(m_2) = 0 \quad \text{when } k \to +\infty.$$

By passing to the limit when $k \to +\infty$ in (4.5), we obtain $\lambda_1(m_2) = 0 = +\infty$ which is absurd. Then $0 \le L < +\infty$. Thus, to show that $L = 0$, suppose by contradiction that $L > 0$. For this we consider a sequence $(\alpha_k)$ such that $\alpha_k > 0$, $\lim_{k\to+\infty} \alpha_k = 0$ and $\lim_{k\to+\infty} \beta_1(\alpha_k) = L$.

Using the continuity of $\lambda_1$ with respect to the weight we deduce that

$$1 = \lim_{k\to+\infty} \lambda_1(\alpha_k m_1 + \beta_1(\alpha_k)m_2) = \lambda_1(L\, m_2) = 0$$

which is absurd. Hence

$$L = \limsup_{\alpha\to0^+} \beta_1(\alpha) \leq 0. \tag{4.6}$$

On the other hand, we have $\beta_1(\alpha) > 0$ for all $\alpha \in ]0, \lambda_1(m_1)[$. Then

$$\liminf_{\alpha\to0^+} \beta_1(\alpha) \geq 0. \tag{4.7}$$

From (4.6) and (4.7) we conclude that

$$0 \leq \liminf_{\alpha\to0^+} \beta_1(\alpha) \leq \limsup_{\alpha\to0^+} \beta_1(\alpha) \leq 0.$$

Hence

$$\lim_{\alpha\to0^+} \beta_1(\alpha) = \liminf_{\alpha\to0^+} \beta_1(\alpha) = \limsup_{\alpha\to0^+} \beta_1(\alpha) = 0. \tag{4.8}$$

Now, we will prove that $\lim_{\alpha\to0^-} \beta_1(\alpha) = 0$. First we recall that for $\alpha < 0$ we have $\beta_1(\alpha) < 0$ (see Remark 4.1). Then $\limsup_{\alpha\to0^-} \beta_1(\alpha) \leq 0$. This last limit is zero, because otherwise there exists $\delta < 0$ such that $\limsup_{\alpha\to0^-} \beta_1(\alpha) < \delta < 0$, and then there exists a sequence $(\alpha_k)$ such that $\alpha_k < 0$, $\lim_{k\to+\infty} \alpha_k = 0$ and $\lim_{k\to+\infty} \beta_1(\alpha_k) < \delta$. Thus

$$\forall\varepsilon > 0\, \exists N_\varepsilon \in \mathbb{N}\, \forall k \geq N_\varepsilon : \beta_1(\alpha_k) < \delta + \varepsilon < 0,$$

which gives

$$\alpha_k m_1 + \beta_1(\alpha_k)m_2 \leq \alpha_k m_1 + (\delta + \varepsilon)m_2.$$

By using the monotony of $\lambda_1$ with respect to the weight, we obtain

$$\lambda_1(\alpha_k m_1 + (\delta + \varepsilon)m_2) \leq 1. \tag{4.9}$$

As $\lim_{k\to+\infty} \alpha_k = 0$ and $(\delta + \varepsilon)m_2 \leq 0$, then by the continuity of $\lambda_1$ with respect to the weight, we obtain a contradiction. Hence

$$\limsup_{\alpha\to0^-} \beta_1(\alpha) = 0. \tag{4.10}$$

As $\beta_1(\alpha) < 0$ for all $\alpha < 0$, we have

$$\liminf_{\alpha\to0^-} \beta_1(\alpha) \leq 0. \tag{4.11}$$

Then from (4.10) and (4.11) we obtain

$$0 \leq \limsup_{\alpha\to0^-} \beta_1(\alpha) \leq \liminf_{\alpha\to0^-} \beta_1(\alpha) \leq 0.$$

We conclude that

$$\lim_{\alpha\to 0^-} \beta_1(\alpha) = \liminf_{\alpha\to 0^-} \beta_1(\alpha) = \limsup_{\alpha\to 0^-} \beta_1(\alpha) = 0. \tag{4.12}$$

Finally, by combining (4.8) with (4.12), we obtain

$$\lim_{\alpha\to 0} \beta_1(\alpha) = \lim_{\alpha\to 0^-} \beta_1(\alpha) = \lim_{\alpha\to 0^+} \beta_1(\alpha) = 0.$$

This completes the proof of (i).

(ii) Let $\alpha_0$ be a non-zero real number. We will show that $\lim_{\alpha\to\alpha_0} \beta_1(\alpha) = \beta_1(\alpha_0)$. Indeed, we suppose by contradiction that $\lim_{\alpha\to\alpha_0} \beta_1(\alpha) \neq \beta_1(\alpha_0)$. So there exists a sequence $(\alpha_k)_{k\geq 1}$ such that $\lim_{k\to+\infty} \alpha_k = \alpha_0$ and there exists $\varepsilon > 0$ such that

$$\big\{k \in \mathbb{N} : \beta_1(\alpha_k) \notin [\beta_1(\alpha_0) - \varepsilon,\ \beta_1(\alpha_0) + \varepsilon]\big\} \text{ is infinite.}$$

We distinguish two cases.

*Case 1.* $\big\{k \in \mathbb{N} : \beta_1(\alpha_k) < \beta_1(\alpha_0) - \varepsilon\big\}$ *is infinite.*

So there exists a subsequence of $(\alpha_k)$, still noted $(\alpha_k)$, such that

$$\alpha_k m_1 + \beta_1(\alpha_k) m_2 \leq \alpha_k m_1 + \beta_1(\alpha_0) m_2 - \varepsilon\, m_2.$$

By monotony of $\lambda_1$ with respect to the weight, we obtain

$$1 = \lambda_1(\alpha_k m_1 + \beta_1(\alpha_k) m_2) \geq \lambda_1(\alpha_k m_1 + \beta_1(\alpha_0) m_2 - \varepsilon\, m_2).$$

Passing to the limit when $k \to +\infty$ in the above inequality we have

$$1 \geq \lambda_1(\alpha_0 m_1 + \beta_1(\alpha_0) m_2 - \varepsilon\, m_2).$$

The strict monotony of $\lambda_1$ with respect to the weight gives

$$1 \geq \lambda_1(\alpha_0 m_1 + \beta_1(\alpha_0) m_2 - \varepsilon\, m_2) > \lambda_1(\alpha_0 m_1 + \beta_1(\alpha_0) m_2) = 1,$$

which is a contradiction.

*Case 2.* $\big\{k \in \mathbb{N} : \beta_1(\alpha_k) > \beta_1(\alpha_0) + \varepsilon\big\}$ *is infinite.*

The proof is based on similar arguments as in the first case. □

*Proof of Theorem 3.3.* Let $\alpha$ be a non-zero real, we consider $(\alpha, \beta_1(\alpha)) \in \mathcal{C}_1$, then $\lambda_1(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$, namely

$$\inf_{u\in\mathcal{A}(\alpha m_1 + \beta_1(\alpha) m_2)} \int_\Omega |\nabla u|^p = 1. \tag{4.13}$$

Fix a real number $\alpha$. Then for any $u \in W^*$, we study the two following cases.

*Case 1.* $\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p>0$.

We put

$$v=\frac{u}{\left(\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p\right)^{\frac{1}{p}}}.$$

Then we have

$$\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|v|^p=1,$$

which implies that

$$v\in\mathcal{A}(\alpha\, m_1+\beta_1(\alpha)\, m_2).$$

Thus, according to (4.13) we obtain

$$1\leq\int\limits_{\Omega}|\nabla v|^p,$$

then

$$1\leq\frac{\int_{\Omega}|\nabla u|^p}{\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p},$$

hence

$$\int\limits_{\Omega}|\nabla u|^p\geq\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p.$$

*Case 2.* $\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p\leq 0$.

In this case it is clear that

$$\int\limits_{\Omega}|\nabla u|^p\geq 0.$$

Then

$$\int\limits_{\Omega}|\nabla u|^p\geq\int\limits_{\Omega}(\alpha\, m_1+\beta_1(\alpha)\, m_2)|u|^p.$$

Therefore in both cases we get

$$\int\limits_{\Omega}|\nabla u|^p-\alpha\int\limits_{\Omega}m_1|u|^p\geq\beta_1(\alpha)\int\limits_{\Omega}m_2|u|^p.$$

Since $u \in W^*$, we obtain

$$\beta_1(\alpha) \leq \frac{\int\limits_\Omega |\nabla u|^p - \alpha \int\limits_\Omega m_1 |u|^p}{\int\limits_\Omega m_2 |u|^p},$$

which yields

$$\beta_1(\alpha) \leq \inf_{u \in W^*} \frac{\int\limits_\Omega |\nabla u|^p - \alpha \int\limits_\Omega m_1 |u|^p}{\int\limits_\Omega m_2 |u|^p} = \theta(\alpha).$$

It follows that

$$\beta_1(\alpha) \leq \theta(\alpha). \tag{4.14}$$

On the other hand, for $\alpha \neq 0$, we consider an eigenfunction $\varphi \in W^{1,p}(\Omega)$ associated to $\lambda_1(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$ such that $\varphi > 0$, so we have

$$\int\limits_\Omega |\nabla \varphi|^p = \alpha \int\limits_\Omega m_1 |\varphi|^p + \beta_1(\alpha) \int\limits_\Omega m_2 |\varphi|^p.$$

Since $m_2 \geq 0$ and $m_2 \neq 0$ a.e. in $\Omega$, we deduce that $\int_\Omega m_2 |\varphi|^p > 0$. Then we can write

$$\beta_1(\alpha) = \frac{\int\limits_\Omega |\nabla \varphi|^p - \alpha \int\limits_\Omega m_1 |\varphi|^p}{\int\limits_\Omega m_2 |\varphi|^p},$$

thus

$$\theta(\alpha) = \inf_{u \in W^*} \frac{\int\limits_\Omega |\nabla u|^p - \alpha \int\limits_\Omega m_1 |u|^p}{\int\limits_\Omega m_2 |u|^p} \leq \frac{\int\limits_\Omega |\nabla \varphi|^p - \alpha \int\limits_\Omega m_1 |\varphi|^p}{\int\limits_\Omega m_2 |\varphi|^p} = \beta_1(\alpha). \tag{4.15}$$

Hence from (4.14) and (4.15) we deduce that for all $\alpha \neq 0$, we have $\theta(\alpha) = \beta_1(\alpha)$.

If $\alpha = 0$, then we readily see that

$$\beta_1(0) = \inf_{u \in W^*} \frac{\int\limits_\Omega |\nabla u|^p}{\int\limits_\Omega m_2 |u|^p} = 0,$$

because $1 \in W^*$.

Finally, for all $\alpha \in \mathbb{R}$, if $(\alpha, \beta_1(\alpha)) \in \mathcal{C}_1$, then $\beta_1(\alpha)$ must be expressed as

$$\beta_1(\alpha) = \inf_{u \in W^*} \frac{\int\limits_\Omega |\nabla u|^p - \alpha \int\limits_\Omega m_1 |u|^p}{\int\limits_\Omega m_2 |u|^p}.$$

This completes the proof. □

*Proof of Theorem 3.4.* Let us verify assertion (i). We consider $\alpha > \lambda_1(m_1)$. Then by Theorem 3.1 there exists $\beta_1(\alpha) \in \mathbb{R}^*$ such that $(\alpha, \beta_1(\alpha)) \in \mathcal{C}_1$, i.e.

$$\lambda_1(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1.$$

Since $\alpha\, m_1 + \beta_1(\alpha)\, m_2 = \alpha(m_1 + \frac{\beta_1(\alpha)}{\alpha}\, m_2)$, then

$$\lambda_1\Big(m_1 + \frac{\beta_n(\alpha)}{\alpha}\, m_2\Big) = \alpha,$$

which is a finite quantity and positive, so

$$m_1 + \frac{\beta_1(\alpha)}{\alpha}\, m_2 \in M^+(\Omega).$$

Thus there exists a subset $\Omega_\alpha$ such that

$$\mathrm{meas}(\Omega_\alpha) \neq 0 \quad \text{and} \quad m_1 + \frac{\beta_1(\alpha)}{\alpha}\, m_2 > 0 \quad \text{a.e.} x \in \Omega_\alpha.$$

Since $\frac{\beta_1(\alpha)}{\alpha} < 0$, then $m_1 > 0$ a.e. $x \in \Omega_\alpha$. Therefore, in view of (A2) we have $\Omega_\alpha \subset \Omega^*_{m_2}$, which yields

$$\frac{-\beta_1(\alpha)}{\alpha} < \frac{m_1}{m_2} \quad \text{a.e.} \quad x \in \Omega_\alpha \subset \Omega^*_{m_2}.$$

It follows that

$$\frac{-\beta_1(\alpha)}{\alpha} < \operatorname*{ess\,sup}_{\Omega^*_{m_2}} \frac{m_1}{m_2}.$$

Thus

$$\limsup_{\alpha \to +\infty} \frac{-\beta_1(\alpha)}{\alpha} \leq \operatorname*{ess\,sup}_{\Omega^*_{m_2}} \frac{m_1}{m_2}. \tag{4.16}$$

On the other hand, if we denote $l = \liminf_{\alpha \to +\infty} \frac{-\beta_1(\alpha)}{\alpha}$, then for a sequence $(\alpha_k)_k$ such that $\alpha_k \to +\infty$ we have

$$\lim_{k \to +\infty} \frac{-\beta_1(\alpha_k)}{\alpha_k} = l \quad \text{and} \quad \lambda_1\Big(m_1 + \frac{\beta_1(\alpha_k)}{\alpha_k}\, m_2\Big) = \alpha_k.$$

Since

$$m_1 + \frac{\beta_1(\alpha_k)}{\alpha_k}\, m_2 \to m_1 - l\, m_2 \quad \text{in} \;\; L^\infty(\Omega) \quad \text{and} \quad \alpha_k \to +\infty, \tag{4.17}$$

it follows from Proposition 2.3 and (4.17) that $m_1 - l\, m_2 \leq 0$ for all $x \in \Omega$. Hence

$$\operatorname*{ess\,sup}_{\Omega^*_{m_2}} \frac{m_1}{m_2} \leq l = \liminf_{\alpha \to +\infty} \frac{-\beta_1(\alpha)}{\alpha}. \tag{4.18}$$

Finally, combining (4.16) and (4.18) we get the equality of the first assertion.

We can prove the assertion (ii) in the same way as in (i). Thus, the proof of Theorem 3.4 is completed. □

*Proof of Theorem 3.5.* (i) For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $t \in ]0,1[$, we have

$$\begin{aligned}
\beta_1(t\alpha_1 + (1-t)\alpha_2) &= \inf_{u \in W^*} \frac{\displaystyle\int_\Omega |\nabla u|^p dx - (t\alpha_1 + (1-t)\alpha_2) \int_\Omega m_1 |u|^p dx}{\displaystyle\int_\Omega m_2 |u|^p dx} \\
&\geq t \inf_{u \in W^*} \frac{\displaystyle\int_\Omega |\nabla u|^p dx - \alpha_1 \int_\Omega m_1 |u|^p dx}{\displaystyle\int_\Omega m_2 |u|^p dx} \\
&\quad + (1-t) \inf_{u \in W^*} \frac{\displaystyle\int_\Omega |\nabla u|^p dx - \alpha_2 \int_\Omega m_1 |u|^p dx}{\displaystyle\int_\Omega m_2 |u|^p dx} \\
&\geq t\beta_1(\alpha_1) + (1-t)\beta_1(\alpha_2).
\end{aligned}$$

This shows that $\alpha \to \beta_1(\alpha)$ is a concave function.

(ii) For any $\alpha, \alpha_0 \in \mathbb{R}$ such that $\alpha \neq \alpha_0$, by the variational characterization of $\beta_1(\alpha)$ and $\beta_1(\alpha_0)$, we have

$$\beta_1(\alpha_0) = \frac{\displaystyle\int_\Omega |\nabla \varphi_{\alpha_0}|^p - \alpha_0 \int_\Omega m_1 |\varphi_{\alpha_0}|^p}{\displaystyle\int_\Omega m_2 |\varphi_{\alpha_0}|^p} \leq \frac{\displaystyle\int_\Omega |\nabla \varphi_\alpha|^p - \alpha_0 \int_\Omega m_1 |\varphi_\alpha|^p}{\displaystyle\int_\Omega m_2 |\varphi_\alpha|^p}$$

and

$$\beta_1(\alpha) = \frac{\int\limits_\Omega |\nabla \varphi_\alpha|^p - \alpha \int\limits_\Omega m_1 |\varphi_\alpha|^p}{\int\limits_\Omega m_2 |\varphi_\alpha|^p} \leq \frac{\int\limits_\Omega |\nabla \varphi_{\alpha_0}|^p - \alpha \int\limits_\Omega m_1 |\varphi_{\alpha_0}|^p}{\int\limits_\Omega m_2 |\varphi_{\alpha_0}|^p}.$$

Then

$$(\alpha_0 - \alpha) \frac{\int\limits_\Omega m_1 |\varphi_\alpha|^p}{\int\limits_\Omega m_2 |\varphi_\alpha|^p} \leq \beta_1(\alpha) - \beta_1(\alpha_0) \leq (\alpha_0 - \alpha) \frac{\int\limits_\Omega m_1 |\varphi_{\alpha_0}|^p}{\int\limits_\Omega m_2 |\varphi_{\alpha_0}|^p}.$$

Finally, we get the desired result by dividing by $\alpha_0 - \alpha$ and passing to the limit as $\alpha \to \alpha_0$. $\square$

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