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ONE BOUNDARY VALUE PROBLEM INCLUDING A SPECTRAL PARAMETER IN ALL BOUNDARY CONDITIONS

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Abstract. In this paper, asymptotic formulae for solutions and Green's function of a boundary value problem are investigated when the equation and the boundary conditions contain a spectral parameter.

Keywords: boundary value problems, asymptotics, Green's functions.

Mathematics Subject Classification: 34B08, 34B27, 34L10.

1. INTRODUCTION

Boundary value problems for differential equations of the second order arise upon separation of variables in the one-dimensional wave and heat equations. These types of problems have an important role in solving many problems of mathematical physics such as vibrational modes of different systems, i.e., the vibration of a string or the energy eigenfunctions of a quantum mechanical oscillator, where the eigenvalues correspond to the resonant frequencies of vibration and energy levels. Usually, the eigenvalue parameter λ appears linearly only in the differential equation of the classic problems (see $[5,7,11,18,20,21]$). However, such problems are encountered in mathematical physics, which contain eigenvalue parameters not only in the differential equation, but also in the boundary conditions. For instance, we shall consider the cooling of a thin solid bar one end of which is placed in contact with a finite amount of liquid at time zero. Assuming that heat flows only into the liquid and is convected only from the liquid into the surrounding medium, the initial-boundary value problem for a bar of length one takes form [16]:

$$\begin{aligned} &u_t = \alpha^2 u_{xx}, \\ &u_x(0,t) = 0 \quad \text{for all } t, \\ &-kAu_x(1^-,t) = qM(dv/dt) + k_1 Bv(t), \\ &u(x,0) = u_0(x) \quad \text{for } x \in [0,1], \\ &v(0) = v_0 \end{aligned} \tag{1.1}$$

after factoring out the steady-state solution. Applying Newton's law of cooling and Fourier's law of heat conduction at $x = 1$, we get

$$v(t) = u(1,t) + kc^{-1}u_x(1^-,t) \quad \text{for } t > 0, \tag{1.2}$$

where $c > 0$ is the coefficient of heat transfer for the liquid. Putting (1.2) in (1.1) and using the method of separation of variables, the problem can be written as the boundary value problem including a spectral parameter in one boundary condition:

$$\begin{aligned}
&X''(x) + \lambda X(x) = 0,\\
&X'(0) = 0,\\
&(k_1B/\alpha^2 qM)X(1) + [(1 + k_1B/cA)/\sigma]X'(1^-) = \lambda[X(1) + (k/c)X'(1)],
\end{aligned}$$

where $\sigma := \alpha^2 qM/kA$.

Recently, there is a substantial literature on this field (see [2–4, 6, 8–10, 12, 19, 22, 23]). In general, eigenvalues and eigenfunctions of the problem have been investigated under the different boundary conditions. Additionally, there are several studies on Green's functions (see [13, 14]). Possibility of a transition from the problems in mathematical physics to integral equations is based on the fundamental concept of Green's function. Therefore, the powerful and unifying formalism of Green's functions finds applications not only in standard physics subjects such as perturbation and scattering theory, bound-state formation etc., but also at the forefront of current and, most likely, future developments [1, 15].

In this work, we seek the asymptotic solutions and Green's function of the boundary value problem, where eigenparameters take part both in equation and in boundary conditions. Namely, we consider the boundary value problem for the differential equation

$$y''(t) + [\lambda - q(t)]y(t) = 0 \tag{1.3}$$

on the segment $[0, 1]$, with boundary conditions

$$\left[\alpha_0 + \alpha_1\lambda^{1/2} + \alpha_2\lambda\right] y(0) + y'(0) = 0, \tag{1.4}$$

$$\left[\beta_0 + \beta_1\lambda^{1/2} + \beta_2\lambda\right] y(1) + y'(1) = 0, \tag{1.5}$$

where λ is a real parameter, $q(t) \in L^1[0, 1]$ and $\alpha_i, \beta_i \in \mathbb{R}, i = 0, 1, 2$. It was Kerimov and Mamedov's article [22] that motivated the author to investigate asymptotic formulae for solutions of this problem. It is assumed that the following conditions are satisfied for real eigenvalues:

$$\alpha_0 < 0, \quad \alpha_2 > 0, \quad \beta_0 > 0, \quad \beta_2 < 0, \quad |\alpha_1| + |\beta_1| \neq 0.$$

Also, q has a mean value zero, i.e., $\int_0^1 q(t)dt = 0$.

2. ASYMPTOTIC APPROXIMATIONS FOR THE SOLUTIONS

Let $y(t,\lambda)$ denote a solution of the equation (1.3) which is complex-valued in the sense that neither the real nor the imaginary part is identically zero. If $v(t,\lambda)=\frac{y'(t,\lambda)}{y(t,\lambda)}$ transform is applied to (1.3), we have the Riccati equation

$$v' = -\lambda + q - v^2. \tag{2.1}$$

We set

$$S(t,\lambda) := \mathrm{Re}\{v(t,\lambda)\}, \quad T(t,\lambda) := \mathrm{Im}\{v(t,\lambda)\}, \tag{2.2}$$

where $v(t,\lambda)$ is a complex-valued solution of (2.1). It was shown by Harris in [17] that any nontrivial real-valued solution, z, of (1.3) can be expressed as

$$z(t,\lambda) = c_1 \exp\left(\int_0^t S(x,\lambda)dx\right)\cos\left\{c_2 + \int_0^t T(x,\lambda)dx\right\} \tag{2.3}$$

with

$$\begin{aligned} z'(t,\lambda) = {} & c_1 S(t,\lambda)\exp\left(\int_0^t S(x,\lambda)dx\right)\cos\left\{c_2 + \int_0^t T(x,\lambda)dx\right\} \\ & - c_1 T(t,\lambda)\exp\left(\int_0^t S(x,\lambda)dx\right)\sin\left\{c_2 + \int_0^t T(x,\lambda)dx\right\}. \end{aligned} \tag{2.4}$$

With a view to determining the asymptotic approximations, we define two solutions, $\Theta(t,\lambda)$ and $\Omega(t,\lambda)$ of (1.3) satisfying the following initial conditions

$$\Theta(0,\lambda) = 1, \quad \Theta'(0,\lambda) = -\alpha_0 - \alpha_1\lambda^{1/2} - \alpha_2\lambda \tag{2.5}$$

and

$$\Omega(1,\lambda) = 1, \quad \Omega'(1,\lambda) = -\beta_0 - \beta_1\lambda^{1/2} - \beta_2\lambda. \tag{2.6}$$

Theorem 2.1. *For $\Theta(t,\lambda)$ and $\Omega(t,\lambda)$, we have*

(i)

$$\begin{aligned} \Theta(t,\lambda) = {} & \frac{1}{\cos(\cot^{-1}\Gamma_0)}\exp\left(\int_0^t S(x,\lambda)dx\right) \\ & \times \cos\left\{\cot^{-1}\Gamma_0 + \int_0^t T(x,\lambda)dx\right\}, \end{aligned} \tag{2.7}$$

where

$$\Gamma_0 := \frac{T(0,\lambda)}{\alpha_2\lambda + \alpha_1\lambda^{1/2} + \alpha_0 + S(0,\lambda)}.$$

(ii)

$$\Omega(t,\lambda) = \frac{1}{\cos(\cot^{-1}\Gamma_1)} \exp\left(-\int_t^1 S(x,\lambda)dx\right) \times \cos\left\{\cot^{-1}\Gamma_1 - \int_t^1 T(x,\lambda)dx\right\},$$

where

$$\Gamma_1 := \frac{T(1,\lambda)}{\beta_2\lambda + \beta_1\lambda^{1/2} + \beta_0 + S(1,\lambda)}.$$

Proof. (i) By using (2.3), (2.4) and (2.5) we obtain

$$\Theta(0,\lambda) = c_1\cos(c_2) = 1, \tag{2.8}$$

$$\begin{aligned}\Theta'(0,\lambda) &= c_1 S(0,\lambda)\cos(c_2) - c_1 T(0,\lambda)\sin(c_2) \\ &= -\alpha_0 - \alpha_1\lambda^{1/2} - \alpha_2\lambda.\end{aligned} \tag{2.9}$$

From (2.8)

$$c_1 = \frac{1}{\cos(c_2)}. \tag{2.10}$$

Using this in (2.9),

$$c_2 = \cot^{-1}\left[\frac{T(0,\lambda)}{\alpha_2\lambda + \alpha_1\lambda^{1/2} + \alpha_0 + S(0,\lambda)}\right]. \tag{2.11}$$

We substitute the values of c_1 and c_2 into (2.3). Thus, the proof is completed.

(ii) From (2.3), (2.4) and (2.6)

$$\Omega(1,\lambda) = c_1 \exp\left(\int_0^1 S(x,\lambda)dx\right)\cos\left\{c_2 + \int_0^1 T(x,\lambda)dx\right\} = 1, \tag{2.12}$$

$$\begin{aligned}\Omega'(1,\lambda) = c_1 S(1,\lambda) &\exp\left(\int_0^1 S(x,\lambda)dx\right)\cos\left\{c_2 + \int_0^1 T(x,\lambda)dx\right\} \\ &- c_1 T(1,\lambda)\exp\left(\int_0^1 S(x,\lambda)dx\right)\sin\left\{c_2 + \int_0^1 T(x,\lambda)dx\right\} \\ = -\beta_0 - \beta_1\lambda^{1/2} - \beta_2\lambda.\end{aligned} \tag{2.13}$$

By (2.12), we can write

$$c_1 = \frac{1}{\exp\left(\int_0^1 S(x,\lambda)dx\right)\cos\left\{c_2 + \int_0^1 T(x,\lambda)dx\right\}}.$$

Using this in (2.13), we have

$$c_2 = \cot^{-1}\left[\frac{T(1,\lambda)}{\beta_2\lambda + \beta_1\lambda^{1/2} + \beta_0 + S(1,\lambda)}\right] - \int_0^1 T(x,\lambda)dx.$$

Substituting the values of c_1 and c_2 into (2.3) proves the result. □

Now we seek asymptotic approximations for the solutions, $\Theta(t,\lambda)$ and $\Omega(t,\lambda)$, of (1.3) satisfying initial conditions (2.5) and (2.6), respectively. That's why the asymptotic formulae of $S(t,\lambda)$ and $T(t,\lambda)$ defined by (2.2) are needed. We seek approximate solution of (2.1) of the form

$$v(t,\lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t,\lambda), \tag{2.14}$$

where $\lim_{t\to\infty} v_n(t,\lambda) = 0$ for $n = 1, 2, \ldots$ ([17]). Substitution of (2.14) into (2.1) and rearrangement yield

$$\begin{aligned} & v_1' + 2i\lambda^{1/2}v_1 + v_2' + 2i\lambda^{1/2}v_2 + \sum_{n=3}^{\infty}(v_n' + 2i\lambda^{1/2}v_n) \\ & = q - v_1^2 - \sum_{n=3}^{\infty}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right). \end{aligned} \tag{2.15}$$

We choose

$$\begin{aligned} v_1' + 2i\lambda^{1/2}v_1 &= q, \\ v_2' + 2i\lambda^{1/2}v_2 &= -v_1^2, \\ v_n' + 2i\lambda^{1/2}v_n &= -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right), \quad n \geq 3. \end{aligned}$$

Then,

$$\begin{aligned} v_1(t,\lambda) &= -e^{-2i\lambda^{1/2}t}\int_t^1 e^{2i\lambda^{1/2}x} q(x)dx, \\ v_2(t,\lambda) &= e^{-2i\lambda^{1/2}t}\int_t^1 e^{2i\lambda^{1/2}x} v_1^2(x,\lambda)dx, \\ v_n(t,\lambda) &= e^{-2i\lambda^{1/2}t}\int_t^1 e^{2i\lambda^{1/2}x}\left[v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right]dx, \quad n \geq 3. \end{aligned}$$

It is shown in [17] that $\sum_{n=1}^{\infty} v_n(t,\lambda)$ is uniformly absolutely convergent for all $\lambda \geq \lambda_0$ and for all $t \in [0,1]$. It also follows from the choice of v_ns that $\sum_{n=1}^{\infty} v_n'(t,\lambda)$ is uniformly absolutely convergent. The series $i\lambda^{1/2}+\sum_{n=1}^{\infty} v_n(t,\lambda)$ is therefore a solution of (2.1). So, we get

$$S(t,\lambda) = \mathrm{Re}\sum_{n=1}^{\infty} v_n(t,\lambda), \tag{2.16}$$

$$T(t,\lambda) = \lambda^{1/2} + \mathrm{Im}\sum_{n=1}^{\infty} v_n(t,\lambda). \tag{2.17}$$

For the asymptotic approximations, we suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^1 e^{2i\lambda^{1/2}x} q(x)dx\right| \le A(t)\eta(\lambda), \tag{2.18}$$

where

(i) $A(t) := \int_t^1 |q(x)|dx$ is a decreasing function of t,

(ii) $A(t) \in L^1[0,1]$,

(iii) $\eta(\lambda) \to 0$ as $\lambda \to \infty$.

The existence of these functions are established in [17]. For completeness, let define

$$F(t,\lambda) := \begin{cases} |\int_t^1 e^{2i\lambda^{1/2}x} q(x)dx| / \int_t^1 |q(x)|dx, & \text{if } \int_t^1 |q(x)|dx \neq 0, \\ 0, & \text{if } \int_t^1 |q(x)|dx = 0. \end{cases} \tag{2.19}$$

Then $0 \le F(t,\lambda) \le 1$ and we set $\eta(\lambda) := sup_{0\le t\le 1} F(t,\lambda)$. Note that $\eta(\lambda)$ is well-defined by (2.19) and $\eta(\lambda) \to 0$ as $\lambda \to \infty$ [17]. In this manner, it was proven in [9] that the following approximations are satisfied as $\lambda \to \infty$:

$$S(t,\lambda) = -\sin(2\lambda^{1/2}t + \xi_t) + O(\eta^2(\lambda)) \tag{2.20}$$

and

$$T(t,\lambda) = \lambda^{1/2} - \cos(2\lambda^{1/2}t + \xi_t) + O(\eta^2(\lambda)), \tag{2.21}$$

where

$$\sin\xi_t := \int_t^1 q(x)\cos 2\lambda^{1/2}x dx, \quad \cos\xi_t := \int_t^1 q(x)\sin 2\lambda^{1/2}x dx.$$

By using them, we have the following results.

Theorem 2.2. *For the solutions of* (1.3) *having the initial conditions* (2.5) *and* (2.6) *respectively, we obtain as* $\lambda \to \infty$

(i)

$$\Theta(t,\lambda) = -\alpha_2\lambda^{1/2}\sin(\lambda^{1/2}t) - \alpha_1\sin(\lambda^{1/2}t) + \left(1 + \frac{\alpha_2}{2}\int_0^t q(x)dx\right) \times \cos(\lambda^{1/2}t) + O(\eta(\lambda)), \tag{2.22}$$

(ii)

$$\Omega(t,\lambda) = \beta_2\lambda^{1/2}\sin\left[\lambda^{1/2}(1-t)\right] + \beta_1\sin\left[\lambda^{1/2}(1-t)\right] + \left(1 - \frac{\beta_2}{2}\int_t^1 q(x)dx\right)\cos\left[\lambda^{1/2}(1-t)\right] + O(\eta(\lambda)). \tag{2.23}$$

Proof. (i) The terms in (2.7) are calculated as $\lambda \to \infty$. By using (2.20), (2.21) and the series expansion method, it is obtained that

$$\begin{aligned}
\Gamma_0 &= \frac{\lambda^{1/2} - \cos\xi_0 + O(\eta^2(\lambda))}{\alpha_2\lambda + \alpha_1\lambda^{1/2} + \alpha_0 - \sin\xi_0 + O(\eta^2(\lambda))} \\
&= \frac{\lambda^{1/2} - \cos\xi_0 + O(\eta^2(\lambda))}{\alpha_2\lambda[1 + \frac{\alpha_1}{\alpha_2}\lambda^{-1/2} + \frac{\alpha_0}{\alpha_2}\lambda^{-1} - \frac{1}{\alpha_2}\lambda^{-1}\sin\xi_0 + O(\lambda^{-1}\eta^2(\lambda))} \\
&= \left\{\frac{1}{\alpha_2}\lambda^{-1/2} - \frac{1}{\alpha_2}\lambda^{-1}\cos\xi_0 + O(\lambda^{-1}\eta^2(\lambda))\right\} \\
&\quad \times \left\{1 - \frac{\alpha_1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_0}{\alpha_2}\lambda^{-1} + \frac{1}{\alpha_2}\lambda^{-1}\sin\xi_0 + O(\lambda^{-1}\eta^2(\lambda))\right\} \\
&= \frac{1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda)),
\end{aligned} \tag{2.24}$$

$$\int_0^t S(x,\lambda)dx = \frac{1}{2}\lambda^{-1/2}\left\{\cos(2\lambda^{1/2}t + \xi_t) - \cos\xi_0\right\} + O(\lambda^{-1/2}\eta^2(\lambda)), \tag{2.25}$$

$$\int_0^t T(x,\lambda)dx = \lambda^{1/2}t - \frac{1}{2}\lambda^{-1/2}\left\{\sin(2\lambda^{1/2}t + \xi_t) - \sin\xi_0 + \int_0^t q(x)dx\right\} + O(\lambda^{-1/2}\eta^2(\lambda)). \tag{2.26}$$

From (2.24)

$$\cot^{-1}\Gamma_0 = \frac{\pi}{2} - \frac{1}{\alpha_2}\lambda^{-1/2} + \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda)).$$

With this last equality, we find the following results:

$$\begin{aligned}\cos(\cot^{-1}\Gamma_0) &= \sin\left\{\frac{1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda))\right\} \\ &= \frac{1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda)),\end{aligned} \tag{2.27}$$

$$\begin{aligned}\sin(\cot^{-1}\Gamma_0) &= \cos\left\{\frac{1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda))\right\} \\ &= 1 - \frac{1}{2(\alpha_2)^2}\lambda^{-1} + \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-3/2} + O(\lambda^{-3/2}\eta(\lambda)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}\frac{1}{\cos(\cot^{-1}\Gamma_0)} &= \frac{1}{\frac{1}{\alpha_2}\lambda^{-1/2} - \frac{\alpha_1}{(\alpha_2)^2}\lambda^{-1} + O(\lambda^{-1}\eta(\lambda))} \\ &= \alpha_2\lambda^{1/2} + \alpha_1 + O(\eta(\lambda)).\end{aligned} \tag{2.29}$$

Using trigonometric expansions together with (2.25), (2.26), (2.27) and (2.28) we have

$$\begin{aligned}\cos\left\{\cot^{-1}\Gamma_0 + \int_0^t T(x,\lambda)dx\right\} &= -\sin(\lambda^{1/2}t) + \left(\frac{1}{\alpha_2} + \frac{1}{2}\int_0^t q(x)dx\right)\lambda^{-1/2} \\ &\quad \times \cos(\lambda^{1/2}t) + O(\lambda^{-1/2}\eta(\lambda)).\end{aligned} \tag{2.30}$$

Finally, by substituting (2.25), (2.29) and (2.30) into (2.7), we complete the proof.
For (ii) the proof is similar. □

We have also some approximations for the derivatives of $\Theta(t,\lambda)$ and $\Omega(t,\lambda)$. They will be used in calculation of the Green's function.

Lemma 2.3. *As $\lambda \to \infty$,*

(i)

$$\begin{aligned}\Theta'(t,\lambda) &= -\alpha_2\lambda\cos(\lambda^{1/2}t) - \alpha_1\lambda^{1/2}\cos(\lambda^{1/2}t) \\ &\quad - \left(1 + \frac{\alpha_2}{2}\int_0^t q(x)dx\right)\lambda^{1/2}\sin(\lambda^{1/2}t) + O(\lambda^{1/2}\eta(\lambda)),\end{aligned} \tag{2.31}$$

(ii)

$$\begin{aligned}\Omega'(t,\lambda) &= -\beta_2\lambda\cos\left[\lambda^{1/2}(1-t)\right] - \beta_1\lambda^{1/2}\cos\left[\lambda^{1/2}(1-t)\right] \\ &\quad + \left(1 - \frac{\beta_2}{2}\int_t^1 q(x)dx\right)\lambda^{1/2}\sin\left[\lambda^{1/2}(1-t)\right] + O(\lambda^{1/2}\eta(\lambda)).\end{aligned} \tag{2.32}$$

Proof. Here the proof of part (i) will be shown. The proof of (ii) is similar to that.

We use $z'(t,\lambda)$ given by the equality (2.4). If we substitute (2.10) and (2.11) into (2.4), it is obtained that

$$\begin{aligned}\Theta'(t,\lambda) = {} & \frac{1}{\cos(\cot^{-1}\Gamma_0)} \exp\left(\int_0^t S(x,\lambda)dx\right)\left\{S(t,\lambda)\cos\left[\cot^{-1}\Gamma_0\right.\right. \\ & \left.\left. + \int_0^t T(x,\lambda)dx\right] - T(t,\lambda)\sin\left[\cot^{-1}\Gamma_0 + \int_0^t T(x,\lambda)dx\right]\right\}.\end{aligned} \tag{2.33}$$

The terms in (2.33) are evaluated as $\lambda \to \infty$. So, the lemma is proved by (2.25), (2.26), (2.28) (2.29) and (2.30). ☐

3. ASYMPTOTIC APPROXIMATIONS FOR THE GREEN'S FUNCTION

Green's function of the problem (1.3)-(1.5) has the form [16]

$$G(t,\xi,\lambda) = \begin{cases} \dfrac{\Theta(\xi,\lambda)\Omega(t,\lambda)}{w(\lambda)}, & 0 \le \xi \le t \le 1, \\ \dfrac{\Theta(t,\lambda)\Omega(\xi,\lambda)}{w(\lambda)}, & 0 \le t \le \xi \le 1, \end{cases} \tag{3.1}$$

where $w(\lambda)$ is the Wronskian determinant of Θ and Ω which is independent of t, i.e.,

$$w(\lambda) := \Theta(t,\lambda)\Omega'(t,\lambda) - \Theta'(t,\lambda)\Omega(t,\lambda). \tag{3.2}$$

Theorem 3.1. *The following asymptotic formula is valid:*

$$\begin{aligned} G(t,\xi,\lambda) = {} & -\lambda^{-1/2}\frac{\sin(\lambda^{1/2}t)\sin\left[\lambda^{1/2}(1-\xi)\right]}{\sin(\lambda^{1/2})} + \frac{\lambda^{-1}}{\sin(\lambda^{1/2})} \\ & \times\left\{\left(\frac{1}{\beta_2} - \frac{1}{\alpha_2}\right)\cot(\lambda^{1/2})\sin(\lambda^{1/2}t)\sin\left[\lambda^{1/2}(1-\xi)\right]\right. \\ & + \left(\frac{1}{\alpha_2} + \frac{1}{2}\int_0^t q(x)dx\right)\cos(\lambda^{1/2}t)\sin\left[\lambda^{1/2}(1-\xi)\right] \\ & \left. - \left(\frac{1}{\beta_2} - \frac{1}{2}\int_\xi^1 q(x)dx\right)\sin(\lambda^{1/2}t)\cos\left[\lambda^{1/2}(1-\xi)\right]\right\} \\ & + O(\lambda^{-1}\eta(\lambda)), \quad 0 \le \xi \le t \le 1. \end{aligned}$$

A similar result holds for $0 \le t \le \xi \le 1$ *changing the role of* t *and* ξ.

Proof. Firstly, we calculate the Wronskian, $w(\lambda)$. Therefore (2.22), (2.23), (2.31) and (2.32) are applied in (3.2). We find

$$\begin{aligned} w(\lambda) &= \alpha_2\beta_2\lambda^{3/2}\sin(\lambda^{1/2}) + (\alpha_2\beta_1+\alpha_1\beta_2)\lambda\sin(\lambda^{1/2}) + (\alpha_2-\beta_2)\,\lambda\cos(\lambda^{1/2}) \\ &\quad + O(\lambda\eta(\lambda)). \end{aligned}$$

And from that,

$$\begin{aligned} \frac{1}{w(\lambda)} &= \frac{1}{\alpha_2\beta_2\lambda^{3/2}\sin(\lambda^{1/2})\left\{\begin{array}{c} 1+\left(\frac{\beta_1}{\beta_2}+\frac{\alpha_1}{\alpha_2}\right)\lambda^{-1/2}+\left(\frac{1}{\beta_2}-\frac{1}{\alpha_2}\right)\lambda^{-1/2} \\ \times\cot(\lambda^{1/2})+O(\lambda^{-1/2}\eta(\lambda)) \end{array}\right\}} \\ &= \frac{1}{\alpha_2\beta_2\sin(\lambda^{1/2})}\lambda^{-3/2} - \frac{\alpha_2\beta_1+\alpha_1\beta_2}{(\alpha_2\beta_2)^2\sin(\lambda^{1/2})}\lambda^{-2} + \frac{\beta_2-\alpha_2}{(\alpha_2\beta_2)^2}\lambda^{-2} \\ &\quad \times\frac{\cos(\lambda^{1/2})}{\sin^2(\lambda^{1/2})} + O(\lambda^{-2}\eta(\lambda)). \end{aligned} \tag{3.3}$$

Finally, (2.22), (2.23) and (3.3) together with (3.1) give the result. □

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