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NONEXISTENCE OF GLOBAL SOLUTIONS FOR A NONLINEAR PARABOLIC EQUATION WITH A FORCING TERM

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Abstract. The purpose of this work is to analyze the blow-up of solutions of a nonlinear parabolic equation with a forcing term depending on both time and space variables

$$u_t - \Delta u = |x|^\alpha |u|^p + \mathsf{a}(t)\,\mathbf{w}(x) \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

where $\alpha \in \mathbb{R}$, $p > 1$, and $\mathsf{a}(t)$ as well as $\mathbf{w}(x)$ are suitable given functions. We generalize and somehow improve earlier existing works by considering a wide class of forcing terms that includes the most common investigated example $t^\sigma\,\mathbf{w}(x)$ as a particular case. Using the test function method and some differential inequalities, we obtain sufficient criteria for the nonexistence of global weak solutions. This criterion mainly depends on the value of the limit $\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathsf{a}(s)\,ds$. The main novelty lies in our treatment of the nonstandard condition on the forcing term.

Keywords: nonlinear heat equation, forcing term, blow-up, test-function, differential inequalities.

Mathematics Subject Classification: 35K05, 35A01, 35B44.

1. INTRODUCTION

This paper is concerned with the finite time blow-up of solutions of the following inhomogeneous parabolic equation

$$\begin{cases} u_t - \Delta u = |x|^\alpha |u|^p + \mathsf{a}(t)\,\mathbf{w}(x), & (t,x) \in \mathbb{S} := (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\mathbf{w} : \mathbb{R}^N \to \mathbb{R}$ is a continuous and globally integrable function, $\mathsf{a} : (0,\infty) \to [0,\infty)$ is a continuous locally integrable function, $\alpha \in \mathbb{R}$, and $p > 1$.

It is well known that many biological processes and chemical reactions are frequently described by reaction-diffusion equations, consisting of the heat equation modified by a reaction term:

$$u_t - \Delta u = f(t, x, u), \tag{1.2}$$

where $u(t, x)$ stands for the density function at time t and position x in a diffusion medium, and $f(t, x, u)$ represents the rate of change due to reaction; see [25] for more details and examples. For a survey on physical models arising from various fields and their mathematical studies, we refer to [7, 9, 10, 21–23, 28, 30, 30, 33] and the references therein.

In the case $\mathsf{a}(t)\mathbf{w}(x) = 0$, problem (1.1) reduces to

$$\begin{cases} u_t - \Delta u = |x|^{\alpha}|u|^p, & (t, x) \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

In [5, 6] Fujita considered the initial value problem (1.3) with $\alpha = 0$ and $u_0(x) \geq 0$. He established the following results:

– If $p < 1 + \frac{2}{N}$, then all nontrivial solutions of (1.3) blow up in finite time.
– If $p > 1+\frac{2}{N}$, then (1.3) has both bounded global solutions and solutions which blow up in finite time. More precisely, for initial values $u_0(x)$ bounded by a sufficiently small Gaussian $\varepsilon e^{-|x|^2}$ the solution is global. See, for instance, [30, Theorem 20.1, p. 129].

In the borderline case, $p = 1 + \frac{2}{N}$, it was shown by Hayakawa [8] for dimensions $N = 1, 2$ and by Kobayashi, Sino, and Tanaka [16] and Aronson and Weinberger [1] for all $N \geq 1$ that all nontrivial solutions to (1.3) blow up in finite time. The number $p_F := 1 + \frac{2}{N}$ is nowadays usually known as the *Fujita exponent* for problem (1.3).

The case $\alpha \neq 0$ was investigated in [29] (see also [2, 18]). It is shown that the Cauchy problem (1.3) has no global solution if $p < 1 + \frac{2+\alpha}{N}$, but global solutions exist when $p > 1 + \frac{2+\alpha}{N}$ provided that $\alpha > -1$ for $N = 1$ and $\alpha > -2$ for $N \geq 2$. See [29, Theorem 1.1, p. 125] for a precise statement. Nonetheless, the critical case $p = 1 + \frac{2+\alpha}{N}$ is open.

Note that the case $\alpha = 0$ and $\mathsf{a}(t) = 1$ was investigated by Bandle, Levine and Zhang in [3]. They showed that (1.1) has no global solutions provided that $p < \frac{N}{N-2}$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$. In [11], the authors consider (1.1) with $\alpha = 0$ and $\mathsf{a}(t) = t^{\sigma}$ where $\sigma > -1$. They showed that the critical exponent is given by

$$p^*(\sigma) = \begin{cases} \frac{N-2\sigma}{N-2\sigma-2} & \text{if } -1 < \sigma < 0, \\ \infty & \text{if } \sigma > 0. \end{cases}$$

Recently in [19], the author shows that blow-up depends on the behavior of $\mathsf{a}(t)$ at infinity by considering the case

$$\mathsf{a}(t) \sim \begin{cases} \mathsf{a}_0\, t^{\sigma} & \text{as } t \to 0, \\ \mathsf{a}_{\infty}\, t^{m} & \text{as } t \to \infty, \end{cases}$$

where $\mathsf{a}_0, \mathsf{a}_\infty > 0$, $\sigma > -1$ and $m \in \mathbb{R}$. Here the notation $u(t) \sim v(t)$ as $t \to t_0$ means that $\lim_{t\to t_0} \frac{u(t)}{v(t)} = 1$. It was shown that the critical exponent is given by

$$p^*(m,\alpha) = \begin{cases} \frac{N-2m+\alpha}{N-2m-2} & \text{if } m \le 0,\ \alpha > -2, \\ \infty & \text{if } m > 0,\ \alpha > -2, \end{cases}$$

provided that $\int_{\mathbb{R}^N} \mathbf{w}(x)\, dx > 0$. In addition, some local and global existence results was obtained in [19]. Let us mention that the fractional counterpart was considered in [20].

There exist several well-known methods of the study of the blow-up effect, which have their specific domain of applicability to corresponding problems of mathematical physics. For a survey of blow-up results for solutions of first-order nonlinear evolution inequalities and related Cauchy problems we refer to [3, 4, 7, 12–15, 17, 26, 27] and the references therein. See also the books [10, 25, 30, 34].

We adopt the following definition of solution.

Definition 1.1. We say that u is a global weak solution of (1.1) if it satisfies the conditions

$$u_0 \in L^1_{loc}(\mathbb{R}^N), \quad |x|^\alpha |u|^p \in L^1_{loc}((0,\infty)\times\mathbb{R}^N),$$

and, for all $\psi \in C_0^\infty((0,\infty)\times\mathbb{R}^N)$,

$$\int_0^\infty \int_{\mathbb{R}^N} u(-\partial_t\psi - \Delta\psi)dxdt = \int_{\mathbb{R}^N} u_0(x)\psi(0,x)dx + \int_0^\infty \int_{\mathbb{R}^N} |x|^\alpha |u|^p \psi dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} \mathsf{a}(t)\mathbf{w}(x)\psi\, dx\, dt.$$

As is a standard practice, (1.1) is equivalent in an appropriate framework to the Duhamel formulation

$$u(x,t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left(|\cdot|^\alpha |u(s)|^p\right) ds + \int_0^t \mathsf{a}(s)\, e^{(t-s)\Delta}\mathbf{w}\, ds, \tag{1.4}$$

where $e^{t\Delta}$ is the linear semi-group generated by the Laplace operator Δ. A solution to the integral equation (1.4) is often called mild solution of (1.1). Let us mention that the existence and regularity for some parabolic and ultraparabolic problems was carried out in [31, 32].

In the present article, we generalize and somehow improve earlier existing works [3, 11, 19] by considering a wide class of functions $\mathsf{a}(t)$ that includes the most common investigated example $\mathsf{a}(t) = t^\sigma$. One of the novelties here is the non-standard assumptions on the time-dependent term $\mathsf{a}(t)$ which cover the hypotheses in earlier works [3, 11, 19].

Roughly speaking, our main results say that the blow-up occurs if one of the following conditions is satisfied:

(a) $0 < \ell < \infty$, $p_*(N,\alpha) < p < p^*(N,\alpha)$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$,
(b) $\ell = \infty$, $p > p_*(N,\alpha)$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$,
(c) $\ell = 0$, $\mathbf{J} \neq \emptyset$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$,

where ℓ, $p_*(N,\alpha)$, $p^*(N,\alpha)$, $\mathbf{J}$ are respectively given by (2.1), (2.2) and (2.3) below.

We conclude the introduction with an outline of the paper. The next section contains our main results, that is Theorem 2.1, Theorem 2.4 and Theorem 2.8. In Section 3, we give the proofs of Theorems 2.1-2.4. The fourth section is devoted to the proof of Theorem 2.8. In the sequel, C will be used to denote a constant which may vary from line to line.

2. MAIN RESULTS

In order to present our main results clearly, we define the function

$$\mathtt{A}(t) = \frac{1}{t}\int_0^t \mathtt{a}(s)\,ds,$$

and assume that

$$\lim_{t\to\infty} \mathtt{A}(t) = \ell \in [0,\infty]. \tag{2.1}$$

We also define

$$p_*(N,\alpha) = 1 + \frac{\alpha}{N}, \quad p^*(N,\alpha) = \frac{N+\alpha}{N-2} \quad \Big(p^*(N,\alpha) = \infty \text{ if } N = 1,2\Big). \tag{2.2}$$

Theorem 2.1. *Suppose* $0 < \ell < \infty$, $p_*(N,\alpha) < p < p^*(N,\alpha)$ *and* $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$. *Then* (1.1) *has no global solutions on* $\mathbb{S}$ *in the sense of Definition 1.1.*

Remark 2.2. Clearly, the condition $p > p_*(N,\alpha)$ makes sense only for $\alpha > 0$.

Examples 2.3. Some examples of functions $\mathtt{a}(t)$ satisfying (2.1) with $\ell \in (0,\infty)$ are listed below.

(i) Constant at infinity: $\mathtt{a}(t) = \mathtt{a}_\infty + o(1)$ as $t \to \infty$, where $\mathtt{a}_\infty > 0$.
(ii) Oscillating: $\mathtt{a}(t) = \cos^2(t)$ or $\mathtt{a}(t) = \sin^2(t)$. Other combinations are allowed.
(iii) Periodic: More generally, if $\mathtt{a}(t)$ is ϑ-periodic then by (5.4) $\mathtt{a}(t)$ satisfies (2.1) with

$$\ell = \frac{1}{\vartheta}\int_0^\vartheta \mathtt{a}(s)\,ds.$$

Theorem 2.4. *Suppose* $\ell = \infty$, $p > p_*(N,\alpha)$ *and* $\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$. *Then* (1.1) *has no global solutions on* $\mathbb{S}$ *in the sense of Definition 1.1.*

Examples 2.5. Some examples of functions $\mathsf{a}(t)$ satisfying (2.1) with $\ell = \infty$ are listed below.

(i) Power functions: $\mathsf{a}(t) = \mathsf{a}_\infty t^m + o(1)$ as $t \to \infty$, where $\mathsf{a}_\infty, m > 0$.
(ii) Power-Log functions: $\mathsf{a}(t) = \mathsf{a}_\infty t^m (\ln t)^q + o(1)$ as $t \to \infty$, where $\mathsf{a}_\infty > 0$ and either $m > 0, q \in \mathbb{R}$ or $m = 0, q > 0$.
(iii) Oscillating: $\mathsf{a}(t) = t^m\, \psi(t) + o(1)$ as $t \to \infty$ with $m > 0$ and ψ is a continuous ϑ-periodic function such that $\int_0^{\vartheta} \psi(s) ds > 0$. It follows by (5.4) that $\mathsf{a}(t)$ satisfies (2.1) with $\ell = \infty$.

As it will be clear in the proofs, our method doesn't cover the case $\ell = 0$. To handle this case, we introduce the set

$$\mathbf{J} = \left\{ q \in \mathbb{R} \,:\, \lim_{T\to\infty} T^q \int_{\frac{T}{2}}^{\frac{2}{3}T} \mathsf{a}(t)\, dt = \infty \right\}. \tag{2.3}$$

Note that the choice of $(\frac{T}{2}, \frac{2T}{3})$ is technical and is related to the test function method used here. It can be any interval $(\lambda T, \mu T)$ with $0 < \lambda < \mu < 1$.

The following description of the set $\mathbf{J}$ is straightforward.

Proposition 2.6. *Suppose that $\mathbf{J} \neq \emptyset$. Then, we have $J = (q_0, \infty)$, where*

$$q_0 = \inf \mathbf{J} \in [-\infty, \infty).$$

Remark 2.7.

1. If $\mathsf{a}(t) = \frac{1}{t}$ then $\mathbf{J} = (0, \infty)$.
2. If $\mathsf{a}(t) = e^t$ then $\mathbf{J} = \mathbb{R}$.
3. We may have $\mathbf{J} = \emptyset$ if we take, for example, $\mathsf{a}(t) = e^{-t}$.

The following theorem covers the case $\ell = 0$ and can be seen as a general statement for blow-up.

Theorem 2.8. *Suppose $\int_{\mathbb{R}^N} \mathbf{w}(x) dx > 0$ and $\mathbf{J} \neq \emptyset$.*

1. *If $\mathbf{J} = \mathbb{R}$ then (1.1) has no global solutions on $\mathbb{S}$ in the sense of Definition 1.1 provided that $p > p_*(N, \alpha)$.*
2. *If $\mathbf{J} = (q_0, \infty)$ with $q_0 \in \mathbb{R}$ then (1.1) has no global solutions on $\mathbb{S}$ in the sense of Definition 1.1 provided that*

$$\frac{2p+\alpha}{2(p-1)} - \frac{N}{2} - 1 > q_0. \tag{2.4}$$

Remark 2.9. Theorem 2.8 partially covers earlier results given in [3, 11, 19] as explained below.

For $\mathsf{a}(t) \equiv 1$, we have $q_0 = -1$. The condition (2.4) with $\alpha = 0$ translates to $p < \frac{N}{N-2}$ ($p < \infty$ for $N = 1, 2$). Hence, we recover [3, Theorem 2.1, Part (a)].

Let $\mathsf{a}(t) = t^\sigma$ with $\sigma > -1$ and $\alpha = 0$. The condition $\sigma > -1$ ensure the local integrability of $\mathsf{a}(t)$ on $(0, \infty)$. Clearly $q_0 = -1 - \sigma$. Owing to (2.4), we see that blow-up occurs for $\frac{2p}{p-1} > N - 2\sigma$. We conjecture that

$$p^*(\sigma) = \begin{cases} \frac{N-2\sigma}{N-2\sigma-2} & \text{if } -1 < \sigma < \frac{N}{2} - 1, \\ \infty & \text{if } \sigma \geq \frac{N}{2} - 1. \end{cases}$$

Suppose now that $\mathsf{a}(t) \in L^1_{loc}(0, \infty)$ and $\mathsf{a}(t) \sim \mathsf{a}_\infty\, t^m$ as $t \to \infty$ for some constants $\mathsf{a}_\infty > 0$ and $m \in \mathbb{R}$. Clearly $q_0 = -1 - m$. Thanks to (2.4), we conjecture that

$$p^*(m, \alpha) = \begin{cases} \frac{N-2m+\alpha}{N-2m-2} & \text{if } m < \frac{N}{2} - 1, \\ \infty & \text{if } m \geq \frac{N}{2} - 1. \end{cases}$$

Remark 2.10. While Theorem 2.1 and Theorem 2.4 cover the cases $0 < \ell < \infty$ and $\ell = \infty$ respectively, Theorem 2.8 treats the remainder case $\ell = 0$ and can be seen as a more general statement for blow-up.

Open Problem 2.11. The validity of Theorem 2.8 in the case $\mathbf{J} = \emptyset$ remains an interesting open problem. For example, we have no clue how to handle the case $\mathsf{a}(t) = t^\sigma e^{-t}$, $\sigma > -1$. This will be investigated in a forthcoming paper.

3. PROOFS OF THEOREM 2.1 AND THEOREM 2.4

As one will see, our proof borrows some arguments from [3]. Let $\varphi \in C^2(\mathbb{R}^N)$ be a smooth function such that:

$$\varphi = 1 \text{ in } B_1(0), \quad \varphi = 0 \text{ in } B_2^C(0) \text{ and } 0 \leq \varphi \leq 1 \quad \text{everywhere}, \tag{3.1}$$

$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial\big(B_2(0) - B_1(0)\big), \tag{3.2}$$

$$|\Delta \varphi| \leq C_\theta \varphi^\theta \text{ in } B_2(0) - B_1(0) \text{ for all } \theta \in (0,1), \tag{3.3}$$

where, for $r > 0$, $B_r(0)$ stands for the euclidean ball in $\mathbb{R}^N$ centered at 0 and with radius r.

An example of a function φ satisfying (3.1)–(3.3) is given by (see [3])

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \exp\left(1 - \frac{1}{1-(1-|x|)^4}\right) & \text{if } 1 < |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Define the functions

$$F_R(t) = \int_{\mathbb{R}^N} u(t,x)\varphi_R(x)dx,$$

$$G_R(t) = \int_{\mathbb{R}^N} |x|^\alpha\, |u(t,x)|^p\, \varphi_R(x)dx,$$

$$H_R(t) = \int_{\mathbb{R}^N} |u(t,x)|\varphi_R(x)dx,$$

$$K_R(t) = \int_{t_0}^{t} H_R^p(s)ds,$$

where $\varphi_R(x) = \varphi(\frac{x}{R})$, $R > 0$ and $t_0 > 0$. The following propositions summarize the main properties of functions F_R, G_R, H_R, K_R that will be crucial in our proofs.

Proposition 3.1. *For $R > 0$ we have*

$$F_R'(t) \geq G_R(t) - C\, R^{-2+\frac{N}{p}(p-p_*(N,\alpha))}\, G_R^{\frac{1}{p}}(t) + \mathsf{a}(t) \int_{\mathbb{R}^N} \mathbf{w}(x)\varphi_R(x)\, dx, \tag{3.4}$$

where $p_*(N,\alpha)$ is given by (2.2)*.*

Proof. Multiply the first equation in (1.1) by φ_R and making an integration by parts, we get

$$F_R'(t) = \int_{\mathbb{R}^N} u\Delta\varphi_R dx + \int_{\mathbb{R}^N} |x|^\alpha |u|^p \varphi_R dx + \mathsf{a}(t) \int_{\mathbb{R}^N} \mathbf{w}(x)\varphi_R dx.$$

Using Hölder's inequality, we infer

$$\begin{aligned} \Big| \int_{\mathbb{R}^N} u\Delta\varphi_R dx \Big| &\leq \Big(\int_{\mathbb{R}^N} |x|^\alpha |u|^p \varphi_R\, dx\Big)^{\frac{1}{p}} \Big(\int_{R\leq|x|\leq 2R} |x|^{-\frac{\alpha q}{p}}\, \varphi_R^{-\frac{q}{p}}\, |\Delta\varphi_R|^q\, dx\Big)^{\frac{1}{q}} \\ &\leq \big(G_R(t)\big)^{\frac{1}{p}} \Big(\int_{R\leq|x|\leq 2R} |x|^{-\frac{\alpha q}{p}}\, \varphi_R^{-\frac{q}{p}}\, |\Delta\varphi_R|^q\, dx\Big)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

From (3.3), one easily verifies that $\varphi_R^{-\frac{q}{p}}\, |\Delta\varphi_R|^q \leq C\, R^{-2q}$. Hence,

$$\int_{R\leq|x|\leq 2R} |x|^{-\frac{\alpha q}{p}}\, \varphi_R^{-\frac{q}{p}}\, |\Delta\varphi_R|^q\, dx \leq C\, R^{-2q}\, R^{N-\frac{\alpha\, q}{p}},$$

which in turn completes the proof of (3.4). □

Proposition 3.2. *For $R>0$, we have*

$$F_R'(t) \geqslant \mathtt{a}(t) \int\limits_{\mathbb{R}^N} \mathbf{w}(x)\varphi_R(x)dx - C\,R^{\frac{p(N-2)-(N+\alpha)}{p-1}}. \tag{3.5}$$

Proof. The proof of (3.5) immediately follows from (5.1) by choosing $Z = G_R(t)$, $\lambda = C\,R^{-2+\frac{N}{p}(p-p_*(N,\alpha))}$ and $\theta = \frac{1}{p}$. Note that

$$\frac{1}{1-\theta}\left(-2+\frac{N}{p}(p-p_*(N,\alpha))\right) = \frac{p(N-2)-(N+\alpha)}{p-1},$$

where $p_*(N,\alpha)$ is given by (2.2). □

Proposition 3.3. *There exist $R_0>0$ and $\delta>0$ such that, for all $R \geq R_0$, one has*

$$F_R'(t) \geqslant \delta\mathtt{a}(t) - C\,R^{\frac{p(N-2)-(N+\alpha)}{p-1}}. \tag{3.6}$$

Proof. Taking into consideration $\mathbf{w} \in L^1(\mathbb{R}^N)$, one obtains thanks to Lebesgue theorem $\int_{\mathbb{R}^N} \mathbf{w}(x)\varphi_R(x)\,dx \to \int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0$ as $R\to\infty$. This obviously leads to (3.6). □

Proposition 3.4. *Let $R \geq R_0$ where R_0 is as in Proposition 3.3. Suppose either*

$$\ell = \infty, \tag{3.7}$$

or

$$\ell \in (0,\infty) \quad \textit{and} \quad p < p^*(N,\alpha). \tag{3.8}$$

Then, we have

$$\lim_{t\to\infty} F_R(t) = \infty. \tag{3.9}$$

Proof. Integrating (3.6) with respect to t yields

$$F_R(t) \geq F_R(0) + t\left(\delta\mathtt{A}(t) - C\,R^{\frac{p(N-2)-(N+\alpha)}{p-1}}\right). \tag{3.10}$$

Observe that

$$\frac{p(N-2)-(N+\alpha)}{p-1} = \frac{N-2}{p-1}\left(p-p^*(N,\alpha)\right), \quad N\geq 3,$$

so that (3.9) follows from (3.10) under the assumption (3.8). The case (3.7) is easier. This finishes the proof of Proposition 3.4. □

Proposition 3.5. *Let $p>p_*(N,\alpha)$ and $R\geq R_0$ where R_0 is as in Proposition 3.3. Then, we have*

$$F_R(t) \leqslant H_R(t) \leqslant CR^{\frac{N(p-1)-\alpha}{p}}\Big(G_R(t)\Big)^{\frac{1}{p}}. \tag{3.11}$$

In particular

$$\lim_{t\to\infty} G_R(t) = \infty. \tag{3.12}$$

Proof. Clearly $F_R(t) \leqslant H_R(t)$. Next, by invoking Hölder's inequality together with (3.1), we get

$$\begin{aligned} H_R(t) &= \int\limits_{\mathbb{R}^N} \big(|u(t,x)||x|^{\frac{\alpha}{p}} \varphi_R^{\frac{1}{p}}(x)\big)\big(|x|^{-\frac{\alpha}{p}} \varphi_R^{1-\frac{1}{p}}(x)\big)dx \\ &\leqslant \Bigg(\int\limits_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} \varphi_R(x)dx\Bigg)^{1-\frac{1}{p}} \Bigg(\int\limits_{\mathbb{R}^N} |u(t,x)|^p |x|^{\alpha} \varphi_R(x)dx\Bigg)^{\frac{1}{p}} \\ &\leqslant \Bigg(\int\limits_{\{|x|\leq 2R\}} |x|^{-\frac{\alpha}{p-1}}\, dx\Bigg)^{1-\frac{1}{p}} \Big(G_R(t)\Big)^{\frac{1}{p}} \\ &\leqslant C R^{\frac{N(p-1)-\alpha}{p}} \Big(G_R(t)\Big)^{\frac{1}{p}}. \end{aligned}$$

The proof of Proposition 3.5 is now complete. □

Proposition 3.6. *There exists $t_0 > 0$ such that, for all $t \geqslant t_0$ and $R \geqslant R_0$,*

$$F_R'(t) \geqslant \delta\, \mathsf{a}(t) + \frac{1}{2} G_R(t), \tag{3.13}$$

where R_0 is as in Proposition 3.3.

Proof. Owing to (3.4) and granted (3.12), one can write

$$\begin{aligned} F_R'(t) &\geq \delta \mathsf{a}(t) + G_R(t) - C\, R^{-2+\frac{N}{p}(p-p_*(N,\alpha))}\, G_R^{\frac{1}{p}}(t) \\ &\geq \delta \mathsf{a}(t) + G_R(t)\Big[1 - C\, R^{-2+\frac{N}{p}(p-p_*(N,\alpha))}\, G_R^{\frac{1}{p}-1}(t)\Big] \\ &\geq \delta\, \mathsf{a}(t) + \frac{1}{2} G_R(t), \end{aligned}$$

for $t \geq t_0 > 0$ large enough. This finishes the proof. □

Proposition 3.7. *For $t \geqslant t_0$ and $R \geqslant R_0$, we have*

$$K_R'(t) \geqslant C\, R^{p(\alpha - N(p-1))} \Big(K_R(t)\Big)^p. \tag{3.14}$$

Proof. Using (3.13) and (3.11), we find that

$$F_R'(t) \geqslant \delta \mathsf{a}(t) + C\, R^{\alpha - N(p-1)}\, H_R^p(t). \tag{3.15}$$

Upon integrating (3.15) on $[t_0, t]$, we obtain

$$F_R(t) \geqslant F_R(t_0) + \delta \int\limits_{t_0}^{t} \mathsf{a}(s)ds + C\, R^{\alpha-N(p-1)}\, K_R(t) \geqslant C\, R^{\alpha-N(p-1)}\, K_R(t).$$

Recalling (3.11) yields (3.14) as desired. □

End of proofs of Theorems 2.4 and 2.8. Suppose that assumptions of Theorem 2.4 (respectively Theorem 2.8) are fulfilled. Since the differential inequality (3.14) blows up in finite time, we deduce that the solution of (1.1) cannot be global in time. This completes the proofs. □

4. PROOF OF THEOREM 2.8

We will focus in this section on blow-up results stated in Theorems 2.8. Suppose that the maximal solution u of (1.1) is global in time. In order to obtain a contradiction we use the so-called test function method [3, 11, 19, 24]. Pick two cut-off functions $f, g \in C^\infty([0,\infty))$ such that $0 \le f, g \le 1$,

$$f(\tau) = \begin{cases} 1 & \text{if } 1/2 \le \tau \le 2/3, \\ 0 & \text{if } \tau \in [0,1/4] \cup [3/4,\infty), \end{cases} \tag{4.1}$$

and

$$g(\tau) = \begin{cases} 1 & \text{if } 0 \le \tau \le 1, \\ 0 & \text{if } \tau \ge 2. \end{cases} \tag{4.2}$$

For $T, R > 0$, we introduce $\psi_{T,R}(t,x) = f_T(t)\, g_R(x)$, where

$$f_T(t) = \left(f\left(\frac{t}{T}\right)\right)^{\frac{p}{p-1}} \quad \text{and} \quad g_R(x) = \left(g\left(\frac{|x|^2}{R^2}\right)\right)^{\frac{2p}{p-1}}.$$

Multiplying both sides of the differential equation in (1.1) by $\psi_{T,R}$ and integrating over $(0,T) \times \mathbb{R}^N$ yields

$$\int_0^T \int_{\mathbb{R}^N} |x|^\alpha\, |u|^p\, \psi_{T,R}\, dx\, dt + \int_0^T \int_{\mathbb{R}^N} \mathtt{a}(t)\mathbf{w}(x)\, \psi_{T,R}\, dx\, dt + \int_{\mathbb{R}^N} u_0(x)\psi_{T,R}(0,x)\, dx$$
$$= -\int_0^T \int_{\mathbb{R}^N} u\, \Delta\psi_{T,R}\, dx\, dt - \int_0^T \int_{\mathbb{R}^N} u\, \partial_t\psi_{T,R}\, dx\, dt.$$

Granted to $\psi_{T,R}(0,x) = 0$, we obtain that

$$(I) + (II) \le (III) + (IV),$$

where

$$(I) = \int_0^T \int_{\mathbb{R}^N} |x|^\alpha \, |u|^p \, f_T(t) g_R(x) \, dx \, dt,$$

$$(II) = \int_0^T \int_{\mathbb{R}^N} \mathsf{a}(t)\mathbf{w}(x) \, f_T(t) g_R(x) \, dx \, dt,$$

$$(III) = \int_0^T \int_{\mathbb{R}^N} |u| \, f_T(t) |\Delta \, g_R(x)| \, dx \, dt,$$

$$(IV) = \int_0^T \int_{\mathbb{R}^N} |u| \, g_R(x) |\partial_t \, f_T(t)| \, dx \, dt.$$

Next, applying Young's inequality as in [19], we infer

$$(III) \leq \frac{1}{2}(I) + \mathbf{A}(T,R),$$
$$(IV) \leq \frac{1}{2}(I) + \mathbf{B}(T,R),$$

where

$$\mathbf{A}(T,R) = C \int_0^T \int_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} f_T(t) g_R^{-\frac{1}{p-1}}(x) |\Delta g_R(x)|^{\frac{p}{p-1}} \, dx \, dt,$$

$$\mathbf{B}(T,R) = C \int_0^T \int_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} f_T^{-\frac{1}{p-1}}(t) g_R(x) |\partial_t f_T(t)|^{\frac{p}{p-1}} \, dx \, dt,$$

where we have used (4.1), (4.2) and the fact that

$$|\Delta \, g_R(x)| \lesssim R^{-2} \, g_R^{1/p}(x).$$

Arguing as in [19], we find that

$$\mathbf{A}(T,R) \lesssim TR^{N-\frac{2p+\alpha}{p-1}},$$
$$\mathbf{B}(T,R) \lesssim T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}},$$

provided that $p > p_*(N, \alpha)$. Plugging all estimates above together, we find that

$$\int_0^T \int_{\mathbb{R}^N} \mathsf{a}(t)\mathbf{w}(x) \, \psi_{T,R}(t,x) \, dx \, dt \lesssim TR^{N-\frac{2p}{p-1}-\frac{\alpha}{(p-1)}} + T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{(p-1)}}.$$

Now, since $\mathbf{w} \in L^1(\mathbb{R}^N)$ and $g_R(x) \to g(0) = 1$ as $R \to \infty$, we obtain by the Lebesgue theorem that

$$\int_{\mathbb{R}^N} \mathbf{w}(x) g_R(x)\,dx \underset{R\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} \mathbf{w}(x)\,dx > 0.$$

It follows that there exists $R_0 > 0$ such that for all $R \geq R_0$ and $T > 0$, we have

$$\left(\int_0^T \mathsf{a}(t)\left(f\left(\frac{t}{T}\right)\right)^{\frac{p}{p-1}} dt\right)\left(\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx\right) \lesssim TR^{N-\frac{2p}{p-1}-\frac{\alpha}{(p-1)}} + T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}.$$

Owing to (4.1), the above estimate translates to

$$\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx \lesssim \frac{R^{N-\frac{2p+\alpha}{p-1}} + T^{-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}}{\frac{1}{T}\int_{\frac{T}{2}}^{\frac{2T}{3}} \mathsf{a}(t)dt}. \tag{4.3}$$

End of the proof of Theorem 2.8. Taking $R = \sqrt{T}$ in (4.3), we deduce that for large enough $T > 0$, one has

$$\int_{\mathbb{R}^N} \mathbf{w}(x)\,dx \lesssim \frac{1}{T^q \int_{\frac{T}{2}}^{\frac{2T}{3}} \mathsf{a}(t)dt}, \tag{4.4}$$

where $q = \frac{\alpha+2p}{2(p-1)} - 1 - \frac{N}{2}$.

If $\mathbf{J} = \mathbb{R}$, we obtain a contradiction by letting $T \to \infty$ in (4.4). This proves Part (i) in Theorem 2.8. Next, if $\mathbf{J} = (q_0, \infty)$ for some $q_0 \in \mathbb{R}$, and $\frac{\alpha+2p}{2(p-1)} - 1 - \frac{N}{2} > q_0$, we obtain a contradiction by letting $T \to \infty$ in (4.4). This proves Part (ii) in Theorem 2.8. This finishes the proof of Theorem 2.8. □

Remark 4.1. Note that we use in our proof (4.3) with $R = \sqrt{T}$, that is (4.4), instead of (4.3). Nevertheless, (4.3) is useful to give an alternative proof of Theorems 2.1–2.4.

5. APPENDIX

The following technical results are classical. For the sake of completeness, we give their proofs here.

Lemma 5.1. *Let $\lambda > 0$ and $\theta \in (0, 1)$. Then*

$$\min_{Z\geqslant 0}\left(Z - \lambda Z^{\theta}\right) = (\theta - 1)\,\theta^{\frac{\theta}{1-\theta}}\,\lambda^{\frac{1}{1-\theta}}. \tag{5.1}$$

Proof. Define $F(Z) = Z - \lambda Z^{\theta}$. Then F is differentiable on $(0,\infty)$ and $F'(Z) = 1-\theta\lambda\, Z^{\theta-1}$. It follows that F is decreasing on $(0, Z_0)$ and increasing on (Z_0,∞), where $Z_0 = (\theta\lambda)^{\frac{1}{1-\theta}}$. Consequently, F achieves its minimum at $Z = Z_0$, that is

$$\min_{Z\geq 0} F(Z) = F(Z_0) = (\theta\lambda)^{\frac{1}{1-\theta}} - \lambda\,(\theta\lambda)^{\frac{\theta}{1-\theta}} = (\theta-1)\,\theta^{\frac{\theta}{1-\theta}}\,\lambda^{\frac{1}{1-\theta}}.$$

The proof of Lemma 5.1 is now complete. □

Lemma 5.2. *Let $y : [t_0, T) \to (0,\infty)$ be a differentiable function such that*

$$y'(t) \geqslant C\, y^p(t), \quad t_0 \leq t < T, \tag{5.2}$$

where $C > 0$ and $p > 1$. Then

$$T \leq t_0 + \frac{y^{1-p}(t_0)}{C(p-1)} < \infty.$$

Proof. From (5.2) we get

$$\frac{d}{dt}\left[\frac{y^{1-p}(t)}{1-p}\right] = y'(t)y^{-p}(t) \geq C. \tag{5.3}$$

By integrating (5.3) in time, we infer

$$\frac{y^{1-p}(t_0)}{p-1} - \frac{y^{1-p}(t)}{p-1} \geq C(t-t_0), \quad t_0 \leq t < T.$$

Keeping in mind that $y(t) > 0$ and $p > 1$, we obtain that

$$C(t-t_0) \leq \frac{y^{1-p}(t_0)}{p-1}.$$

Therefore,

$$t \leq t_0 + \frac{y^{1-p}(t_0)}{C(p-1)}.$$

Since $t_0 \leq t < T$ is arbitrary, we get

$$T \leq t_0 + \frac{y^{1-p}(t_0)}{C(p-1)}.$$

This finishes the proof of Lemma 5.2. □

Lemma 5.3. *Let $g : [a,b] \to \mathbb{R}$ be a C^1–function and $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period $\vartheta > 0$. Then*

$$\lim_{\lambda\to\infty} \int_a^b g(s)\,\psi(\lambda\, s)\,ds = \left(\frac{1}{\vartheta}\int_0^{\vartheta} \psi(s)\,ds\right)\left(\int_a^b g(s)\,ds\right). \tag{5.4}$$

Proof. Define

$$\mathcal{I}(\lambda) = \int_a^b g(s)\,\psi(\lambda\, s)\,ds - \Big(\frac{1}{\vartheta}\int_0^\vartheta \psi(s)\,ds\Big)\Big(\int_a^b g(s)\,ds\Big).$$

Obviously, one can write

$$\mathcal{I}(\lambda) = \int_a^b g(s)\,f(\lambda\, s)\,ds,$$

where

$$f(s) = \psi(s) - \frac{1}{\vartheta}\int_0^\vartheta \psi(s)\,ds.$$

We also define

$$F(s) = \int_0^s f(\tau)\,d\tau.$$

Note that f is ϑ-periodic and $\int_0^\vartheta f(\tau)\,d\tau = 0$. Hence, F is also ϑ-periodic and consequently bounded, that is

$$M = \sup_{s\in\mathbb{R}} |F(s)| < \infty. \tag{5.5}$$

By performing an integration by parts, we infer

$$\begin{aligned}\mathcal{I}(\lambda) &= \int_a^b g(s)\,f(\lambda\, s)\,ds = \int_a^b g(s)\,\frac{1}{\lambda}\frac{d}{ds}\left[F(\lambda\, s)\right]\,ds \\ &= \frac{1}{\lambda}\Big(g(b)F(\lambda\, b) - g(a)F(\lambda\, a)\Big) - \frac{1}{\lambda}\int_a^b g'(s)\,F(\lambda\, s)\,ds.\end{aligned}$$

Owing to (5.5), we deduce that

$$|\mathcal{I}(\lambda)| \le \frac{M}{\lambda}\Big(|g(b)| + |g(a)| + \int_a^b |g'(s)|\,ds\Big). \tag{5.6}$$

From (5.6) we easily deduce that $\mathcal{I}(\lambda) \to 0$ as $\lambda \to \infty$ which is exactly the desired conclusion (5.4). This ends the proof of Lemma 5.3. □

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