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REGULARITY AND EXISTENCE OF SOLUTIONS TO PARABOLIC EQUATIONS WITH NONSTANDARD $p(x,t), q(x,t)$-GROWTH CONDITIONS

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Abstract. We study the Cauchy–Dirichlet problem for a class of nonlinear parabolic equations driven by nonstandard $p(x,t), q(x,t)$-growth condition. We prove theorems of existence and uniqueness of weak solutions in suitable Orlicz–Sobolev spaces, derive global and local in time L^∞ bounds for the weak solutions.

Keywords: existence theory, nonlinear parabolic problems, nonstandard growth, regularity theory.

Mathematics Subject Classification: 35K55, 35K65.

1. INTRODUCTION

In this paper, we are concerned with the existence and regularity properties of the following Cauchy–Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} A(x,t,\nabla u) & \\ = -\operatorname{div}\left(|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\right) & \text{in } \Omega_T, \\ u = g & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = g(\cdot,0) & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T)$, Ω an open, bounded Lipschitz domain in $\mathbb{R}^N$ of dimension $N \geq 2$ and $0 < T < \infty$. Throughout the paper we assume that the functions $A : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are such that $A(\cdot,\cdot,\zeta)$ are Lebesgue measurable for all $\zeta \in \mathbb{R}$ and $A(x,t,\cdot)$ are continuous for almost $(x,t) \in \Omega_T$. We also assume that the following structure conditions are satisfied

$$A(x,t,\zeta)\zeta \geq C_1\left(|\zeta|^{p(x,t)} + a(x,t)|\zeta|^{q(x,t)}\right), \tag{1.2}$$

$$|A(x,t,\zeta)| \leq C_2\left(|\zeta|^{p(x,t)-1} + a(x,t)|\zeta|^{q(x,t)-1}\right), \tag{1.3}$$

where C_1 and C_2 are positive constants.

The modulating coefficient $a(\cdot,\cdot)$ is assumed to satisfy the following

$$\exists\alpha_1,\alpha_2\in\mathbb{R}^+ : \alpha_1\leq a(x,t)\leq\alpha_2. \tag{1.4}$$

In addition, we suppose that

$$\left(A(x,t,\zeta)-A(x,t,\eta)\right)(\zeta-\eta)>0,\quad \zeta\neq\eta. \tag{1.5}$$

The exponents p and q are measurable functions in Ω_T satisfying the following conditions

$$\begin{cases} \frac{2N}{N+2}<p^-=\operatorname*{ess\,inf}_{\Omega_T} p(x,t)\leq p(x,t)\leq p^+=\operatorname*{ess\,sup}_{\Omega_T} p(x,t)<\infty,\\ |p(x_1,t_1)-p(x_2,t_2)|\leq\omega(d_{\mathfrak{p}}((x_1,t_1),(x_2,t_2))), \end{cases} \tag{1.6}$$

and

$$\begin{cases} \frac{2N}{N+2}<q^-=\operatorname*{ess\,inf}_{\Omega_T} q(x,t)\leq q(x,t)\leq q^+=\operatorname*{ess\,sup}_{\Omega_T} q(x,t)<\infty,\\ |q(x_1,t_1)-q(x_2,t_2)|\leq\omega(d_{\mathfrak{p}}((x_1,t_1),(x_2,t_2))), \end{cases} \tag{1.7}$$

where $\omega:[0,\infty]\to[0,1]$ denotes a modulus of continuity. More precisely, we shall assume that $\omega(\cdot)$ is a concave non-decreasing function with $\lim_{\rho\to 0}\omega(\rho)=\omega(0)=0$. Moreover, the parabolic distance is given by

$$d_{\mathfrak{p}}((x_1,t_1),(x_2,t_2))=\max\left\{|x_1-x_2|,\sqrt{|t_1-t_2|}\right\}.$$

In addition, for the modulus of continuity $\omega(\cdot)$ we assume the following weak logarithmic continuity condition

$$\limsup_{\rho\to 0}\,\omega(\rho)\log\left(\frac{1}{\rho}\right)<+\infty. \tag{1.8}$$

Equations of the type (1.1) where the modulating coefficient $a(x,t)$ could be degenerate on a set of zero measure are often called the double phase problems. The study of the double-phase problems started in the late 80th with the works of Zhikov in a series of remarkable papers [23–25] who introduced such classes of operators to describe models of strongly anisotropic materials by treating the Euler–Lagrange equation of the functional

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a^i(x,\nabla u)=b(x),\quad \text{a.e. } x\in\Omega, \tag{1.9}$$

where a^i satisfy some nonstandard growth conditions like, for example,

$$\sum_{ij}a^i_{\zeta_j}(x,\zeta)\lambda_i\lambda_j\geq m\left(1+|\zeta|^2\right)^{\frac{p-2}{2}}|\lambda|^2,\quad \forall\zeta,\lambda\in\mathbb{R}^N,\text{a.e. } x\in\Omega,$$

$$|a^i_{\zeta_j}(x,\zeta)|\leq M\left(1+|\zeta|^2\right)^{\frac{q-2}{2}},\quad \forall\zeta\in\mathbb{R}^N,\text{a.e. } x\in\Omega,\ \forall i,j,$$

for some positive constants m, M, and for exponents $q \geq p \geq 2$. Integral functionals of the form (1.9) have been considered by several authors concerning regularity results and non-standard growth, see for example, Baroni–Colombo–Mingione [6], Filippis–Mingione [10], Ragusa–Tachikawa [18] and the references therein. Later on, Marcellini [17] and Bögelein–Duzaar–Marcellini [8] give some existence results of a class of parabolic equations of the type

$$u_t = \mathrm{div} D_\zeta f(x, u, \nabla u) - D_u f(x, u, \nabla u), \tag{1.10}$$

with a convex integrand $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfying the p, q-growth condition of the type

$$\nu |\zeta|^p \leq f(x, u, \zeta) \leq M(1 + |u|^q + |\zeta|^q),$$

with some positive constants ν, M. It is worth mentioning that Arora–Shmarev [5] recently studied the variable exponent case of (1.10). In the parabolic setting, the case $p = q$ is well understood. Whereas, existence and regularity results for parabolic systems with p-growth in the cases of constant or variable exponents can be found in [13, 14, 20] and references therein.

In this framework, a particularly relevant class of interest is given by equations where the modulating coefficient $a(\cdot, \cdot)$ is bounded away from zero and thus the growth of the flux is controlled by operators with distinct exponents. Those types of equations are of a (p, q)-phase which is a special case of the double phase problems. Such equations arise in many mathematical models of physical processes. An important example where equation (1.1) arises is the study of the following nonlinear Schrödinger equation

$$i\psi_t = -\Delta \psi + q(x)\psi - \lambda f(x)|\psi|^{\gamma - 2}\psi - \Delta_q \psi + W'(x, \psi),$$

where $\Delta_q \psi = \mathrm{div}(|\nabla \psi|^{q-2} \nabla \psi)$ is a q-Laplacian. This class of equations was introduced by Derrick [11] and later by Benci-D'Avenia-Fortunato-Pisani [7] for the elliptic case, where $(p, 2)$-equations were used as a model for elementary particles in order to produce soliton-type solutions. We also mention the works of Cherfils–Il'yasov [9], where the authors studied the steady state solutions of reaction-diffusion systems, and of Zhikov [23] who studied problems related to nonlinear elasticity theory. It is worth noting that the existence and the regularity properties of the elliptic case of (1.1) has been studied by Ambrosio–Rădulescu [2], Zhang–Rădulescu [22] and references therein. Moreover, the boundedness of the solutions to (1.1) with the homogeneous Dirichlet boundary conditions can be derived from Theorem 1 in [13].

The aim of the present paper is also to develop a variational approach in the parabolic setting in the spirit of papers [4, 15, 16, 21]. The approach we used to prove the existence and the uniqueness of (1.1) is to construct a family of solutions by Galerkin approximation, which solves the homogeneous case of (1.1). Next, we deduce some energy bounds. These estimates with the compact embedding yield the desired convergence of the approximate solutions to general solutions. Afterward, by using the well known Moser's iteration technique which is essentially based on a combination of a Sobolev and a Caccioppoli type inequalities, the question of local boundedness of the solution to (1.1) is proved. Finally, for sufficiently regular data and by using the approach presented in [4], we derive the global boundedness of the weak solution to (1.1).

2. PRELIMINARY AND MAIN RESULTS

2.1. THE FUNCTION SPACES

We collect here the background information on the variable Lebesgue and Sobolev spaces used throughout the paper. We refer to the monograph [3] for further information.

Let Ω be a bounded domain with Lipschitz-continuous boundary $p : \Omega \to [p^-, p^+] \subset (1, +\infty)$ be a measurable function. The set

$$L^{p(x)}(\Omega) = \left\{ u \,:\, u \text{ is a measurable real-valued function }, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 \,:\, \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \le 1 \right\}.$$

is a reflexive and separable Banach space and $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$. The norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ can be estimated as follows:

$$-1 + \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\, dx \le 1 + \|u\|^{p^+}_{L^{p(x)}(\Omega)}, \quad \forall u \in L^{p(x)}(\Omega). \tag{2.1}$$

Moreover, if $p_1(x) \ge p_2(x)$ a.e. in Ω, then $L^{p_1(x)}(\Omega)$ is continuously embedded in $L^{p_2(x)}(\Omega)$ and

$$\|u\|_{L^{p_2(x)}(\Omega)} \le C\|u\|_{L^{p_1(x)}(\Omega)}, \quad \forall u \in L^{p_1(x)}(\Omega).$$

Let $W^{1,p(x)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(x)}(\Omega)$. The norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$$

makes $W^{1,p(x)}(\Omega)$ a Banach space. It is well known that if $p(x)$ satisfies the log-Hölder condition $(1.6)_2$, then $C^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$. Moreover, we can define the Sobolev space with zero boundary value $W_0^{1,p(x)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$, with respect to the norm of $W^{1,p(x)}(\Omega)$. It is known that for the elements of $W_0^{1,p(x)}(\Omega)$, the Poincaré inequality holds

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)},$$

and an equivalent norm of $W_0^{1,p(x)}(\Omega)$ can be defined by

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

For the study of parabolic problem (1.1), we need the spaces of functions depending on $(x,t) \in \Omega_T$. With a slight abuse of the notation, we consider more general nonstandard parabolic Sobolev. Then, by $W_g^{p(x,t)}(\Omega_T)$ we denote the Banach space

$$W_g^{p(x,t)}(\Omega_T) := \left\{ u \in L^{p(x,t)}(\Omega_T) : \nabla u \in L^{p(x,t)}(\Omega_T), u - g \in L^1(0,T; W_0^{1,1}(\Omega)) \right\}$$

equipped by the norm

$$\|u\|_{W^{p(x,t)}(\Omega_T)} := \|u\|_{L^{p(x,t)}(\Omega_T)} + \|\nabla u\|_{L^{p(x,t)}(\Omega_T)}.$$

If $g = 0$ we write $W_0^{p(x,t)}(\Omega_T)$ instead of $W_g^{p(x,t)}(\Omega_T)$. $W^{p(x,t)}(\Omega_T)'$ is the dual of $W_0^{p(x,t)}(\Omega_T)$ such that

$$\omega \in W^{p(x,t)}(\Omega_T)' \Leftrightarrow \begin{cases} \exists(\omega_0, \dots, \omega_N),\ \omega_0 \in L^{p'(x,t)}(\Omega_T),\ \omega_i \in L^{p'(x,t)}(\Omega_T), \\ \forall \phi \in W_0^{p(x,t)}(\Omega_T), \\ \langle \omega, \phi \rangle = \int\limits_{\Omega} \left(\omega_0 \phi + \sum\limits_{i=1}^{N} \omega_i D_i \phi \right) dxdt. \end{cases}$$

Let us now define

$$W(\Omega_T) := \left\{ \omega \in W^{p(x,t)}(\Omega_T) \,:\, \omega_t \in W^{p(x,t)}(\Omega_T)' \right\},$$

such that if $\omega \in W(\Omega_T)$ then there exists $\omega_t \in W^{p(x,t)}(\Omega_T)'$ satisfying

$$< \omega_t, \varphi >= -\int\limits_{\Omega_T} \omega \varphi_t \; dxdt, \quad \forall \varphi \in C_0^\infty(\Omega_T).$$

The previous equality makes sense due to the inclusions

$$W^{p(x,t)}(\Omega_T) \hookrightarrow L^2(\Omega_T) \cong (L^2(\Omega_T))' \hookrightarrow W^{p(x,t)}(\Omega_T)',$$

which allow us to identify ω as an element of $W^{p(x,t)}(\Omega_T)'$. For more results about spaces $W^{p(x,t)}(\Omega_T)$ and $W^{p(x,t)}(\Omega_T)'$ see for instance [4, 16] and references therein.

2.2. IMBEDDING AND TECHNICAL LEMMAS

To derive our existence and regularity results, we will need the following

Lemma 2.1 ([16, Lemma 2.3]). *Assume that $u \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(x,t)}(\Omega_T)$ and the exponent p satisfies the conditions* (1.6) *and* (1.8). *Then there exists a constant $C = C(N, p^-, p^+, \mathrm{diam}(\Omega))$, such that the following estimate*

$$\|u\|^{p^-}_{L^{p(x,t)}(\Omega_T)} \leq C \left(\|u\|^{\frac{4p^+}{N+2}}_{L^\infty(0,T;L^2(\Omega))} + 1 \right) \left(\int\limits_{\Omega_T} |\nabla u|^{p(x,t)} + 1 \; dxdt \right) \tag{2.2}$$

holds.

Theorem 2.2 ([16, Theorem 1.3]). *Let $\Omega \subset \mathbb{R}^N$ an open, bounded Lipschitz domain with $N \geq 2$ and $p(x,t) > \frac{2N}{N+2}$ satisfying* (1.6) *and* (1.8)*. Furthermore, define*

$$\hat{p}(x,t) = \max\{2, p(x,t)\}.$$

Then the inclusion $W(\Omega_T) \hookrightarrow L^{\hat{p}(x,t)}(\Omega_T)$ is compact.

Proposition 2.3. *If $u \in L^\infty(0,T; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$ and $p > \frac{2N}{N+2}$, then we have the following estimation*

$$\begin{aligned} &\left(\int\limits_{\Omega_T} |u(x,t)|^l \, dxdt \right)^{\frac{N}{N+p}} \\ &\leq C \left\{ \operatorname*{ess\,sup}_{t \in (0,T)} \int\limits_{\Omega} |u(x,t)|^2 \, dxdt + \int\limits_{\Omega_T} |\nabla u(x,t)|^p \, dxdt \right\}, \end{aligned} \tag{2.3}$$

where $l = p\frac{N+2}{N}$.

Proof. Clearly $l > 2$ for $p > \frac{2N}{N+2}$. Then, by using Proposition 3.1 in Chapter I of [12], we get

$$\int\limits_{\Omega_T} |u(x,t)|^l \, dxdt \leq C \left(\operatorname*{ess\,sup}_{t \in (0,T)} \int\limits_{\Omega} |u(x,t)|^2 \, dxdt \right)^{\frac{p}{N}} \int\limits_{\Omega_T} |\nabla u(x,t)|^p \, dxdt.$$

Therefore, by applying Young's inequality we get that

$$\left(\int\limits_{\Omega_T} |u(x,t)|^l \, dxdt \right)^{\frac{N}{N+p}} \leq C \left\{ \operatorname*{ess\,sup}_{t \in (0,T)} \int\limits_{\Omega} |u(x,t)|^2 \, dxdt + \int\limits_{\Omega_T} |\nabla u(x,t)|^p \, dxdt \right\}.$$

This completes the proof. □

2.3. MOLLIFICATION IN TIME

It would be technically convenient to have at hand a formulation of weak solution involving the time derivative u_t. Unfortunately, solutions of (1.1), whenever they exist, possess a modest degree of time regularity, and, in general, u_t has a meaning only in the sense of distributions. In order to be nevertheless able to test properly, there are several possibilities to smooth the solution with respect to the time direction. To overcome these faculties, we consider the Friedrichs mollifier as was done in [1]. Indeed, taking the kernel

$$\rho \geq 0, \quad \rho \in C_0^\infty(\mathbb{R}^N), \quad \rho(x) \equiv 0 \quad \text{for} \quad |x| \geq 1, \quad \int\limits_{\mathbb{R}^N} \rho(x) \, dx = 1,$$

we introduce regularization of $f \in L^1(\mathbb{R}^{N+1})$ by

$$\begin{aligned} I^h f = f_h(x,t) = h^{-1} \int_t^{t+h} \int_{\mathbb{R}^N} f(y,\tau)\rho_h(x-y)\, dy d\tau, \\ \rho_h(x) = h^{-N}\rho(h^{-1}x) \end{aligned} \tag{2.4}$$

The basic property of the mollification (2.4), which can be retrieved from [1, Lemma 3.1], is summarized in the following:

Lemma 2.4. *If the exponent p satisfies the conditions* (1.6) *and* (1.8), *then $f_h \to f$ in $L^{p(x,t)}(\Omega_T)$ as $h \to 0$, for any $f \in L^{p(x,t)}(\Omega_T)$.*

2.4. FORMULATION OF THE PROBLEM AND MAIN RESULTS

We consider a space-time cylinder $\Omega_T \equiv \Omega \times (0,T)$, where $\Omega \in \mathbb{R}^N$ is a bounded domain with $N \geq 2$. On the lateral boundary $\partial\Omega \times (0,T)$, we consider the Cauchy–Dirichlet boundary data given by

$$\begin{cases} g \in C^0([0,T]; L^2(\Omega)) \cap W^{p(x,t)}(\Omega_T) \cap W^{q(x,t)}(\Omega_T), \\ \partial_t g \in L^{(p^-)'}(0,T; W^{-1,(p^-)'}(\Omega)). \end{cases} \tag{2.5}$$

As for the right-hand side of (1.1), we assume that

$$F \in L^{p(x,t)}(\Omega_T) \cap L^{q(x,t)}(\Omega_T). \tag{2.6}$$

In the following, we describe the concept of weak solutions to Cauchy–Dirichlet problems as for instance those considered in (1.1).

Definition 2.5. *Assume that g and F fulfill* (2.5) *and* (2.6). *We define a measurable map $u : \Omega_T \to \mathbb{R}$ in the class*

$$u \in C^0([0,T]; L^2(\Omega)) \cap W_g^{p(x,t)}(\Omega_T) \cap W_g^{q(x,t)}(\Omega_T) \quad \text{with } \partial_t u \in W^{s(x,t)}(\Omega_T)',$$

where $s(x,t) = \max\{p(x,t), q(x,t)\}$ as a weak solution to the parabolic double phase associated to (1.1) *if and only if the variational equality*

$$\begin{aligned} &\left[\int_\Omega u\phi dx \right]_0^T + \int_{\Omega_T} -u\phi_t + A(x,t,\nabla u).\nabla\phi \, dxdt \\ &= \int_{\Omega_T} \left[|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F \right] \nabla\phi \, dxdt \end{aligned} \tag{2.7}$$

holds for every test function $\phi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ with $\partial_t\phi \in W^{s(x,t)}(\Omega_T)'$.

We can write (2.7) in a way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Friedrichs mollifier of a function. Then, we get the following lemma.

Lemma 2.6. *If u is a solution of equation* (1.1) *in the sense of Definition 2.5, then for u_h defined in* (2.4)*, and for any $0 \leq t_1 \leq t_2 \leq T$, the following relation*

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega}\Big\{ & u_{h,t}\varphi + [A(x,t,\nabla u)]_h \cdot\nabla\varphi \\ & - \Big[|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\Big]\nabla\varphi\Big\}\, dxdt = 0,\end{aligned} \tag{2.8}$$

holds for any tested function $\varphi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ with $\partial_t\varphi \in W^{s(x,t)}(\Omega_T)'$.

Proof. We introduce the following regularization operator:

$$I^{-h}f = f_{-h}(x,t) = h^{-1}\int_{t-h}^{t}\int_{\mathbb{R}^N} f(y,\tau)\rho_h(x-y)\; dyd\tau.$$

Consider equation (2.7) with

$$\phi = I^{-h}(\varphi\chi), \quad \varphi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T) \quad \text{with } \partial_t\varphi \in W^{s(x,t)}(\Omega_T)'.$$

Since

$$-\int_{\Omega_T} u\frac{\partial I^{-h}(\varphi\chi)}{\partial t}\; dxdt = \int_{t_1}^{t_2}\int_{\Omega} u_{h,t}\varphi\chi\; dxdt,$$

it follows that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega}\Big[& u_{h,t}\varphi\chi + [A(x,t,\nabla u)]_h \cdot\nabla(\varphi\chi) \\ & - \Big[|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\Big]\nabla(\varphi\chi)\Big]\; dxdt = 0.\end{aligned}$$

Passing here from $\chi \in C_0^\infty(t_1,t_2)$ to characteristic function of the segment $[t_1,t_2]$, we obtain the desired relation (2.8). □

The main results are given in the following theorems.

Theorem 2.7. *Assume that g and F fulfill the assumptions* (2.5) *and* (2.6)*. Then, there exists a unique solution*

$$u \in C^0([0,T];L^2(\Omega)) \cap W_g^{p(x,t)}(\Omega_T) \cap W_g^{q(x,t)}(\Omega_T) \quad \textit{with } \partial_t u \in W^{s(x,t)}(\Omega_T)'$$

of the problem (1.1) *in the sense of Definition 2.5 such that*

$$\begin{aligned}&\sup_{0\le t\le T}\int_{\Omega}|u(\cdot,t)|^2\,dx+\int_{\Omega_T}|\nabla u|^{p(x,t)}\,dxdt+\int_{\Omega_T}|\nabla u|^{q(x,t)}\,dxdt\\&\le C\Big\{\|g(\cdot,0)\|^2_{L^2(\Omega)}+\|g\|_{L^\infty(0,T;L^2(\Omega))}+\|\partial_t g\|^{(p^-)'}_{L^{(p^-)'}(0,T;W^{-1,(p^-)'}(\Omega))}\\&\qquad+\int_{\Omega_T}|\nabla g|^{p(x,t)}+|\nabla g|^{q(x,t)}\,dxdt+\int_{\Omega_T}|F|^{p(x,t)}+|F|^{q(x,t)}\,dxdt\Big\}\end{aligned}$$

with a positive constant

$$C=C(N,\alpha_1,p^\pm,q^\pm,\mathrm{diam}(\Omega)).$$

Theorem 2.8. *Let the assumptions* (1.2)-(1.8), (2.5) *and* (2.6) *be satisfied. Let u be the solution to* (1.1) *in the sense of Definition 2.5. Then u is locally bounded in Ω_T.*

Theorem 2.9. *Let the conditions of Theorem 2.8 be fulfilled. Additionally, we assume that $g\in C^1([0,T];W^{1,\infty}(\Omega))$ and*

$$\left|\mathrm{div}(|F|^{p(x,t)-2}F+a(x,t)|F^{q(x,t)-2}F|)\right|\le h(x,t), \tag{2.9}$$

where h is a nonnegative function and $h\in L^1(0,T;L^\infty(\Omega))$. Then, if u is a weak solution to (1.1) *in the sense of Definition 2.5, u is globally bounded in Ω_T.*

3. EXISTENCE AND UNIQUENESS

In this section, we will establish some existence results to problem (1.1). These results will be used to prove our main existence Theorem. The starting point is to consider the following homogeneous case of equation (1.1).

$$\begin{cases}u_t-\mathrm{div}A(x,t,\nabla u)\\=-\mathrm{div}\left(|F|^{p(x,t)-2}F+a(x,t)|F|^{q(x,t)-2}F\right) & \text{in } \Omega_T,\\u=0 & \text{on } \partial\Omega\times(0,T),\\u(\cdot,0)=g(\cdot,0) & \text{on } \Omega\times\{0\}.\end{cases} \tag{3.1}$$

Furthermore, initial values $g(\cdot,0)\in L^2(\Omega)$ are given and the vector field $A(x,t,\nabla u)$ satisfies (1.3)–(1.5) and $F\in L^{p(x,t)}(\Omega_T)\cap L^{q(x,t)}(\Omega_T)$, also the exponents p and q are complied with (1.6)–(1.8). Then, we have the following lemma.

Lemma 3.1. *There exists at least one weak solution*

$$u \in C^0([0,T];L^2(\Omega)) \cap W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T) \quad \textit{with } \partial_t u \in W^{s(x,t)}(\Omega_T)'$$

to the equation (3.1) *in the sense of Definition 2.5 such that*

$$\begin{aligned} &\sup_{0\le t\le T} \int_\Omega |u(\cdot,t)|^2 \, dx + \int_{\Omega_T} |\nabla u|^{p(x,t)} \, dxdt + \int_{\Omega_T} |\nabla u|^{q(x,t)} \, dxdt \\ &\le C\Big\{ \|g(\cdot,0)\|^2_{L^2(\Omega)} + \int_{\Omega_T} |F|^{p(x,t)} + |F|^{q(x,t)} \, dxdt \Big\} \end{aligned} \tag{3.2}$$

with a positive constant

$$C = C(N, \alpha_1, p^\pm, q^\pm, \operatorname{diam}(\Omega))$$

and

$$s(x,t) = \max(p(x,t), q(x,t)).$$

Proof. Let $\{\phi_k\}_{i=1}^\infty \subset W_0^{1,p^+} \cap W_0^{1,q^+}$ be an orthonormal basis in $L^2(\Omega)$. Next, we fix a positive integer m and define the approximate solution to (3.1) as follows

$$u^m(x,t) = \sum_{i=1}^m C_i^{(m)}(t)\phi_i(x),$$

where the coefficients $C_i^{(m)}(t)$ are defined via the identity

$$\int_\Omega \Big\{ u_t^m \phi_i(x) + \Big[A(x,t,\nabla u^m) - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F \Big] \nabla\phi_i(x) \Big\} \, dx = 0 \tag{3.3}$$

for $i = 0, \dots, m$ and $t \in (0,T)$ with the initial condition

$$\begin{cases} (C_i^m)'(t) = F_i(t, C_1^{(m)}(t), \dots, C_m^{(m)}(t)), \\ C_i^{(m)}(0) = \int_\Omega g(\cdot,0)\phi_i(x) \, dx, \ i = 1, \dots, m, \end{cases} \tag{3.4}$$

where we abbreviated

$$\begin{aligned} F_i(t,\cdot) = -\int_\Omega \Big[&A(\cdot,t,\nabla u^{(m)}) \\ &- |F(\cdot,t)|^{p(\cdot,t)-2}F(\cdot,t) - a(x,t)|F(\cdot,t)|^{q(\cdot,t)-2}F(\cdot,t) \Big] \nabla\phi_i(x) \, dx, \end{aligned}$$

since $\{\phi_i(x)\}$ is orthonormal in $L^2(\Omega)$. Therefore, by Theorem 1.44 in [19], we assume that for every finite system (3.4), there exists a solution $C_i^{(m)}(t)$, $i = 1, \dots, m$, on the interval $(0, T_m)$ for some $T_m > 0$. Next, we multiply (3.3) by $C_i^{(m)}(t)$, we integrate the resulting equation over $(0,\tau)$ for an arbitrary $\tau \in (0,T_m)$, and we sum up the resulting equation over $i = 1,\ldots,m$. Then, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \int_{\Omega_\tau} u_t^{(m)}\phi_i(x)C_i^{(m)}(t) + \Big[A(x,t,\nabla u^{(m)}) \\ &\qquad - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big].\nabla\phi_i(x)C_i^{(m)}(t)\ dxdt \\ &= \int_{\Omega_\tau} u_t^{(m)}u^{(m)} + \Big[A(x,t,\nabla u^{(m)}) \\ &\qquad - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big].\nabla u^{(m)}\ dxdt. \end{aligned} \tag{3.5}$$

Since $g(\cdot,0) \in L^2(\Omega)$ and $\{\phi_i\}_{i=1}^{\infty} \in L^2(\Omega)$, we get the following estimate

$$\begin{aligned} &\int_\Omega |u^{(m)}(\cdot,0)|^2\ dx \\ &= \int_\Omega \left|\sum_{i=1}^{m} C_i^{(m)}(0)\phi_i(x)\right|^2\ dx = \int_\Omega \left|\sum_{i=1}^{m}\int_\Omega g(\cdot,0)\phi_i(x)\ dx\ \phi_i(x)\right|^2\ dx \\ &\le \int_\Omega \left|\sum_{i=1}^{\infty}\int_\Omega g(\cdot,0)\phi_i(x)\ dx\ \phi_i(x)\right|^2\ dx = \int_\Omega |g(\cdot,0)|^2\ dx, \end{aligned}$$

where we used the fact that

$$g(\cdot,0) = \sum_{i=1}^{\infty}\int_\Omega g(x,0)\phi_i(x)\ dx\ \phi_i(x),$$

since $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$. Therefore, the first term in the right-hand side of (3.5) reads

$$\begin{aligned} &\int_{\Omega_\tau} u_t^{(m)}u^{(m)}\ dxdt \\ &= \frac{1}{2}\int_{\Omega_\tau} \partial_t\left[u^{(m)}\right]^2\ dxdt = \frac{1}{2}\int_\Omega \left|u^{(m)}(\cdot,\tau)\right|^2\ dx - \frac{1}{2}\int_\Omega \left|u^{(m)}(\cdot,0)\right|^2\ dx \\ &\ge \frac{1}{2}\int_\Omega \left|u^{(m)}(\cdot,\tau)\right|^2\ dx - \frac{1}{2}\left\|g(\cdot,0)\right\|_{L^2(\Omega)}^2, \end{aligned}$$

for all $\tau \in (0,T_m)$.

Therefore, (3.5) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \; dx + \int_{\Omega_\tau} A(x,t,\nabla u^{(m)}).\nabla u^{(m)} \; dxdt \\
&\leq \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + \int_{\Omega_\tau} \left[|F|^{p(x,t)}F + a(x,t)|F|^{q(x,t)-2}F\right] \nabla u^{(m)} \; dxdt,
\end{aligned}
\tag{3.6}
$$

for all $\tau \in (0, T_m)$. Next, by using (1.2)–(1.4) on the left-hand side of (3.6) and estimating the right-hand side of (3.6) by the absolute value, we obtain the following

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \; dx + C_1\int_{\Omega_\tau} |\nabla u^{(m)}|^{p(x,t)} \; dxdt + \alpha_1 C_1 \int_{\Omega_\tau} |\nabla u^{(m)}|^{q(x,t)} \; dxdt \\
&\leq \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + \int_{\Omega_\tau} \frac{p(x,t)-1}{p(x,t)} \varepsilon^{\frac{1}{1-p(x,t)}} |F|^{p(x,t)} \; dxdt \\
&\quad + \int_{\Omega_\tau} \frac{1}{p(x,t)} \varepsilon |\nabla u^{(m)}|^{p(x,t)} \; dxdt + \int_{\Omega_\tau} \frac{q(x,t)-1}{q(x,t)} \varepsilon^{\frac{1}{1-q(x,t)}} |F|^{q(x,t)} \; dxdt \\
&\quad + \int_{\Omega_\tau} \frac{1}{q(x,t)} \varepsilon |\nabla u^{(m)}|^{q(x,t)} \; dxdt \\
&\leq \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + C\int_{\Omega_\tau} |F|^{p(x,t)} + |F|^{q(x,t)} \; dxdt \\
&\quad + \frac{\varepsilon}{p^-}\int_{\Omega_\tau} |\nabla u^{(m)}|^{p(x,t)} \; dxdt + \frac{\varepsilon}{q^-}\int_{\Omega_\tau} |\nabla u^{(m)}|^{q(x,t)} \; dxdt,
\end{aligned}
$$

for a.e. $\tau \in (0, T_m)$, where we used Young's inequality for $\varepsilon \in (0, 1)$. Then, we obtain

$$
\begin{aligned}
&\sup_{0\leq\tau\leq T_m} \int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \; dx + \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(x,t)} \; dxdt + \int_{\Omega_\tau} |\nabla u^{(m)}|^{q(x,t)} \; dxdt \\
&\leq \|g(\cdot,0)\|^2_{L^2(\Omega)} + C \int_{\Omega_{T_m}} |F|^{p(x,t)} + |F|^{q(x,t)} \; dxdt.
\end{aligned}
\tag{3.7}
$$

Therefore, $u^{(m)}$ is uniformly bounded in $L^\infty(0, T_m; L^2(\Omega))$ and $\nabla u^{(m)}$ is uniformly bounded in $L^{p(x,t)}(\Omega_{T_m}) \cap L^{q(x,t)}(\Omega_{T_m})$. Next, by using (2.2) and (3.7) we get the following estimate

$$
\begin{aligned}
&\|u^{(m)}\|_{L^{p(x,t)}(\Omega_{T_m})}\\
&\leq C\Bigg\{\bigg[\Big\|u^{(m)}\Big\|^{\frac{4p^+}{N+2}}_{L^\infty(0,T_m;L^2(\Omega))}+1\bigg]\|g(\cdot,0)\|^2_{L^2(\Omega)}\\
&\qquad +C\int\limits_{\Omega_{T_m}} 1+|F|^{p(x,t)}+|F|^{q(x,t)}\ dxdt\Bigg\}^{\frac{1}{p^-}}\\
&\leq C\Bigg\{\|g(\cdot,0)\|^2_{L^2(\Omega)}+\int\limits_{\Omega_{T_m}} 1+|F|^{p(x,t)}+|F|^{q(x,t)}\ dxdt\Bigg\}^{\frac{1}{p^-}\left(\frac{4p^+}{N+2}+1\right)}.
\end{aligned}
\tag{3.8}
$$

By the same method, we get also

$$
\begin{aligned}
&\|u^{(m)}\|_{L^{q(x,t)}(\Omega_{T_m})}\\
&\leq C\Bigg\{\|g(\cdot,0)\|^2_{L^2(\Omega)}+\int\limits_{\Omega_{T_m}} 1+|F|^{p(x,t)}+|F|^{q(x,t)}\ dxdt\Bigg\}^{\frac{1}{q^-}\left(\frac{4q^+}{N+2}+1\right)}.
\end{aligned}
\tag{3.9}
$$

Consequently, we have shown that $u^{(m)}$ is uniformly bounded in $W^{p(x,t)}(\Omega_{T_m})\cap W^{q(x,t)}(\Omega_{T_m})$ and $L^\infty(0,T_m;L^2(\Omega))$ independently of m. As a result, the solution of system (3.4) can be continued to the maximal interval $(0,T)$.

Our next aim is to derive the uniform boundedness of $u_t^{(m)}$ over $W^{s(x,t)}(\Omega_T)'$. For this reason, we introduce the following subspace

$$
W_m(\Omega_T)=\left\{\eta:\ \eta=\sum_{i=1}^m d_i\phi_i,\ d_i\in C^1([0,T])\right\}\subset W_0^{p(x,t)}(\Omega_T)\cap W_0^{q(x,t)}(\Omega_T).
$$

Next, we choose

$$
\varphi(x,t)=\sum_{i=1}^m d_i(t)\phi_i(x)\in W_m(\Omega_T)\quad \text{with } d_i(0)=d_i(T)=0
$$

as a test function in (3.3). Then, we get

$$
\begin{aligned}
&-\int\limits_{\Omega_T} u^{(m)}\varphi_t\ dxdt=\int\limits_{\Omega_T} u_t^m\varphi\ dxdt\\
&=-\int\limits_{\Omega_T}\Big[A(x,t,\nabla u^{(m)})-|F|^{p(x,t)-2}F-a(x,t)|F|^{q(x,t)-2}F\Big]\nabla\varphi\ dxdt.
\end{aligned}
$$

Note that $\partial_t\varphi$ exists since $d_i \in C([0,T])$. Then, we get the following estimate

$$
\begin{aligned}
&\left|\int\limits_{\Omega} u^{(m)}\varphi_t \ dxdt\right| \\
&\leq \int\limits_{\Omega_T} \left[|A(x,t,\nabla u^{(m)})| + |F|^{p(x,t)-1} + a(x,t)|F|^{q(x,t)-1}\right] |\nabla\varphi| \ dxdt \\
&\leq C\int\limits_{\Omega_T} \Big[|\nabla u^{(m)}|^{p(x,t)-1} + |\nabla u^{(m)}|^{q(x,t)-1} \\
&\qquad + |F|^{p(x,t)-1} + |F|^{q(x,t)-1}\Big] \left(|\nabla\varphi| + |\varphi|\right) \ dxdt \\
&\leq C\Big\{ \left\| |\nabla u^{(m)}|^{p(x,t)-1} + |F|^{p(x,t)-1} \right\|_{L^{p'(x,t)}(\Omega_T)} \|\varphi\|_{W^{1,p(x,t)}(\Omega_T)} \\
&\qquad + \left\| |\nabla u^{(m)}|^{q(x,t)-1} + |F|^{q(x,t)-1} \right\|_{L^{q'(x,t)}(\Omega_T)} \|\varphi\|_{W^{1,q(x,t)}(\Omega_T)} \Big\}.
\end{aligned}
\tag{3.10}
$$

Next, we are going to prove the boundedness of terms in the right-hand side of (3.1). Therefore, by using (2.1) and (3.7), we obtain

$$
\begin{aligned}
&\left\| |\nabla u^{(m)}|^{p(x,t)-1} + |F|^{p(x,t)-1} \right\|_{L^{p'(x,t)}(\Omega_T)} \\
&\leq C\left\{ \int\limits_{\Omega_T} |\nabla u^{(m)}|^{p(x,t)} \ dxdt + \int\limits_{\Omega_T} |F|^{p(x,t)} \ dxdt + 1 \right\}^{\frac{1}{p^-}} \\
&\leq C(N, p^+, p^-, \operatorname{diam}(\Omega)).
\end{aligned}
$$

Also, by the same method we get

$$
\left\| |\nabla u^{(m)}|^{q(x,t)-1} + |F|^{q(x,t)-1} \right\|_{L^{q'(x,t)}(\Omega_T)} \leq C(N, q^+, q^-, \operatorname{diam}(\Omega)).
$$

Thus, by combining all the previous estimates into (3.10), we obtain

$$
\left|\int\limits_{\Omega_T} u_t^{(m)}\varphi \ dxdt\right| \leq C(N, \alpha_1, p^{\pm}, q^{\pm}, \operatorname{diam}(\Omega))\|\varphi\|_{W^{s(x,t)}(\Omega_T)}
$$

where $s(x,t) = \max\{p(x,t), q(x,t)\}$. This shows that $u_t^{(m)} \in W^{s(x,t)}(\Omega_T)'$. Accordingly, by (3.7), (3.8), and (3.9) we get that

$$
\begin{cases}
u^{(m)} \in W_0^{p(x,t)}(\Omega_T) \subseteq L^{p^-}(0,T; W_0^{1,p^-}(\Omega)), \\
u^{(m)} \in W_0^{q(x,t)}(\Omega_T) \subseteq L^{q^-}(0,T; W_0^{1,q^-}(\Omega)), \\
u^{(m)} \in L^{\infty}(0,T; L^2(\Omega)), \\
u_t^{(m)} \in W^{s(x,t)}(\Omega_T)'.
\end{cases}
$$

Therefore, there exists a subsequence $\{u^{(m)}\}$ and a limit map u such that

$$\begin{cases} u^{(m)} \rightharpoonup^* u & \text{in } L^\infty(0,T;L^2(\Omega)), \\ \nabla u^{(m)} \rightharpoonup \nabla u & \text{in } L^{p(x,t)}(\Omega_T), \\ \nabla u^{(m)} \rightharpoonup \nabla u & \text{in } L^{q(x,t)}(\Omega_T), \\ u_t^{(m)} \rightharpoonup u_t & \text{in } W^{s(x,t)}(\Omega_T)'. \end{cases}$$

Consequently, by using Theorem 2.2, we get

$$\begin{cases} u^{(m)} \to u & \text{strongly in } L^{\hat{s}(x,t)}(\Omega_T), \\ u^{(m)} \to u & \text{a.e in } \Omega_T \end{cases}$$

with $\hat{s}(x,t) = \max\{2, s(x,t)\}$. Further, the growth condition (1.3) of $A(x,t,\cdot)$ and the energy estimate (3.7) imply that sequence $\{A(x,t,\nabla u^{(m)})\}_{m\in\mathbb{N}}$ is bounded $L^{s'(x,t)}(\Omega_T)$. Then, for another subsequence there exists $A_0 \in L^{s'(x,t)}(\Omega_T)$ such that

$$A(x,t,\nabla u^{(m)}) \rightharpoonup A_0 \quad \text{in } L^{s'(x,t)}(\Omega_T).$$

We claim that $A(x,t,\nabla u) = A_0$ for almost every $(x,t) \in \Omega_T$. Indeed, we have for every $s \leq m$, where $m \in \mathbb{N}$ is fixed, that

$$-\int\limits_{\Omega_T} u_t^{(m)}\varphi + \Big[A(x,t,\nabla u^{(m)}) - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big]\nabla\varphi \, dxdt = 0, \tag{3.11}$$

for all test functions $\varphi \in W_s(\Omega_T)$. Then, by passing to the limit $m \to \infty$, we have

$$-\int\limits_{\Omega_T} u_t\varphi + \Big[A_0 - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big]\nabla\varphi \, dxdt = 0, \tag{3.12}$$

for all $\varphi \in W_s(\Omega_T) \subset W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$. According to the monotonicity assumption (1.5), we get

$$\int\limits_{\Omega_T} \Big(A(x,t,\nabla u^{(m)}) - A(x,t,\nabla\omega)\Big)\nabla(u^{(m)} - \omega) \, dxdt \geq 0, \tag{3.13}$$

for all $\omega \in W_s(\Omega_T)$. Therefore, by adding (3.11) to (3.13) with $\varphi = u^{(m)} - \omega$, we get

$$\begin{aligned} 0 \leq & -\int_{\Omega_T} u_t^{(m)}(u^{(m)} - \omega) + \Big[A(x,t,\nabla u^{(m)}) \\ & \quad - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big] \nabla(u^{(m)} - \omega)\ dxdt \\ & + \int_{\Omega_T} \Big(A(x,t,\nabla u^{(m)}) - A(x,t,\nabla\omega)\Big) \nabla(u^{(m)} - \omega)\ dxdt \\ = & -\int_{\Omega_T} u_t^{(m)}(u^{(m)} - \omega) + \Big[A(x,t,\nabla\omega) \\ & \quad - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big] \nabla(u^{(m)} - \omega)\ dxdt. \end{aligned} \tag{3.14}$$

As a result, by testing (3.12) with $\varphi = u^{(m)} - \omega$, subtracting the resulting from (3.14) and passing to the limit $m \to \infty$, we arrive at

$$-\int_{\Omega_T} [A_0 - A(x,t,\nabla u)]\, \nabla(u - \omega)\ dxdt \geq 0,$$

for all $\omega \in W_s(\Omega_T)$. Since $W_s(\Omega_T) \subset W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ is dense, we are allowed to choose $\omega \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$. On that account we choose $\omega = u \pm \varepsilon\xi$ with $\xi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ and ε is an arbitrary constant such that

$$-\varepsilon \int_{\Omega_T} [A_0 - A(x,t,\nabla(u \pm \varepsilon\xi))]\, \nabla\xi\ dxdt \geq 0.$$

Therefore,

$$\int_{\Omega_T} [A_0 - A(x,t,\nabla(u \pm \varepsilon\xi))]\, \nabla\xi\ dxdt = 0.$$

Finally, after passing to the limit $\varepsilon \to 0$ we obtain

$$\int_{\Omega_T} [A_0 - A(x,t,\nabla u)]\, \nabla\xi\ dxdt = 0, \quad \forall \xi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T).$$

Hence, $A_0 = A(x,t,\nabla u)$ a.e. in Ω_T.

To complete the proof, we need to show that $u(\cdot,0) = g(\cdot,0)$. First of all, we should mention that we get from (3.12) and integration by parts the following

$$\begin{aligned} & \int_{\Omega_T} u\varphi_t - \Big[A(x,t,\nabla u) - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\Big] \nabla\varphi\ dxdt \\ & = \int_{\Omega} (u\varphi)(\cdot,0)\ dx, \end{aligned} \tag{3.15}$$

for all $\varphi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ with $\varphi(\cdot, T) = 0$. Moreover, as in (3.11) we have that

$$\begin{aligned} &\int_{\Omega_T} u^{(m)}\varphi_t - \left[A(x,t,\nabla u^{(m)}) - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\right]\nabla\varphi \; dxdt \\ &= \int_{\Omega} (u^{(m)}\varphi)(\cdot,0) \; dx, \end{aligned} \tag{3.16}$$

for all $\varphi \in W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$ with $\varphi(\cdot, T) = 0$. Also, from the definition of $u^{(m)}$ we get

$$\begin{aligned} u^{(m)}(\cdot,0) &= \sum_{i=1}^{m} C_i^m(0)\phi_i(x) = \sum_{i=1}^{m} \int_{\Omega} g(\cdot,0)\phi_i(x) \; dx \; \phi_i(x) \\ &\underset{m\to\infty}{\rightarrow} \sum_{i=1}^{\infty} \int_{\Omega} g(\cdot,0)\phi_i(x) \; dx \; \phi_i(x) = g(\cdot,0). \end{aligned}$$

That being so, after passing to the limit $m \to \infty$ in (3.16), we have

$$\begin{aligned} &\int_{\Omega_T} u\varphi_t - \left[A(x,t,\nabla u) - |F|^{p(x,t)-2}F - a(x,t)|F|^{q(x,t)-2}F\right]\nabla\varphi \; dxdt \\ &= \int_{\Omega} (g\varphi)(\cdot,0) \; dx. \end{aligned} \tag{3.17}$$

Hence, by comparing (3.15) and (3.17) we get the desired result. □

Now, the existence of solutions to the initial value problem (3.1) can be extended to our main problem as follows

Proof. As a consequence of Lemma 3.1, there exists at least one solution

$$v \in L^\infty(0,T;L^2(\Omega)) \cap W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$$

to the following problem

$$\begin{cases} v_t - \operatorname{div}\tilde{A}(x,t,\nabla v) & \\ = -\operatorname{div}\left(|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\right) - \partial_t g & \text{in } \Omega_T, \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot,0) = g(\cdot,0) - g & \text{on } \Omega \times \{0\}, \end{cases}$$

where

$$\tilde{A}(x,t,\nabla v) = A(x,t,\nabla(v+g)).$$

Knowing that $\partial_t g \in L^{(p^-)'}(0,T;W^{-1,(p^-)'})$, we have

$$\begin{aligned}\int\limits_{\Omega_T} \partial_t g\ v\ dxdt &\leq \|\partial_t g\|_{L^{(p^-)'}(0,T;W^{-1,(p^-)'})} \|v\|_{L^{p^-}(0,T;W^{1,p^-})} \\ &\leq C(\varepsilon)\, \|\partial_t g\|^{(p^-)'}_{L^{(p^-)'}(0,T;W^{-1,(p^-)'})} + \varepsilon \int\limits_{\Omega} |\nabla v|^{p^-}\ dxdt \\ &\leq C(\varepsilon)\, \|\partial_t g\|^{(p^-)'}_{L^{(p^-)'}(0,T;W^{-1,(p^-)'})} + \varepsilon \int\limits_{\Omega} |\nabla v|^{p(x,t)}\ dxdt + 1,\end{aligned}$$

where we used (2.1) and Young's inequality. Therefore, by using the proof of Lemma 3.1 with $u = v + g$, it is easy to show that u is the desired solution to (1.1) with the following energy estimate

$$\begin{aligned}&\sup_{0\leq t\leq T} \int\limits_{\Omega} |u(\cdot,t)|^2\ dx + \int\limits_{\Omega_T} |\nabla u|^{p(x,t)}\ dxdt + \int\limits_{\Omega_T} |\nabla u|^{q(x,t)}\ dxdt \\ &\leq C\Bigg\{ \|g(\cdot,0)\|^2_{L^2(\Omega)} + \|g\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t g\|^{(p^-)'}_{L^{(p^-)'}(0,T;W^{-1,(p^-)'}(\Omega))} \\ &\qquad + \int\limits_{\Omega_T} |\nabla g|^{p(x,t)} + |\nabla g|^{q(x,t)}\ dxdt + \int\limits_{\Omega_T} |F|^{p(x,t)} + |F|^{q(x,t)}\ dxdt \Bigg\},\end{aligned}$$

where we used the fact that

$$|\nabla u|^{p(x,t)} \leq 2^{p^+-1}\left[|\nabla v|^{p(x,t)} + |\nabla g|^{p(x,t)}\right],$$

which implies that

$$|\nabla u|^{p(x,t)} - 2^{p^+-1}|\nabla g|^{p(x,t)} \leq 2^{p^+-1}|\nabla v|^{p(x,t)},$$

and also by the same method that

$$|\nabla u|^{q(x,t)} - 2^{q^+-1}|\nabla g|^{q(x,t)} \leq 2^{q^+-1}|\nabla v|^{q(x,t)} \quad \text{and} \quad |u|^2 - 2|g|^2 \leq 2|v|^2.$$

Finally, we show the uniqueness of the weak solution. Let u_1 and u_2 be the solutions of (1.1). We consider $\varphi = u_1 - u_2$ as a test function in the weak formulation of both solutions. Then, by subtracting we obtain

$$\int\limits_{\Omega_T} \left[(u_1-u_2)(u_1-u_2)_t - (A(x,t,\nabla u_1) - A(x,t,\nabla u_2)\nabla(u_1-u_2))\right]\ dxdt = 0.$$

Therefore, by using (1.5) we arrive at

$$0 \geq \frac{1}{2}\int\limits_{\Omega_T} \partial_t (u_1-u_2)^2\ dxdt = \int\limits_{\Omega_T} (u_1-u_2)(u_1-u_2)_t\ dxdt.$$

Then, we get

$$0 \le \frac{1}{2}\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \le 0, \quad \forall t \in (0, T],$$

since $u_1(\cdot, 0) = u_2(\cdot, 0) = g(\cdot, 0)$. Hence, we get the desired result. $\square$

4. LOCAL BOUNDEDNESS OF THE SOLUTION

Let $K(\rho) = \{x \in \Omega : |x| < \rho^{\frac{\beta}{\alpha}}\}$ and $0 < \rho < 1$ be small enough such that

$$Q_{\text{loc}} = K(\rho^{\frac{\beta}{\alpha}}) \times (t_0 - \rho^\beta, t_0) \subset \Omega_T,$$

and

$$\frac{2N}{N+2} < p^-_{\text{loc}} = \operatorname*{ess\,inf}_{Q_{\text{loc}}} p(x,t) \le p(x,t) \le l = \beta\frac{2+N}{N}, \quad \forall (x,t) \in Q_{\text{loc}}, \tag{4.1}$$

and

$$\frac{2N}{N+2} < q^-_{\text{loc}} = \operatorname*{ess\,inf}_{Q_{\text{loc}}} q(x,t) \le q(x,t) \le l = \beta\frac{2+N}{N}, \quad \forall (x,t) \in Q_{\text{loc}}, \tag{4.2}$$

where $\beta = \max(p^-_{\text{loc}}, q^-_{\text{loc}})$ and $\alpha = \max\{p^+, q^+\}$. We claim that u is bounded in $K(\frac{1}{2}\rho^{\frac{\beta}{\alpha}}) \times (t_0 - \frac{1}{2}\rho^\beta, t_0)$. To prove this, we take $\rho_0,\ \rho_1,\ \tau_0$ and τ_1 such that

$$\frac{1}{2}\rho^{\frac{\beta}{\alpha}} \le \rho_1 < \rho_0 \le \rho^{\frac{\beta}{\alpha}}, \quad t_0 - \rho^\beta \le \tau_0 < \tau_1 \le t_0 - \frac{1}{2}\rho^\beta.$$

Let $\xi(x)$ and $\psi(t)$ be piecewise linear continuous functions depend respectively on x and t such that

$$\xi(x) = \begin{cases} 1, & \text{for } |x| \le \rho_1, \\ 0, & \text{for } |x| \ge \rho_0, \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} 1, & \text{for } t \ge \tau_1, \\ 0, & \text{for } t \le \tau_0. \end{cases}$$

Then, we have

$$0 \le |\xi'| \le \frac{1}{\rho_0 - \rho_1} \quad \text{and} \quad 0 \le \psi' \le \frac{1}{\tau_1 - \tau_0}.$$

In the weak formulation (2.8) we take $\varphi = \xi^\alpha \psi^\alpha (u_h - k)_+$, where u_h are regularizations of the form (2.4), k a positive constant and $(u_h - k)_+ = \max\{u_h - k, 0\}$. Then, for all $t \in (0, T]$ we have

$$
\begin{aligned}
0 = & \int\limits_{\Omega_t} u_{h,t}\varphi + [A(x,t,\nabla u)]_h . \nabla\varphi \\
& - \left[|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F \right] \nabla\varphi \; dxdt \\
\underset{h\to 0}{\rightarrow} & - \int\limits_{\Omega_t} u\varphi_t \; dxdt + \int\limits_{\Omega} u\varphi \; dx \Big|_0^t + \int\limits_{\Omega_t} A(x,t,\nabla u).\nabla\varphi \; dxdt \\
& - \int\limits_{\Omega_t} \left[|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F \right] \nabla\varphi \; dxdt \\
\geq & \frac{-\alpha}{2} \int\limits_{\Omega_t} \xi^{\alpha}\psi^{\alpha-1}\psi'(u-k)_+^2 \; dxdt + \frac{1}{2}\int\limits_{\Omega} \varphi(x,t)(u-k)_+(x,t) \; dxdt \\
& + C \int\limits_{\Omega_t} \xi^{\alpha}\psi^{\alpha} \left[|\nabla u|^{p(x,t)} + a(x,t)|\nabla u|^{q(x,t)} \right] \; dxdt \\
& - C \int\limits_{\Omega_t} \alpha\xi^{\alpha-1}\xi'\psi^{\alpha}(u-k)_+ \left[|\nabla u|^{p(x,t)-1} + a(x,t)|\nabla u|^{q(x,t)-1} \right] \; dxdt \\
& - \int\limits_{\Omega_t} \left[|F|^{p(x,t)-1}F + |F|^{q(x,t)-1}F \right] \nabla\varphi \; dxdt.
\end{aligned}
\tag{4.3}
$$

Next, by using Young's inequality we get the following estimates

$$
\begin{aligned}
& \int\limits_{\Omega_t} \xi^{\alpha-1}\xi'\psi^{\alpha}(u-k)_+ \left[|\nabla u|^{p(x,t)-1} + a(x,t)|\nabla u|^{q(x,t)-1} \right] \; dxdt \\
& \leq \varepsilon \int\limits_{K(k,\rho_0,\tau_0)} \xi^{\frac{(\alpha-1)p(x,t)}{p(x,t)-1}} \psi^{\alpha}|\nabla u|^{p(x,t)} \; dxdt \\
& \quad + C(\varepsilon) \int\limits_{K(k,\rho_0,\tau_0)} \psi^{\alpha}(u-k)^{p(x,t)}|\nabla\xi|^{p(x,t)} \; dxdt \\
& \quad + \upsilon \int\limits_{K(k,\rho_0,\tau_0)} \xi^{\frac{(\alpha-1)q(x,t)}{q(x,t)-1}} \psi^{\alpha}|\nabla u|^{q(x,t)} \; dxdt \\
& \quad + C(\upsilon) \int\limits_{K(k,\rho_0,\tau_0)} \psi^{\alpha}(u-k)^{q(x,t)}|\nabla\xi|^{q(x,t)} \; dxdt \\
& \leq \varepsilon \int\limits_{K(k,\rho_0,\tau_0)} \xi^{\alpha}\psi^{\alpha}|\nabla u|^{p(x,t)} \; dxdt + \frac{C}{(\rho_0-\rho_1)^{p^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{p(x,t)} \; dxdt \\
& \quad + \upsilon \int\limits_{K(k,\rho_0,\tau_0)} \xi^{\alpha}\psi^{\alpha}|\nabla u|^{q(x,t)} \; dxdt + \frac{C}{(\rho_0-\rho_1)^{q^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{q(x,t)} \; dxdt,
\end{aligned}
\tag{4.4}
$$

and also,

$$
\begin{aligned}
&\int\limits_{\Omega_t} \left[|F|^{p(x,t)-1} + |F|^{q(x,t)-1} \right] \nabla\varphi \; dxdt \\
&\leq C(\varepsilon) \Bigg\{ \int\limits_{K(k,\rho_0,\tau_0)} |F|^{p(x,t)} \; dxdt \\
&\quad + \frac{1}{(\rho_0-\rho_1)^{p^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{p(x,t)} \; dxdt \\
&\quad + \int\limits_{K(k,\rho_0,\tau_0)} |F|^{q(x,t)} \; dxdt + \frac{1}{(\rho_0-\rho_1)^{q^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{q(x,t)} \; dxdt \Bigg\} \\
&\quad + \varepsilon \Bigg\{ \int\limits_{K(k,\rho_0,\tau_0)} \xi^\alpha \psi^\alpha |\nabla u|^{p(x,t)} \; dxdt + \int\limits_{K(k,\rho_0,\tau_0)} \xi^\alpha \psi^\alpha |\nabla u|^{q(x,t)} \; dxdt \Bigg\},
\end{aligned} \tag{4.5}
$$

where we used the fact that $0 \leq \xi \leq 1$, $\frac{p(x,t)}{p(x,t)-1} \geq \frac{\alpha}{\alpha-1}$ and $\frac{q(x,t)}{q(x,t)-1} \geq \frac{\alpha}{\alpha-1}$ which imply that $\xi^{\frac{(\alpha-1)p(x,t)}{p(x,t)-1}} \leq \xi^\alpha$ and $\xi^{\frac{(\alpha-1)q(x,t)}{q(x,t)-1}} \leq \xi^\alpha$. Also, we took

$$
K(k,\rho_0,\tau_0) = \{K(\rho_0) \times (\tau_0, t_0)\} \cap \{u > k\}
$$

as the effective domain of integration. Therefore, by combining (4.4) and (4.5) into (4.3) we arrive at

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,t_0)} \int\limits_{\Omega} \xi^\alpha \psi^\alpha (u-k)_+^2 \; dx \\
&+ \int\limits_{K(k,\rho_0,\tau_0)} \xi^\alpha \psi^\alpha |\nabla u|^{p^-_{loc}} \; dxdt + \int\limits_{K(k,\rho_0,\tau_0)} \xi^\alpha \psi^\alpha |\nabla u|^{q^-_{loc}} \; dxdt \\
&\leq C \Bigg\{ \frac{1}{\tau_1-\tau_0} \int\limits_{K(k,\rho_0,\tau_0)} |u-k|^2 \; dxdt \\
&\quad + \frac{1}{(\rho_0-\rho_1)^{p^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{p(x,t)} \; dxdt \\
&\quad + \int\limits_{K(k,\rho_0,\tau_0)} |F|^{p(x,t)} + |F|^{q(x,t)} \; dxdt + \int\limits_{K(k,\rho_0,\tau_0)} \xi^\alpha \psi^\alpha \; dxdt \Bigg\},
\end{aligned} \tag{4.6}
$$

where C is a positive constant independent of k, ρ_0, ρ_1, τ_0 and τ_1. For $l = \beta\frac{N+2}{N}$, it follows from Proposition 2.3 that

$$\begin{aligned}
&\left(\int\limits_{K(k,\rho_2,\tau_2)} (u-k)^l \, dxdt\right)^{\frac{N}{N+\beta}} \\
&\leq C\Bigg\{ \int\limits_{K(k,\rho_1,\tau_1)} |\nabla u|^{p^-_{loc}} + |\nabla u|^{q^-_{loc}} \, dxdt + \operatorname*{ess\,sup}_{t\in(0,t_0)} \int\limits_{K(\rho_1)} |(u-k)_+|^2 \, dx \\
&\quad + \frac{1}{(\rho_1-\rho_2)^{p^+}} \int\limits_{K(k,\rho_1,\tau_1)} (u-k)^{p^-_{loc}} \, dxdt \\
&\quad + \frac{1}{(\rho_1-\rho_2)^{q^+}} \int\limits_{K(k,\rho_1,\tau_1)} (u-k)^{q^-_{loc}} \, dxdt \Bigg\},
\end{aligned} \tag{4.7}$$

for all $\frac{1}{2}\rho^{\frac{\beta}{\alpha}} \leq \rho_2 < \rho_1 < \rho_0 \leq \rho^{\frac{\beta}{\alpha}}$ and $t_0 - \rho^\beta \leq \tau_0 < \tau_1 < \tau_2 \leq t_0 - \frac{1}{2}\rho^\beta$. Afterward, by combining (4.6) and (4.7) and taking $\rho_0 - \rho_1 = \rho_1 - \rho_2$ and $\tau_1 - \tau_0 = \tau_2 - \tau_1$, we obtain

$$\begin{aligned}
&\left(\int\limits_{K(k,\rho_2,\tau_2)} (u-k)^l \, dxdt\right)^{\frac{N}{N+\beta}} \\
&\leq C\Bigg\{ \frac{1}{\tau_2-\tau_0} \int\limits_{K(k,\rho_0,\tau_0)} |u-k|^2 \, dxdt + \frac{1}{(\rho_0-\rho_2)^{p^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{p(x,t)} \, dxdt \\
&\quad + \frac{1}{(\rho_0-\rho_2)^{q^+}} \int\limits_{K(k,\rho_0,\tau_0)} (u-k)^{q(x,t)} \, dxdt \\
&\quad + \int\limits_{K(k,\rho_0,\tau_0)} |F|^{p(x,t)} + |F|^{q(x,t)} + 1 \, dxdt \Bigg\},
\end{aligned}$$

where we used (2.1). Since $\beta > \frac{2N}{N+2}$ which implies that $l > 2$ and by using (4.1) and (4.2) we get the following estimates

$$\begin{aligned}
(u-k)^{p(x,t)} &\leq (u-k)u^{p(x,t)-1} = (u-k)u^{l-1}u^{p(x,t)-l} \leq C(u-k)u^{l-1} \\
&= C(u-k)(u-k+k)^{l-1} \leq C\left((u-k)^l + (u-k)k^{l-1}\right) \\
&\leq C\left((u-k)^l + k^l\right),
\end{aligned}$$

and also, by the same method

$$(u-k)^{q(x,t)} \leq C\left((u-k)^l + k^l\right).$$

Therefore, if $k \geq k_0$ (for $k_0 > 0$ large enough) we get

$$\begin{aligned}
&\left(\int\limits_{K(k,\rho_2,\tau_2)} (u-k)^l \, dxdt \right)^{\frac{N}{N+\beta}} \\
&\leq C\Bigg\{ \frac{1}{\tau_2-\tau_0} \Bigg(\int\limits_{K(k,\rho_0,\tau_0)} |u-k|^l \, dxdt \Bigg)^{\frac{2}{l}} |K(k,\rho_0,\tau_0)|^{1-\frac{2}{l}} \\
&\quad + \frac{1}{(\rho_0-\rho_2)^\alpha} \Bigg[\int\limits_{K(k,\rho_0,\tau_0)} (u-k)^l \, dxdt + K^l \, |K(k,\rho_0,\tau_0)| \Bigg] \\
&\quad + |K(k,\rho_0,\tau_0)| \Bigg\}.
\end{aligned} \tag{4.8}$$

If $k > h \geq k_0$, we have

$$|K(k,\rho_0,\tau_0)| \leq \int\limits_{K(k,\rho_0,\tau_0)} \left| \frac{u-h}{k-h} \right|^l \, dxdt \leq \int\limits_{K(h,\rho_0,\tau_0)} \left| \frac{u-h}{k-h} \right|^l \, dxdt.$$

Then (4.8) can be rewritten as

$$\begin{aligned}
&\left(\int\limits_{K(k,\rho_2,\tau_2)} (u-k)^l \, dxdt \right)^{\frac{N}{N+\beta}} \\
&\leq C\Bigg\{ \frac{1}{\tau_2-\tau_0} (k-h)^{2-l} \int\limits_{K(k,\rho_0,\tau_0)} (u-h)^l \, dxdt \\
&\quad + \frac{1}{(\rho_0-\rho_2)^\alpha} \left(1 + \left(\frac{k}{k-h} \right)^l \right) \int\limits_{K(k,\rho_0,\tau_0)} (u-h)^l \, dxdt \\
&\quad + (k-h)^{-l} \int\limits_{K(k,\rho_0,\tau_0)} (u-h)^l \, dxdt \Bigg\},
\end{aligned} \tag{4.9}$$

for all $k > h \geq k_0$, $\frac{1}{2}\rho^{\frac{\beta}{\alpha}} \leq \rho_2 < \rho_0 \leq \rho^{\frac{\beta}{\alpha}}$, $t_0 - \rho^\beta \leq \tau_0 < \tau_2 \leq t_0 - \frac{1}{2}\rho^\beta$. Let $\varepsilon > 0$ be determined. Considering the absolute continuity of the Lebesgue integral, we take $H > k_0$ large enough such that

$$\int\limits_{t_0-\rho^\beta}^{t_0} \int\limits_{K(\rho^{\frac{\beta}{\alpha}})} (u-H)_+^l \, dxdt \leq \varepsilon \rho^{N+\beta}. \tag{4.10}$$

For $m = 0, 1, \ldots$, set

$$k_m = 2H - \frac{H}{2^m}, \quad \rho_m = \left(\frac{1}{2} + \frac{1}{2^{m+1}}\right)\rho^{\frac{\beta}{\alpha}}, \quad \tau_m = t_0 - \frac{1}{2}\rho^\beta - \frac{1}{2^{m+1}}\rho^\beta,$$

and

$$J_m = \int\limits_{K(k_m,\rho_m,\tau_m)} (u - k_m)^l \, dxdt.$$

Since the constant C in (4.9) is independent of $h,\ k,\ \rho_0,\ \rho_2,\ \tau_0$ and τ_2, we substitute the previous data respectively with $k_m,\ k_{m+1},\ \rho_m,\ \rho_{m+1},\ \tau_m$ and τ_{m+1}. Thereby, we get

$$\begin{aligned} J_{m+1}^{\frac{N}{N+\beta}} \le C\Bigg\{ & \frac{2^{m+2}}{\rho^\beta}\left(\frac{2^{m+1}}{H}\right)^{l-1} J_m \\ & + \frac{2^{m+2}}{\rho^\beta}(1 + 2^{(m+2)l})J_m + \left(\frac{2^{(m+1)}}{H}\right)^l J_m \Bigg\}. \end{aligned} \tag{4.11}$$

By taking $H > 1$, we can simplify (4.11) as follows:

$$J_{m+1}^{\frac{N}{N+\beta}} \le C J_m^{\frac{N}{N+\beta}} \left\{ \frac{2^{ml}}{\rho^\beta} J_m^{\frac{\beta}{N+\beta}} + 2^{ml} J_m^{\frac{\beta}{N+\beta}} \right\}. \tag{4.12}$$

Since (4.10) implies that $J_0 \le \varepsilon\rho^{N+\beta}$, we can prove by induction for suitable $\delta \in (0,1)$ that

$$J_m \le \delta^m \varepsilon \rho^{N+\beta}, \quad \text{for } m = 0, 1, \ldots \tag{4.13}$$

In fact, assume that (4.13) holds for m. It follows by combining (4.12) with (4.13) that

$$J_{m+1}^{\frac{N}{N+\beta}} \le C J_m^{\frac{N}{N+\beta}} \left[2^{ml} \delta^{\frac{m\beta}{N+\beta}} \varepsilon^{\frac{\beta}{N+\beta}} + 2^{ml} \delta^{\frac{m\beta}{N+\beta}} \varepsilon^{\frac{\beta}{N+\beta}} \rho^\beta \right].$$

Since $0 < \rho < 1$ and by letting

$$\varepsilon^{\frac{\beta}{N+\beta}} \le \delta^{\frac{N}{N+\beta}},\ 2^l \delta^{\frac{\beta}{N+\beta}} \le 1,$$

we get that

$$J_{m+1} \le C\delta^{m+1} \varepsilon \rho^{N+\beta}.$$

By induction, (4.13) holds for all m. As a result, we obtain that

$$0 = \lim_{m\to\infty} J_m = \int\limits_{K(2H, \frac{1}{2}\rho^{\frac{\beta}{\alpha}}, t_0 - \frac{1}{2}\rho^\beta)} (u - 2H)^l \, dxdt,$$

i.e.

$$\operatorname*{ess\,sup}_{K(\frac{1}{2}\rho^{\frac{\beta}{\alpha}}) \times (t_0 - \frac{1}{2}\rho^\beta, t_0)} u \le 2H.$$

Hence, we proved that u is locally bounded above in Ω_T. Moreover, by substituting $-u$ for u, we obtain similarly that u is locally bounded below. The proof of Theorem 2.8 is completed.

5. GLOBAL BOUNDEDNESS OF THE SOLUTION

We begin our proof by assuming that $g \in C^1([0,T]; W^{1,\infty}(\Omega))$ and F fulfills (2.9). Afterward, we consider the following problem

$$\begin{cases} v_t - \operatorname{div}\tilde{A}(x,t,\nabla v) & \\ = -\operatorname{div}\left(|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\right) - \partial_t g & \text{in } \Omega_T, \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v = g(\cdot,0) - g & \text{on } \Omega \times \{0\}, \end{cases} \tag{5.1}$$

where

$$\tilde{A}(x,t,\nabla v) = A(x,t,\nabla(g+v)).$$

By Lemma 3.1, there exists a solution

$$v \in L^\infty(0,T;L^2(\Omega)) \cap W_0^{p(x,t)}(\Omega_T) \cap W_0^{q(x,t)}(\Omega_T)$$

with $v_t \in W^{s(x,t)}(\Omega_T)'$ of equation (5.1) in the sense of Definition 2.5. Thereafter, we introduce the following

$$v_k = \min\{|v|, k\}\operatorname{sign}(v) = \begin{cases} k & \text{if } v > k, \\ v & \text{if } |v| \leq k, \\ -k & \text{if } v < -k. \end{cases} \tag{5.2}$$

For every $m \in \mathbb{N}$, the function v_k^{2m-1} can be taken for test function in the weak formulation of (5.1) such that we have the following estimate

$$\begin{aligned} \int_0^t \int_\Omega v_t v_k^{2m-1}\, dx d\tau &= \int_\Omega v v_k^{2m-1}\, dx \Big|_{\tau=0}^{\tau=t} - \int_0^t \int_\Omega v \partial_t (v_k^{2m-1})\, dx d\tau \\ &= \int_\Omega v v_k^{2m-1}\, dx \Big|_{\tau=0}^{\tau=t} - \int_0^t \int_\Omega v_k \partial_t (v_k^{2m-1})\, dx d\tau \\ &= \int_\Omega v v_k^{2m-1}\, dx \Big|_{\tau=0}^{\tau=t} - \frac{2m-1}{2m} \int_\Omega v_k^{2m}\, dx \Big|_{\tau=0}^{\tau=t}. \end{aligned} \tag{5.3}$$

Next, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \tilde{A}(x,t,\nabla v_k).\nabla v_k^{2m-1}\ dxd\tau \\
&= (2m-1)\int_0^t \int_\Omega v_k^{2(m-1)}\tilde{A}(x,t,\nabla v_k)\nabla v_k\ dxd\tau \\
&= (2m-1)\Big\{\int_0^t \int_\Omega v_k^{2(m-1)}A(x,t,\nabla(v_k+g))\nabla(v_k+g)\ dxd\tau \\
&\qquad\qquad - \int_0^t \int_\Omega v_k^{2(m-1)}A(x,t,\nabla(v_k+g))\nabla g\ dxd\tau\Big\} \\
&= (2m-1)\{I_1+I_2\}.
\end{aligned}$$

Therefore, by using (1.2)–(1.4), the assumption on g, Young's and Hölder's inequalities we get the following

$$\begin{aligned}
I_1 \geq C_1\Big\{&\int_0^t \int_\Omega |\nabla(v_k+g)|^{p(x,t)} v_k^{2(m-1)}\ dxd\tau \\
&+ \alpha_1 \int_0^t \int_\Omega |\nabla(v_k+g)|^{q(x,t)} v_k^{2(m-1)}\ dxd\tau\Big\},
\end{aligned}$$

and

$$\begin{aligned}
I_2 \leq C_2\Big\{&\int_0^t \int_\Omega |\nabla(v_k+g)|^{p(x,t)-1} v_k^{2(m-1)}|\nabla g|\ dxd\tau \\
&+ \alpha_1 \int_0^t \int_\Omega |\nabla(v_k+g)|^{q(x,t)-1} v_k^{2(m-1)}|\nabla g|\ dxd\tau\Big\} = C_2\{I_3+I_4\},
\end{aligned}$$

with

$$\begin{aligned}
I_3 &\leq \varepsilon \int_0^t \int_\Omega |\nabla(v_k+g)|^{p(x,t)} v_k^{2(m-1)}\ dxd\tau + C(\varepsilon)\int_0^t \int_\Omega |\nabla g|^{p(x,t)} v_k^{2(m-1)}\ dxd\tau \\
&\leq \varepsilon \int_0^t \int_\Omega |\nabla(v_k+g)|^{p(x,t)} v_k^{2(m-1)}\ dxd\tau + C(\varepsilon)\int_0^t |\Omega|^{\frac{1}{m}} \left(\int_\Omega v_k^{2m}\ dx\right)^{\frac{2(m-1)}{2m}} d\tau,
\end{aligned}$$

and

$$I_4 \leq \varepsilon \int_0^t \int_\Omega |\nabla(v_k+g)|^{q(x,t)} v_k^{2(m-1)}\, dxd\tau + C(\varepsilon)\int_0^t |\Omega|^{\frac{1}{m}} \left(\int_\Omega v_k^{2m}\, dx\right)^{\frac{2(m-1)}{2m}} d\tau.$$

Thereafter, since $|v_k| \leq k$, and by using Hölder's inequality and (2.9) we get the following estimate

$$\begin{aligned}
&\int_0^t \int_\Omega \left[-\mathrm{div}\left(|F|^{p(x,t)-2}F + a(x,t)|F|^{q(x,t)-2}F\right) - \partial_t g\right] v_k^{2m-1}\, dxdt \\
&\leq \int_0^t \int_\Omega (h(x,t)+|\partial_t g|)\, v_k^{2(m-1)} v_k\, dxdt \\
&\leq C\int_0^t |\Omega|^{\frac{1}{m}} \left(\int_\Omega v_k^{2m}\, dx\right)^{\frac{2(m-1)}{2m}} d\tau.
\end{aligned} \tag{5.4}$$

Combining (5.3)–(5.4) into the weak formulation of (5.1), we arrive at

$$\begin{aligned}
&\int_\Omega v v_k^{2m-1}\, dx\Big|_{\tau=0}^{\tau=t} - \frac{2m-1}{2m}\int_\Omega v_k^{2m}\, dx\Big|_{\tau=0}^{\tau=t} \\
&+ \int_0^t \int_\Omega |\nabla(v_k+g)|^{p(x,t)} v_k^{2(m-1)}\, dxd\tau + \int_0^t \int_\Omega |\nabla(v_k+g)|^{q(x,t)} v_k^{2(m-1)}\, dxd\tau \\
&\leq C\int_0^t |\Omega|^{\frac{1}{m}} \left(\int_\Omega v_k^{2m}\, dx\right)^{\frac{2(m-1)}{2m}} d\tau.
\end{aligned} \tag{5.5}$$

Next, by introducing the function $y_m(t) = \|v_k\|_{L^{2m}(\Omega)}(t)$ and using the fact that $vv_k^{2m-1} \geq v_k^{2m}$ and $v(\cdot,0)v_k^{2m-1}(\cdot,0) = v^{2m}(\cdot,0)$, we obtain from (5.5) the following

$$\frac{1}{2m} y_m^{2m}(t) \leq \frac{1}{2m} y_m^{2m}(0) + C|\Omega|^{\frac{1}{m}} \int_0^t y_m^{2(m-1)}(\tau)\, d\tau. \tag{5.6}$$

Let $z(t)$ be the solution of the following equation

$$\frac{1}{2m} z^{2m}(t) = \frac{1}{2m} z^{2m}(0) + C|\Omega|^{\frac{1}{m}} \int_0^t z^{2(m-1)}(\tau)\, d\tau, \tag{5.7}$$

with $z(0) = \|v_0\|_{L^{2m}(\Omega)} + \delta$, where $\|v_0\|_{L^{2m}(\Omega)} = \|v(x,0)\|_{L^{2m}(\Omega)}$ and $\delta > 0$ is an arbitrary positive constant. The function $z(t)$ can be constructed as the solution of the Cauchy problem for the ODE obtained from (5.7) by means of differentiation

$$\begin{cases} z'(t) = C|\Omega|^{\frac{1}{m}} z^{-1}, & t > 0, \\ z(0) = \|v_0\|_{L^{2m}(\Omega)} + \delta. \end{cases} \tag{5.8}$$

This equation can be explicitly integrated and the solution of (5.8) has the form

$$z(t) = \sqrt{2C|\Omega|^{\frac{1}{m}} t + \left(\|v_0\|_{L^{2m}(\Omega)} + \delta\right)^2}.$$

By the choice of the initial data $y_m(0) - z(0) = -\delta < 0$, which yields $y_m(t) < y(t)$ for all $t > 0$. Indeed, if the assertion is false then by the monotonicity property of $z(t)$ and the given initial data, we have that

$$t^* = \sup\{t \geq 0 : \ y_m(t) < z(t)\} < +\infty.$$

Since $0 \leq y_m(t) \leq k|\Omega|^{\frac{1}{m}}$, it follows from (5.6) that $t^* > 0$. Therefore, by subtracting (5.6) from (5.7), we find that

$$\begin{aligned} 0 = \frac{1}{2m}\left(y_m^{2m}(t^*) - z^{2m}(t^*)\right) &\leq C|\Omega|^{\frac{1}{m}} \int_0^{t^*} \underbrace{\left(y_m^{2(m-1)}(\tau) - z^{2(m-1)}(\tau)\right)}_{<0} d\tau \\ &+ \frac{1}{2m}\underbrace{\left(y_m^{2m}(0) - z^{2m}(0)\right)}_{<0} < 0, \end{aligned}$$

which is impossible. Thus, $y_m(t) \leq z(t)$ for every m and δ. Letting $m \to \infty$ we conclude that for every $\delta > 0$

$$\|v_k\|_{L^\infty(\Omega)}(t) \leq \sqrt{2C|\Omega|^{\frac{1}{m}} t + \left(\|v_0\|_{L^\infty(\Omega)} + \delta\right)^2} = C(t) \leq C(T).$$

Next, let us choose $k \geq C(T) + 1$. Under this choice of k, we get

$$v_k = \min\{|v|, k\}\operatorname{sign}(v) = v.$$

Hence, since $u = v + g$ is the solution of (1.1) and $g \in L^\infty(\Omega)$ we conclude the proof of Theorem 2.9.

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