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OSCILLATION CONDITIONS FOR DIFFERENCE EQUATIONS WITH SEVERAL VARIABLE DELAYS

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Abstract. A technique is developed to establish a new oscillation criterion for a first-order linear difference equation with several delays and non-negative coefficients. Our result improves recent oscillation criteria and covers the cases of monotone and non-monotone delays. Moreover, the paper is concluded with an illustrative example to show the applicability and strength of our result.

Keywords: oscillation, difference equations, non-monotone delays, first order.

Mathematics Subject Classification: 39A10, 39A21.

1. INTRODUCTION

In this paper, we study the oscillation of the first order delay difference equation with several retarded arguments of the form

$$\Delta y(n) + \sum_{i=1}^{m} p_i(n) y(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $(p_i(n))$, $1 \le i \le m$ are sequences of non-negative real numbers and $(\tau_i(n))$ is a sequence of integers for each $1 \le i \le m$ such that

$$\tau_i(n) \le n-1, \quad n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \le i \le m. \tag{1.2}$$

Here, Δ denotes the forward difference operator $\Delta y(n) = y(n+1) - y(n)$ and $\mathbb{N}_0$ is the set of non-negative integers. In view of (1.2), the number

$$w = -\min_{\substack{n\ge 0\\ 1\le i\le m}} \tau_i(n)$$

is a finite positive integer.

By a solution of Eq. (1.1), we mean a sequence of real numbers $(y(n))_{n\geq -w}$ which satisfies Eq. (1.1) for all $n \geq 0$. Clearly, for each choice of real numbers $c_{-w}, c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(y(n))_{n\geq -w}$ of Eq. (1.1) which satisfies the initial conditions

$$y(-w) = c_{-w}, y(-w+1) = c_{-w+1}, \ldots, y(-1) = c_{-1}, y(0) = c_0.$$

Such a solution is called oscillatory if the terms $y(n)$ of the sequence are neither eventually positive nor eventually negative; otherwise it is called non-oscillatory. The equation is oscillatory if all its solutions oscillate.

Assume that the arguments $\tau_i(n)$, $1 \leq i \leq m$, are not necessarily monotone,

$$\begin{aligned} &\tau(n) := \max_{1\leq i\leq m} \tau_i(n), \quad \gamma_i(n) := \max_{0\leq s\leq n} \tau_i(s) \\ &\text{and } \gamma(n) := \max_{1\leq i\leq m} \gamma_i(n), \quad \text{for all } n \in \mathbb{N}_0. \end{aligned} \tag{1.3}$$

Throughout this work, we will consider the following:

$$\sum_{i=r}^{r-1} A(i) = 0 \quad \text{and} \quad \prod_{i=r}^{r-1} A(i) = 1, \quad \text{where } A(i) \in \mathbb{R}^+,$$

$$\tilde{p}(n) := \sum_{i=1}^{m} p_i(n),$$

$$\alpha := \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tilde{p}(j), \tag{1.4}$$

and

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e, \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0, 1/e]. \end{cases} \tag{1.5}$$

Also, λ_0 stands for the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

Our aim in this work is to improve recent oscillation criteria of the limsup type. The first of this type of conditions appeared in [9] for Eq. (1.1) with $m = 1$ in the form

$$\limsup_{n\to\infty} \sum_{j=\gamma(n)}^{n} \tilde{p}(j) > 1. \tag{1.6}$$

This condition is working also for Eq. (1.1), see [11]. Other interesting nonoscillation criteria can be found in [4]. The reader is referred to [1–3, 5–11, 13, 14] for several improvements of (1.6) for Eq. (1.1) with general delay arguments. One of these improvements (see [6]) states that (1.1) is oscillatory if

$$\limsup_{n\to\infty} \sum_{l=\gamma(n)}^{n} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{\gamma(n)-1} \tilde{p}(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1 - Z_w(u)} \right) > 1 - D(\alpha), \text{ for some } w \in \mathbb{N}, \tag{1.7}$$

where

$$Z_w(n) = \tilde{p}(n)\left[1 + \sum_{l=\tau(n)}^{n-1} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{n-1} \tilde{p}(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1 - Z_{w-1}(u)}\right)\right]$$

with

$$Z_0(n) = \tilde{p}(n)\left[1 + \sum_{l=\tau(n)}^{n-1} \tilde{p}(l) \exp(\lambda_0 \sum_{j=\tau(l)}^{n-1} \tilde{p}(j))\right].$$

In this paper not only an essential improvement of (1.7) is established but also a criterion that works on certain equations when other known criteria fail to do so.

The following lemmas will be used in our proof.

Lemma 1.1 ([5])**.** *Assume that* (1.2) *holds and* α *is defined by* (1.4) *with* $\alpha > 0$*. Then we have*

$$\liminf_{n\to\infty} \sum_{j=\gamma(n)}^{n-1} \tilde{p}(j) = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tilde{p}(j) = \alpha,$$

where $\gamma(n)$, $\tau(n)$ *are defined by* (1.3)*.*

Lemma 1.2 ([5])**.** *Assume that* (1.2) *holds,* α *is defined by* (1.4) *with* $0 < \alpha \leq 1/e$*, and* $y(n)$ *is an eventually positive solution of* (1.1)*. Then we have*

$$\liminf_{n\to\infty} \frac{y(\gamma(n))}{y(n)} \geq \lambda_0, \tag{1.8}$$

where $\gamma(n)$ *is defined by* (1.3)*.*

Let $y(n)$ be an eventually positive solution of (1.1). Then $y(n)$ will be an eventually positive nonincreasing solution of the inequality

$$\Delta y(n) + \tilde{p}(n) y(\gamma(n)) \leq 0.$$

On the other hand, a close look at the proof of [12, Lemma 2.1] shows that it can be carried out verbatim on this inequality. This leads to the following result; see also [6, Lemma 3].

Lemma 1.3. *Assume that* (1.2) *holds,* $\gamma(n)$ *is defined by* (1.3)*,* α *is defined by* (1.4) *with* $0 < \alpha \leq 1/e$ *and* $y(n)$ *is an eventually positive solution of* (1.1)*. Then*

$$\liminf_{n\to\infty} \frac{y(n+1)}{y(\gamma(n))} \geq D(\alpha),$$

where $D(\alpha)$ *is defined by* (1.5)*.*

2. MAIN RESULTS

Lemma 2.1. *Assume that $y(n)$ is an eventually positive solution of* (1.1) *for all $n \geq n_0 > 0$. Then there exists a sub-sequence $\{n_k\}$ such that $n_k \geq n_{k-1}$, $\tau(n) > n_{k-1}$ for $n \geq n_k$, $k = 1, 2, \dots$ and*

$$\frac{y(n+1)}{y(n)} \leq v(n,k) \quad \textit{for all } n \geq n_k, \tag{2.1}$$

where

$$v(n,k) = 1 - \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} v^{-1}(j, k-1) \quad \textit{for all } n \geq n_{k+1},\ k = 1, 2, \dots, \tag{2.2}$$

and $v(n, 0) = 1$ for all $n \geq n_1$.

Proof. Since $y(n) > 0$ for all $n \geq n_0$, then $\Delta y(n) \leq 0$ as long as $\tau_i(n) > n_0$, for all $i = 1, 2, \dots, m$. Since $\lim_{n\to\infty} \tau_i(n) = \infty$, $1 \leq i \leq m$, then there exists $n_1 \geq n_0$ such that $\tau_i(n) > n_0$ for each i and all $n \geq n_1$. Then (1.1) implies that

$$y(n+1) - y(n) + \tilde{p}(n) y(\tau(n)) \leq 0, \quad \text{for all } n \geq n_1. \tag{2.3}$$

Dividing by $y(n)$, and using the product representation of the quotient $\frac{y(\tau(n))}{y(n)}$, we obtain

$$\frac{y(n+1)}{y(n)} - 1 + \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} \frac{y(j)}{y(j+1)} \leq 0, \quad \text{for all } n \geq n_1.$$

Let $u(n) = \frac{y(n+1)}{y(n)}$ for all $n \geq n_0$. Then

$$u(n) \leq 1 - \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} u^{-1}(j), \quad n \geq n_1. \tag{2.4}$$

Since $u(n) < 1 = v(n, 0)$ for $n \geq n_1$, then (2.4) yields

$$u(n) \leq 1 - \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} v^{-1}(j, 0) = v(n, 1), \quad n \geq n_2, \tag{2.5}$$

where $n_2 \geq n_1$ is so large that $\tau(n) \geq n_1$ for all $n \geq n_2$. Substituting from (2.5) in (2.4), we obtain

$$u(n) \leq 1 - \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} v^{-1}(j, 1) = v(n, 2), \quad n \geq n_3,$$

where $n_3 \geq n_2$ and $\tau(n) \geq n_2$ for all $n \geq n_3$. Continuing this way, we arrive at

$$u(n) \leq 1 - \tilde{p}(n) \prod_{j=\tau(n)}^{n-1} v^{-1}(j, k-1) = v(n, k), \quad n \geq n_{k+1}, \quad k = 1, 2, \ldots$$

The proof is complete. □

Theorem 2.2. *Assume that there exist $l, k \in \mathbb{N}_0$ and $\epsilon > 0$ such that*

$$\limsup_{n\to\infty} \left(\sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1 - F_l(i_2, \epsilon)} + A(\gamma(n), \tau(i_1), k) \Big) \right) > 1 - D(\alpha), \tag{2.6}$$

where

$$F_l(n, \epsilon) = \tilde{p}(n) \left(1 + \sum_{i_1=\tau(n)}^{n-1} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1 - F_{l-1}(i_2, \epsilon)} + A(n, \tau(i_1), k) \Big) \right),$$

$$F_0(n, \epsilon) = \tilde{p}(n) \left(1 + \sum_{i_1=\tau(n)}^{n-1} \tilde{p}(i_1) \exp\Big((\lambda_0 - \epsilon) \sum_{j=\tau(i_1)}^{n-1} \tilde{p}(j) + A(n, \tau(i_1), k) \Big) \right).$$

and

$$A(n, r, k) = \sum_{j=r}^{n-1} \big(v(j, k) - \ln v(j, k) - 1 \big), \quad r \geq n_k, \quad k = 1, 2, \ldots,$$

where $v(n, k)$ is defined by (2.2). *Then Eq.* (1.1) *is oscillatory.*

Proof. Assume that $y(n)$ is an eventually positive solution of (1.1) for sufficiently large n. Let $n_0 \geq 0$ be an integer such that $y(n) > 0$, for all $n \geq n_0$. Then, there exists $n_1 \geq n_0$ such that (2.3) holds, which yields

$$\Delta y(n) \leq -\sum_{i=1}^{m} p_i(n) y(\tau(n)) = -\tilde{p}(n) y(\tau(n)), \quad \text{for all } n \geq n_1. \tag{2.7}$$

Now, dividing Eq. (2.7) by $y(n)$ and summing up from r to $n-1$, we get

$$\sum_{j=r}^{n-1} \frac{\Delta y(j)}{y(j)} \leq -\sum_{j=r}^{n-1} \tilde{p}(j) \frac{y(\tau(j))}{y(j)}. \tag{2.8}$$

Notice that $e^x \ge x + e^a - a$, for all $x \le a \le 0$. Moreover, the sequence $\{n_k\}$ of Lemma 2.1 exists. Then using (2.1), we have

$$\begin{aligned}\exp\Big(\ln\frac{y(j+1)}{y(j)}\Big) &\ge \ln\frac{y(j+1)}{y(j)} + \exp(\ln v(j,k)) - \ln v(j,k)\\ &= \ln\frac{y(j+1)}{y(j)} + v(j,k) - \ln v(j,k), \quad j \ge n_k.\end{aligned}$$

Now, using the above inequality, we get

$$\begin{aligned}\sum_{j=r}^{n-1}\frac{\Delta y(j)}{y(j)} &= \sum_{j=r}^{n-1}\Big(\frac{y(j+1)}{y(j)} - 1\Big)\\ &= \sum_{j=r}^{n-1}\Big(\exp\Big(\ln\frac{y(j+1)}{y(j)}\Big) - 1\Big)\\ &\ge \sum_{j=r}^{n-1}\Big(\ln\frac{y(j+1)}{y(j)} + v(j,k) - \ln v(j,k) - 1\Big)\\ &= \ln\frac{y(n)}{y(r)} + A(n,r,k), \quad r \ge n_k.\end{aligned}$$

Combining this together with (2.8), we obtain

$$\ln\frac{y(n)}{y(r)} + A(n,r,k) \le -\sum_{j=r}^{n-1}\tilde p(j)\frac{y(\tau(j))}{y(j)}, \quad r \ge n_k.$$

Hence,

$$y(r) \ge y(n)\exp\Big(\sum_{j=r}^{n-1}\tilde p(j)\frac{y(\tau(j))}{y(j)} + A(n,r,k)\Big), \quad n \ge r \ge n_k. \tag{2.9}$$

Summing up (2.7) from $\tau(n)$ to $n-1$ and rearranging, we obtain

$$y(n) - y(\tau(n)) + \sum_{i_1=\tau(n)}^{n-1}\tilde p(i_1)y(\tau(i_1)) \le 0. \tag{2.10}$$

From (2.9) and the fact that $\tau(i_1) \le n$, the last inequality takes the form

$$\begin{aligned}y(n) - y(\tau(n)) + y(n)\sum_{i_1=\tau(n)}^{n-1}\tilde p(i_1)\exp\Big(&\sum_{j=\tau(i_1)}^{n-1}\tilde p(j)\frac{y(\tau(j))}{y(j)}\\ &+ A(n,\tau(i_1),k)\Big) \le 0, \quad n \ge n_k.\end{aligned}$$

Moreover, multiplying the above inequality by $\tilde p(n)$, for $n \ge n_k$ we get

$$\begin{aligned}\tilde p(n)y(n) - \tilde p(n)y(\tau(n)) + \tilde p(n)y(n)\sum_{i_1=\tau(n)}^{n-1}\tilde p(i_1)\exp\Big(&\sum_{j=\tau(i_1)}^{n-1}\tilde p(j)\frac{y(\tau(j))}{y(j)}\\ &+ A(n,\tau(i_1),k)\Big) \le 0.\end{aligned}$$

This and (2.7) give

$$\Delta y(n)+\tilde{p}(n)\Big(1+\sum_{i_1=\tau(n)}^{n-1}\tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1}\tilde{p}(j)\frac{y(\tau(j))}{y(j)} \\ +A(n,\tau(i_1),k)\Big)\Big)y(n)\le 0,\quad n\ge n_k. \tag{2.11}$$

Assuming that n_0 is sufficiently large that

$$y(\gamma(n))>(\lambda_0-\epsilon)y(n),\quad \text{for all } n\ge n(\epsilon)\ge n_k$$

and each $\epsilon>0$. Then

$$y(\tau(n))>(\lambda_0-\epsilon)y(n),\quad \text{for all } n\ge n_k. \tag{2.12}$$

Combining (2.11) and (2.12), we get

$$\Delta y(n)+F_0(n,\epsilon)y(n)\le 0,\quad n\ge n_k,$$

where

$$F_0(n,\epsilon)=\tilde{p}(n)\left(1+\sum_{i_1=\tau(n)}^{n-1}\tilde{p}(i_1)\exp\Big((\lambda_0-\epsilon)\sum_{j=\tau(i_1)}^{n-1}\tilde{p}(j)+A(n,\tau(i_1),k)\Big)\right).$$

That is, $0<\frac{y(n+1)}{y(n)}\le 1-F_0(n,\epsilon)$, for $n\ge n_k$. Taking the product on both sides, we obtain

$$y(r)\ge y(n)\prod_{j=r}^{n-1}\frac{1}{1-F_0(j,\epsilon)},\quad \text{for all } n\ge r\ge n_k.$$

It follows that $y(\tau(j))\ge y(j)\prod_{i_2=\tau(j)}^{j-1}\frac{1}{1-F_0(i_2,\epsilon)}$, for all $j\ge n_{k+1}$. Substituting into (2.9), we have

$$y(r)\ge y(n)\exp\Big(\sum_{j=r}^{n-1}\tilde{p}(j)\prod_{i_2=\tau(j)}^{j-1}\frac{1}{1-F_0(i_2,\epsilon)}+A(n,r,k)\Big),\quad r\ge n_{k+1}. \tag{2.13}$$

From (2.10) and (2.13), we obtain

$$y(n)-y(\tau(n))+y(n)\sum_{i_1=\tau(n)}^{n-1}\tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1}\tilde{p}(j)\prod_{i_2=\tau(j)}^{j-1}\frac{1}{1-F_0(i_2,\epsilon)} \\ +A(n,\tau(i_1),k)\Big)\le 0,\quad n\ge n_{k+3}.$$

Multiplying the above inequality by $\tilde{p}(n)$, for $n\ge n_{k+3}$ we get

$$\tilde{p}(n)y(n)-\tilde{p}(n)y(\tau(n))+\tilde{p}(n)y(n)\sum_{i_1=\tau(n)}^{n-1}\tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1}\tilde{p}(j)\prod_{i_2=\tau(j)}^{j-1}\frac{1}{1-F_0(i_2,\epsilon)} \\ +A(n,\tau(i_1),k)\Big)\le 0.$$

This and (2.7) give

$$\Delta y(n) + \tilde{p}(n)\Bigg(1 + \sum_{i_1=\tau(n)}^{n-1} \tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-F_0(i_2,\epsilon)} + A(n,\tau(i_1),k)\Big)\Bigg)y(n) \le 0,$$

for $n \ge n_{k+3}$. Then

$$\Delta y(n) + F_1(n,\epsilon)y(n) \le 0, \quad n \ge n_{k+3}, \tag{2.14}$$

where

$$F_1(n,\epsilon) = \tilde{p}(n)\Bigg(1 + \sum_{i_1=\tau(n)}^{n-1} \tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-F_0(i_2,\epsilon)} + A(n,\tau(i_1),k)\Big)\Bigg).$$

That is, $0 < \frac{y(n+1)}{y(n)} \le 1 - F_1(n,\epsilon)$, for all $n \ge n_{k+3}$. Taking the product on both sides, we have

$$y(r) \ge y(n) \prod_{j=r}^{n-1} \frac{1}{1-F_1(j,\epsilon)}, \quad \text{for all } n \ge r \ge n_{k+3}.$$

It follows that

$$y(\tau(j)) \ge y(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-F_1(i_2,\epsilon)}, \quad j \ge n_{k+4}.$$

Substituting into (2.9), we obtain

$$y(r) \ge y(n)\exp\Big(\sum_{j=r}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-F_1(i_2,\epsilon)} + A(n,r,k)\Big), \quad r \ge n_{k+4}. \tag{2.15}$$

Applying the same strategy implying (2.14), we arrive at the inequality

$$\Delta y(n) + F_2(n,\epsilon)y(n) \le 0, \quad n \ge n_{k+6}.$$

Now, by induction, we obtain $\Delta y(n) + F_l(n,\epsilon)y(n) \le 0$, for all $n \ge n_{k+3l}$, where

$$F_l(n,\epsilon) = \tilde{p}(n)\Bigg(1 + \sum_{i_1=\tau(n)}^{n-1} \tilde{p}(i_1)\exp\Big(\sum_{j=\tau(i_1)}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-F_{l-1}(i_2,\epsilon)} + A(n,\tau(i_1),k)\Big)\Bigg).$$

Therefore, in a similar fashion as (2.15), we obtain

$$y(r) \geq y(n) \exp\Big(\sum_{j=r}^{n-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-\mathcal{F}_l(i_2,\epsilon)} + A(n,r,k)\Big), \quad n \geq n_{k+3l+1}. \tag{2.16}$$

Again, summing up (2.7) from $\gamma(n)$ to n and rearranging, we obtain

$$y(n+1) - y(\gamma(n)) + \sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) y(\tau(i_1)) \leq 0.$$

From (2.16), taking into account the fact $\tau(i_1) \leq \gamma(i_1) \leq \gamma(n)$, we obtain

$$\begin{aligned} y(n+1) - y(\gamma(n)) + y(\gamma(n)) \sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-\mathcal{F}_l(i_2,\epsilon)} \\ + A(\gamma(n),\tau(i_1),k)\Big) \leq 0, \end{aligned}$$

that is,

$$\sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-\mathcal{F}_l(i_2,\epsilon)} + A(\gamma(n),\tau(i_1),k)\Big) \leq 1 - \frac{y(n+1)}{y(\gamma(n))},$$

for $n \geq n_{k+2(l+1)}$. Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \Bigg(\sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1-\mathcal{F}_l(i_2,\epsilon)} \\ + A(\gamma(n),\tau(i_1),k)\Big)\Bigg) \leq 1 - D(\alpha). \end{aligned}$$

This contradicts (2.6). The proof is complete. □

Remark 2.3. Since $A(\gamma(n),\tau(i_1),k)$ is positive and $\mathcal{F}_l(n,\epsilon)$ increases in $A(\gamma(n),\tau(i_1),k)$, then its lower bound, with respect to A, equals $Z_w(n)$ for sufficiently small ϵ. This means that the left-hand side of (2.6) is greater than the left-hand side of (1.7). Thus condition (2.6) is a substantial improvement of (1.7).

In the next example we show the strength of our condition over some known oscillation criteria including (1.7). First, we collect those criteria as follows:

(1) Chatzarakis, Horvat-Dmitrović and Pašić [8]:

$$\limsup_{n\to\infty} \sum_{j=\gamma(n)}^{n} \tilde{p}(j) \prod_{i=\tau(j)}^{\gamma(n)-1} \frac{1}{1-\tilde{p}_w(i)} > 1, \tag{2.17}$$

where $w \in \mathbb{N}$ and

$$\tilde{p}_w(n) = \tilde{p}(n)\left[1 + \sum_{i=\tau(n)}^{n-1} \tilde{p}(i) \prod_{j=\tau(i)}^{\gamma(n)-1} \frac{1}{1-\tilde{p}_{w-1}(j)}\right],$$

with $\tilde{p}_0(n) = \tilde{p}(n)$.

(2) Chatzarakis and Pašić [7]:

$$\limsup_{n\to\infty} \sum_{l=\gamma(n)}^{n} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\widetilde{R}_w(i)}\right) > 1, \quad \text{for some } w \in \mathbb{N}, \tag{2.18}$$

where

$$\widetilde{R}_w(n) = \tilde{p}(n)\left[1 + \sum_{l=\tau(n)}^{n-1} \tilde{p}(l) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \tilde{p}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\widetilde{R}_{w-1}(i)}\right)\right]$$

with $\widetilde{R}_0(n) = \tilde{p}(n)\left[1 + \lambda_0 \sum_{l=\tau(n)}^{n-1} \tilde{p}(\ell)\right]$.

(3) Kilic and Ocalan [14]:

$$\liminf_{n\to\infty} \left(\sum_{j=\tau(n)}^{n-1} \left(\sum_{i=1}^{m} \left(p_i(j) \prod_{l=\tau_i(j)}^{\gamma(j)-1} (1-\tilde{p}(l))^{-1}\right)\right)\right) > \frac{1}{\mathrm{e}}. \tag{2.19}$$

Example 2.4. Consider the first order difference equation with several delays

$$\Delta y(n) + p_1(n)y(\tau_1(n)) + p_2(n)y(\tau_2(n)) = 0, \tag{2.20}$$

where $p_1(n) = 0.17$, $p_2(n) = 0.005$,

$$\tau_1(n) = \begin{cases} n-1, & \text{if } n = 3k, \\ n-3, & \text{if } n = 3k+1, \\ n-1, & \text{if } n = 3k+2, \end{cases}$$

and $\tau_2(n) = \tau_1(n) - 1$, where $k \in \mathbb{N}_0$. Clearly,

$$\gamma_1(n) = \begin{cases} n-1, & \text{if } n = 3k, \\ n-2, & \text{if } n = 3k+1, \\ n-1, & \text{if } n = 3k+2, \end{cases}$$

and

$$\alpha := \liminf_{n\to\infty} \sum_{j=\tau(3k)}^{3k-1} \tilde{p}(j) = 0.175 < 1.$$

It follows that $\lambda_0 \approx 1.242995532$ and $1 - D(\alpha) \approx 0.9810019060$. Since $v(n,0) = 1$ and (2.2), we get

$$v(n,1) = 1 - \tilde{p}(n) \prod_{j=\tau(3k)}^{3k-1} v^{-1}(j,0) = 1 - \sum_{j=1}^{2} p_j(n) = 1 - 0.175 = 0.825.$$

Then

$$A(n,r,1) = \sum_{j=r}^{n-1} \big(v(j,1) - \ln v(j,1) - 1\big) = 0.0173718926(n-r).$$

Also, for $\epsilon = 0.0001$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i_1=\gamma(n)}^{n} \tilde{p}(i_1) \exp\Big(\sum_{j=\tau(i_1)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1 - F_1(i_2,\epsilon)} + A(\gamma(n),\tau(i_1),1)\Big) \\
&= \lim_{k\to\infty} \sum_{i_1=\gamma(3k+2)}^{3k+2} \tilde{p}(i_1) \exp\Bigg(\sum_{j=\tau(i_1)}^{\gamma(3k+2)-1} \tilde{p}(j) \prod_{i_2=\tau(j)}^{j-1} \frac{1}{1 - F_1(i_2,\epsilon)} \\
&\qquad\qquad + A(\gamma(3k+2),\tau(i_1),1)\Bigg) \\
&= 1.06 > 1 - D(\alpha).
\end{aligned}$$

Then according to Theorem 2.2, for $l = 1$ the equation is oscillatory.

Now we show that other criteria fail to show that (2.20) is oscillatory. All calculation are done using Mathematica software. First, we see that (2.17) fails, for $w = 2$, since

$$\limsup_{n\to\infty} \sum_{l=\gamma_1(n)}^{n} \tilde{p}(l) \prod_{i=\tau(\ell)}^{\gamma(n)-1} \frac{1}{1 - \tilde{p}_w(i)} = \lim_{k\to\infty} \sum_{l=\gamma_1(3k+2)}^{3k+2} 0.175 \prod_{i=\tau(\ell)}^{\gamma(3k+2)-1} \frac{1}{1 - \tilde{p}_2(i)} < 0.638.$$

Second, (2.18) fails, for $w = 1$, since

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{l=\gamma(n)}^{n} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{\gamma(n)-1} \tilde{p}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1 - \widetilde{R}_w(i)}\right) \\
&= \lim_{k\to\infty} \sum_{l=\gamma_1(3k+2)}^{3k+2} 0.175 \exp\left(\sum_{j=\tau(l)}^{\gamma_1(3k+2)-1} 0.175 \prod_{i=\tau(j)}^{j-1} \frac{1}{1 - \widetilde{R}_1(i)}\right) < 0.814.
\end{aligned}$$

Also, (1.7) fails, for $w = 1$, since

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\ell=\gamma_1(n)}^{n} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{\gamma_1(n)-1} \tilde{p}(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1 - Z_1(u)}\right) \\
&= \lim_{k\to\infty} \sum_{\ell=\gamma_1(3k+1)}^{3k+1} \tilde{p}(l) \exp\left(\sum_{j=\tau(l)}^{\gamma_1(3k+1)-1} \tilde{p}(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1 - Z_1(u)}\right) < 0.908.
\end{aligned}$$

Finally,

$$\begin{aligned}\liminf_{n\to\infty} &\sum_{j=\tau(n)}^{n-1}\left(\sum_{i=1}^{2}\left(p_i(j)\prod_{l=\tau(j)}^{\gamma_1(j)-1}(1-\tilde{p}(l))^{-1}\right)\right)\\ &=\lim_{k\to\infty}\sum_{j=\tau(3\,k)}^{3\,k-1}\left(\sum_{i=1}^{2}\left(p_i(j)\prod_{l=\tau(j)}^{\gamma_1(j)-1}(1-\tilde{p}(l))^{-1}\right)\right)=0.175<\frac{1}{\mathrm{e}}.\end{aligned}$$

Thus, (2.19), which is a limit inferior type criterion, fails. This highlights the broad applicability and strength of our criterion.

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