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# THE CROSSING NUMBERS OF JOIN PRODUCTS OF FOUR GRAPHS OF ORDER FIVE WITH PATHS AND CYCLES

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**Abstract.** The crossing number $\mathrm{cr}(G)$ of a graph $G$ is the minimum number of edge crossings over all drawings of $G$ in the plane. In the paper, we extend known results concerning crossing numbers of join products of four small graphs with paths and cycles. The crossing numbers of the join products $G^* + P_n$ and $G^* + C_n$ for the disconnected graph $G^*$ consisting of the complete tripartite graph $K_{1,1,2}$ and one isolated vertex are given, where $P_n$ and $C_n$ are the path and the cycle on $n$ vertices, respectively. In the paper also the crossing numbers of $H^* + P_n$ and $H^* + C_n$ are determined, where $H^*$ is isomorphic to the complete tripartite graph $K_{1,1,3}$. Finally, by adding new edges to the graphs $G^*$ and $H^*$, we are able to obtain crossing numbers of join products of two other graphs $G_1$ and $H_1$ with paths and cycles.

**Keywords:** graph, crossing number, join product, path, cycle, separating cycle.

**Mathematics Subject Classification:** 05C10, 05C38.

## 1. INTRODUCTION

The crossing number is an important parameter of a graph, as it provides information about the complexity of the graph and the difficulty of visualizing it [28]. In addition, the crossing number is related to many other graph parameters and algorithms, such as planarity testing, graph coloring, and graph embedding. Graphs are widely used to represent complex networks such as social, communication, and transportation networks. Reducing the number of edge crossings in network visualizations can help understand the network's underlying structure and identify important nodes and connections [1]. In electronic circuit design, minimizing the number of edge crossings is important for reducing signal interference and improving circuit performance. Graph drawings with fewer crossings can lead to more efficient and reliable circuit designs [26]. In graph theory, reducing the number of edge crossings is a fundamental problem in planar graph theory. Many important graph algorithms and optimization problems are defined for planar graphs, where the graphs have no edge crossings [10]. Overall, reducing the number of crossings on graph edges can help in visualizing and understanding complex data, improving system performance, and optimizing graph algorithms and optimizations [8]. Examining the number of crossings of simple graphs is a classic, but still challenging problem. Garey and Johnson [9] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Note that the exact values of the crossing numbers are known for some families of graphs, see Clancy *et al.* [4].

The *crossing number* $\mathrm{cr}(G)$ of a simple graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of $G$ in the plane (for the definition of a *drawing* see Klešč [14]). A drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let $D$ be a good drawing of the graph $G$. We denote the number of crossings in $D$ by $\mathrm{cr}_D(G)$. Let $G_i$ and $G_j$ be edge-disjoint subgraphs of $G$. We denote the number of crossings between edges of $G_i$ and edges of $G_j$ by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of $G_i$ in $D$ by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs $G_i$, $G_j$, and $G_k$ of $G$ by [14], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

Throughout this paper, Kleitman's result [13] on the crossing numbers for some complete bipartite graphs $K_{m,n}$ are used in several parts of proofs. He proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if } m \leq 6. \tag{1.1}$$

The join product of two graphs $G_i$ and $G_j$, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of $G_i$ and $G_j$ by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs $G_i$, $G_j$, and the complete bipartite graph $K_{m,n}$. Let $P_n$ and $C_n$ be the *path* and the *cycle* on $n$ vertices, respectively, and let $D_n$ denote the *discrete graph* (sometimes called *empty graph*) on $n$ vertices. The crossings numbers of join products of paths and cycles with all graphs of order at most four have been well-known for a long time by Klešč [15, 16], and Klešč and Schrötter [18]. It is understandable that our immediate aim is to establish exact values for crossing numbers of $G + P_n$ and $G + C_n$ also for all graphs $G$ of order five and six. Of course, the crossing numbers of $G + P_n$ and $G + C_n$ are already known for a lot of graphs $G$ of order five and six [2, 3, 6, 7, 14, 17, 20, 21, 27, 30, 31, 35]. In all these cases, the graph $G$ is connected and usually contains at least one cycle. Note that $\mathrm{cr}(G + P_n)$ and $\mathrm{cr}(G + C_n)$ are known only for some disconnected graphs $G$ on five or six vertices [19, 32–34]. To date, the crossing number of $K_3 \cup 2K_1 + P_n$, $K_3 \cup 2K_1 + C_n$, $K_4 \cup K_1 + C_n$, $K_{1,1,2} \cup K_1 + P_n$ and $K_{1,1,2} \cup K_1 + C_n$ can only be given as a conjecture. The last two open problems will be solved in our paper. The minimal number of crossings in the Cartesian product and in the strong product of paths have been studied by Klešč *et al.* in [22] and [23].

For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known. It is appropriate to combine this idea with possibility of an existence of a separating cycle in some particular drawing of investigated graph. Let $G^*$ be the disconnected graph consisting of the complete tripartite graph $K_{1,1,2}$ and one isolated vertex. The crossing numbers of the join products of $G^*$ with the discrete graphs $D_n$ have been well-known by Klešč *et al.* [25] using a lot of properties of cyclic permutations. This established result will be extended to the same crossing number of $G^* + P_n$ in Corollary 2.2 due to two special drawings in Figures 2 and 3 for $n$ even and odd, respectively. Moreover, the result of $\mathrm{cr}(G^* + C_n)$ in the second main Theorem 4.5 can be estimated for cycles $C_n$ on at least 4 vertices provided by adding the edge $t_1t_3$ only with one additional crossing to the subdrawing of $G^* + P_3$ in Figure 3 offers a drawing of $G^* + C_3$ with just 7 crossings.

Let $H^*$ be the graph isomorphic to the complete tripartite graph $K_{1,1,3}$. The crossing number of $H^* + D_n$ was determined for any $n \geq 1$ by Ho [12], and later also by Staš [29] again thanks to properties of cyclic permutations. One of the main goals of the paper is to establish the crossing numbers of the join products of $H^*$ with paths $P_n$ and cycles $C_n$. The obtained results will be presented in Theorem 3.5 and Theorem 4.9. The paper concludes by giving $\mathrm{cr}(G_1 + P_n)$, $\mathrm{cr}(G_1 + C_n)$, $\mathrm{cr}(H_1 + P_n)$, and $\mathrm{cr}(H_1 + C_n)$ in Corollaries 5.1, 5.2, 5.3, and 5.4, respectively, where the graph $G_1$ is obtained from $G^*$ by adding new edge joining one vertex of degree two with the isolated vertex in $G^*$ and the graph $H_1$ is obtained from $H^*$ by adding new edge joining two vertices of degree two. Note that the result in Theorem 3.5 has already been claimed by Su and Huang [36]. Since this paper does not seem to be available in English, we have not been able to verify this result. Clancy *et al.* [4] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals do not have a sufficiently rigorous peer-review process.

## 2. THE CROSSING NUMBER OF $G^* + P_n$

Let $G^* = (V(G^*), E(G^*))$ be the disconnected graph on five vertices consisting of the complete tripartite graph $K_{1,1,2}$ and one isolated vertex, and let also $V(G^*) = \{v_1, v_2, \ldots, v_5\}$. In the rest of the paper, let $v_5$ be the vertex notation of the isolated vertex of $G^*$ in all considered good subdrawings of the graph $G^*$. Five possible non-isomorphic drawings of $G^*$ were described by Klešč *et al.* [25]. They are presented in Figure 1 with the corresponding vertex notation in two drawings.

We consider the join product of the graph $G^*$ with the discrete graph $D_n$, which yields that $G^* + D_n$ (sometimes used notation $G^* + nK_1$) consists of just one copy of $G^*$ and $n$ vertices $t_1, t_2, \ldots, t_n$. Here, each vertex $t_i$, $i = 1, 2, \ldots, n$, is adjacent to every vertex of the graph $G^*$. Let $T^i$, $1 \leq i \leq n$, denote the subgraph induced by five edges incident with the fixed vertex $t_i$. This means that the graph $T^1 \cup \ldots \cup T^n$ is isomorphic to the complete bipartite graph $K_{5,n}$ and

$$G^* + D_n = G^* \cup \left( \bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

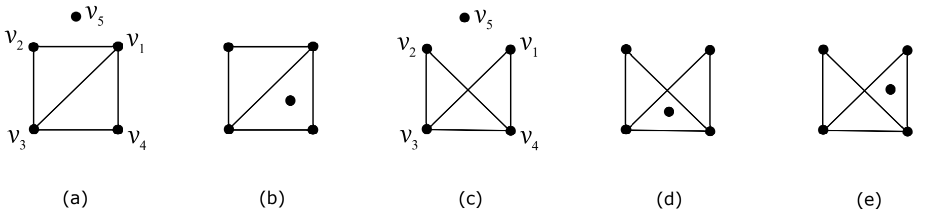


**Fig. 1.** Five possible non-isomorphic drawings of the graph $G^*$

Determining the crossing numbers of $G^* + P_n$ and $G^* + C_n$ will be based on the following theorem presented in [25].

**Theorem 2.1** ([25, Theorem 2]). $\mathrm{cr}(G^* + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

For $n$ even, Figure 2 shows the good drawing of the join product $G^* + P_n$ with exactly $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings provided by edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times and each subgraph $T^i$ crosses edges of $G^*$ just once.

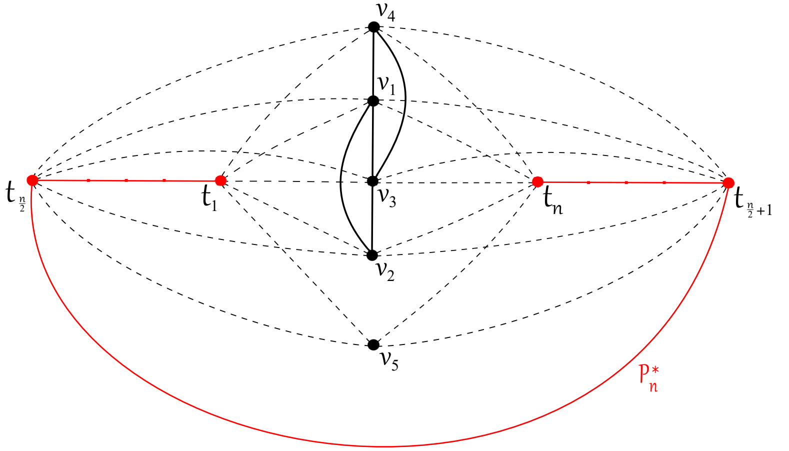


**Fig. 2.** The drawing of $G^* + P_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings for $n$ even

For $n$ odd at least 3, Figure 3 shows the good drawing of $G^* + P_n$ also with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings by adding one subgraph $T^{\frac{n+1}{2}}$ by which edges of each of the $n-1$ subgraphs $T^i$, $i \neq \frac{n+1}{2}$ are crossed exactly twice, that is,

$$4\,\frac{n-1}{2}\,\frac{n-3}{2}+2\,\frac{n-1}{2}+2(n-1)=4\,\frac{n-1}{2}\,\frac{n-1}{2}+2\,\frac{n-1}{2}.$$

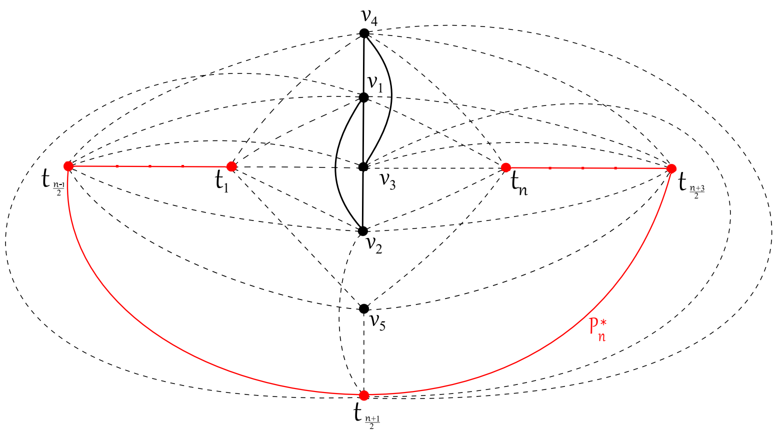


**Fig. 3.** The drawing of $G^*+P_n$ with $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ crossings for $n$ odd at least three

As $G^*+D_n$ is a subgraph of $G^*+P_n$, the lower bound is the same based on Theorem 2.1 and so, the next result is obvious.

**Corollary 2.2.** $\mathrm{cr}(G^*+P_n)=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 2$.

## 3. THE CROSSING NUMBER OF $H^*+P_n$

The graph $H^*+P_n$ contains $H^*+D_n$ as a subgraph. For subgraphs of $H^*+P_n$ which are also subgraphs of $H^*+nK_1$ we use the same notations as above. Let $P_n^*$ denote the path induced on $n$ vertices of $H^*+P_n$ not belonging to the subgraph $H^*$. Hence, $P_n^*$ consists of the vertices $t_1,t_2,\dots,t_n$ and the edges $\{t_i,t_{i+1}\}$ for $i=1,2,\dots,n-1$. One can easily see that

$$H^*+P_n=H^*\cup\left(\bigcup_{i=1}^{n}T^i\right)\cup P_n^*. \tag{3.1}$$

We consider a good drawing $D$ of $H^*+D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex $t_i$ in the drawing $D$ as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave $t_i$ have been defined by Hernández-Vélez et al. [11] or Woodall [37]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex $t_i$ is $t_iv_1$, $t_iv_2$, $t_iv_3$, $t_iv_4$, and $t_iv_5$. We recall that rotation is a cyclic permutation. In the given drawing $D$, it is highly desirable to separate $n$ subgraphs $T^i$ into three mutually disjoint families of subgraphs depending on how many times edges of $H^*$ could be crossed by $T^i$ in $D$. Let us denote by $R_D$ and $S_D$

the set of subgraphs for which $\mathrm{cr}_D(H^*, T^i) = 0$ and $\mathrm{cr}_D(H^*, T^i) = 1$, respectively. Edges of $H^*$ are crossed by each remaining subgraph $T^i$ at least twice in $D$. For $T^i \in R_D \cup S_D$, let $F^i$ denote the subgraph $H^* \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $H^* + D_n$ and let $D(F^i)$ be its subdrawing induced by $D$. Clearly, the idea of dividing the subgraphs $T^i$ into three mentioned families is also retained in all drawings of $H^* + P_n$.

In a good drawing $D$ of some graph $G$, we say that a cycle $C$ *separates* some two different vertices of the subgraph $G \setminus C$ if they are contained in different components of $\mathbb{R}^2 \setminus C$. This considered cycle $C$ is said to be a *separating* cycle of the graph $G$ in $D$. In some proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

**Lemma 3.1.** *For $n \geq 2$, let $D$ be a good drawing of $H^* + P_n$. If there is a separating cycle $C_3$ of $H^*$ in the subdrawing $D(H^*)$ with at least one crossing on some edge of $C_3$ in $D(H^* \cup P_n^*)$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings in $D$.*

*Proof.* By assumption, let us consider a separating cycle $C_3 = v_1v_2v_3v_1$ of $H^*$ in the subdrawing $D(H^*)$. Since two remaining vertices of $H^*$ lie in different regions of $D(C_3)$, there is no subgraph $T^i$ by which the edges of $C_3$ are not crossed. Hence, each subgraph $T^i$ crosses edges of $C_3$ at least once, which yields that $\mathrm{cr}_D(C_3, \bigcup_{i=1}^n T^i) \geq n$. Let $G$ be the graph difference of graphs $H^*$ and $C_3$, i.e., $G$ is isomorphic to the graph $K_{2,2} \cup K_1$. The exact value for the crossing number of $G + P_n$ is given by Staš and Petrillová [33], that is, $\mathrm{cr}(G + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$. This enforces at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 + n + 1$ crossings in $D$ because some edge of the cycle $C_3$ is also crossed in $D(H^* \cup P_n^*)$. □

As the graph $G^*$ is a subgraph of the graph $H^*$, all considered drawings of $H^*$ can be achieved from some drawing of $G^*$ in such a way as shown in Figure 1 by adding two edges joining the isolated vertex with two vertices of degree three in $G^*$. In an effort to reach less than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings in some good drawing $D$ of $H^* + P_n$, it is sufficient to investigate subdrawings of $H^*$ induced by $D$ in which either all five vertices of $H^*$ are placed in one region of $D(H^*)$ on its boundary or there is no crossing on any edge of separating cycle $C_3$ of $H^*$ due to Lemma 3.1. Hence, we obtain only three possible non-isomorphic drawings of $H^*$ given in Figure 4 satisfying these restrictions.

Two vertices $t_i$ and $t_j$ of the graph $H^* + P_n$ are said to be *antipodal* in a drawing of $H^* + P_n$ if the considered subgraphs $T^i$ and $T^j$ do not cross. A drawing with no antipodal vertices is said to be *antipode-free*.

**Lemma 3.2.** *For $n \geq 4$, let $D$ be a good and antipode-free drawing of $H^* + P_n$ with the subdrawing of $H^*$ induced by $D$ given in Figure 4(b). If $|R_D| \geq 1$ or $|S_D| < 2\lceil \frac{n-1}{2} \rceil$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings in $D$.*

*Proof.* Let $T^i$ be a subgraph from the nonempty set $R_D$. It is easy to verify that the subgraph $F^i = H^* \cup T^i$ is represented by $\mathrm{rot}_D(t_i) = (14352)$ and each of the $n - 1$

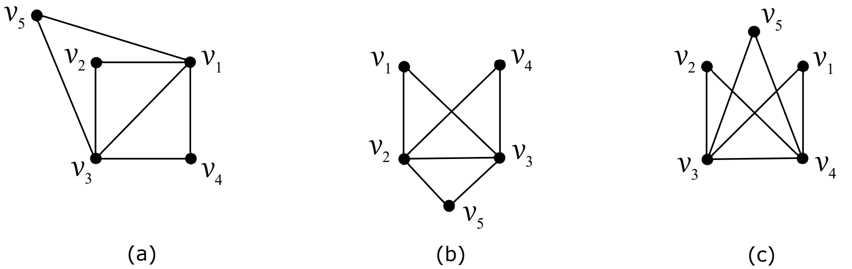


**Fig. 4.** Three possible non-isomorphic drawings of the graph $H^*$: (**a**) the planar drawing of $H^*$ with a separating cycle $C_3$; (**b**) the drawing of $H^*$ with $\mathrm{cr}_D(H^*) = 1$ and all five vertices of $H^*$ are placed in one region; (**c**) the drawing of $H^*$ with $\mathrm{cr}_D(H^*) = 3$ and all five vertices of $H^*$ are placed in one region

remaining subgraphs $T^j$ crosses its edges at least four times over all possible regions of $D(H^* \cup T^i)$. Thus, by fixing the subgraph $H^* \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(H^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, H^* \cup T^i) + \mathrm{cr}_D(H^* \cup T^i)\\ &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4(n-1) + 1\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + \Big\lfloor\frac{n}{2}\Big\rfloor + 2.\end{aligned}$$

If the set $R_D$ is empty, let us assume $|S_D| < 2\lceil\frac{n-1}{2}\rceil$. The edges of $H^* \setminus \{v_2v_3\}$ are crossed once and at least twice by any subgraph $T^j \in S_D$ and $T^j \notin S_D$, respectively. So

$$\begin{aligned}\mathrm{cr}_D\Big(H^* \setminus \{v_2v_3\}, \bigcup_{j=1}^{n} T^j\Big) &\geq |S_D| + 2(n - |S_D|)\\ &= 2n - |S_D| > 2n - 2\Big\lceil\frac{n-1}{2}\Big\rceil = 2\Big\lceil\frac{n}{2}\Big\rceil,\end{aligned}$$

which yields that three edges $v_1v_2$, $v_2v_4$, $v_2v_5$ or $v_1v_3$, $v_3v_4$, $v_3v_5$ are crossed more than $\lceil\frac{n}{2}\rceil$ times by all $n$ subgraphs $T^j$. In the rest of the paper, based on their symmetry, let there be at least $\lceil\frac{n}{2}\rceil + 1$ crossings on the edges $v_1v_2$, $v_2v_4$, and $v_2v_5$ over all subgraphs $T^j$. By removing them from the graph $H^*$, we obtain a subgraph isomorphic to the complete bipartite graph $K_{1,4}$. The exact value for the crossing number of $K_{1,4} + P_n$ is given by Staš [30], that is, $\mathrm{cr}(K_{1,4} + P_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$. This enforces $\mathrm{cr}_D(H^* + P_n) \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + \lceil\frac{n}{2}\rceil + 1 + 1$ because the edge $v_2v_4$ is also crossed in $D(H^*)$. □

In the proofs of the paper, several parts are based on the previous Lemma 3.1 and on Theorem 3.3 presented in [29].

**Theorem 3.3** ([29, Theorem 1]). $\mathrm{cr}(H^* + D_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor$ *for* $n \geq 1$.

In the following, we are able to compute the exact values of crossing numbers of the join products of the graph $H^*$ with both paths $P_2$ and $P_3$ using the algorithm located on the website `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [5]. Unfortunately, the capacity of this system is restricted.

**Lemma 3.4.** $\mathrm{cr}(H^* + P_2) = 5$ *and* $\mathrm{cr}(H^* + P_3) = 10$.

**Theorem 3.5.** $\mathrm{cr}(H^* + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 2$.

*Proof.* In Figure 5, the edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph $T^i$, $i = 1, \ldots, \lceil \frac{n}{2} \rceil$ on the left side crosses edges of the graph $H^*$ once and each subgraph $T^i$, $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n$ on the right side crosses edges of $H^*$ exactly twice. The path $P_n^*$ crosses $H^*$ twice, and so $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings appear among edges of the graph $H^* + P_n$ in this drawing. Thus, $\mathrm{cr}(H^* + P_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$.

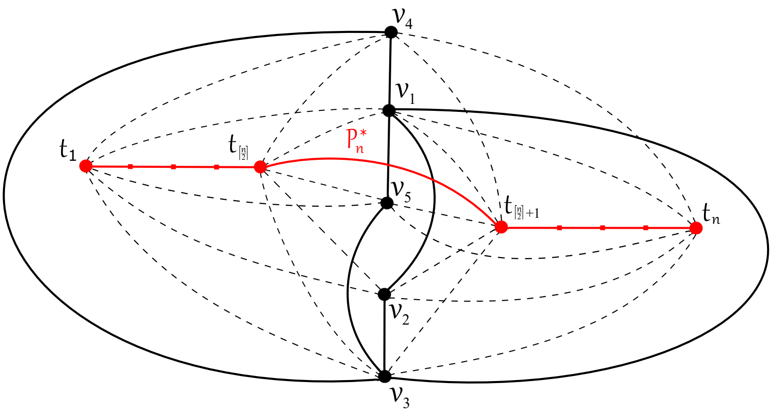


**Fig. 5.** The good drawing of $H^* + P_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings

By Lemma 3.4, the result is true for $n = 2$ and $n = 3$. To prove the reverse inequality by induction on $n$, suppose now that there is a good drawing $D$ of $H^* + P_n$ with

$$\mathrm{cr}_D(H^* + P_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{for some } n \geq 4 \tag{3.2}$$

and that

$$\mathrm{cr}(H^* + P_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor + 2 \quad \text{for any integer } m < n. \tag{3.3}$$

We first show that the considered drawing $D$ must be antipode-free. For this purpose, let $\mathrm{cr}_D(T^i, T^j) = 0$ hold for two different subgraphs $T^i$ and $T^j$. If at least one of $T^i$ and $T^j$, say $T^i$, does not cross $H^*$, it is not difficult to verify in Figure 4(b) and (c) that $T^j$ must cross $H^* \cup T^i$ at least three times. Using positive lower bounds for the number of crossings of two configurations in [29] (they will also be listed in Table 1), one can easily to verify that $\{T^i, T^j\} \not\subseteq S_D$, that is, $\mathrm{cr}_D(H^*, T^i \cup T^j) \geq 3$. We already know using (1.1) that $\mathrm{cr}_D(K_{5,3}) \geq 4$, which yields that edges of $T^i \cup T^j$ are crossed by each other subgraph $T^k$, $k \neq i, j$ at least four times. So, the number of crossings in $D$ satisfies

$$\begin{aligned}\mathrm{cr}_D(H^* + P_n) \geq\ & \mathrm{cr}_D(H^* + P_{n-2}) + \mathrm{cr}_D(T^i \cup T^j) + \mathrm{cr}_D(H^*, T^i \cup T^j) \\ & + \mathrm{cr}_D(K_{5,n-2}, T^i \cup T^j) \\ \geq\ & 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + n - 2 + \left\lfloor \frac{n-2}{2} \right\rfloor + 2 \\ & + 0 + 3 + 4(n-2) \\ =\ & 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2.\end{aligned}$$

The obtained contradiction with the assumption (3.2) does not allow the existence of two antipodal vertices, that is, $D$ is an antipode-free drawing. As $H^* + D_n$ is a subgraph of $H^* + P_n$, there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor$ crossings on edges of $H^* + P_n$ due to Theorem 3.3. The vertices $t_i$ of the path $P_n^*$ must be placed at most in two different regions of $D(H^*)$ because at most one edge of $P_n^*$ can be crossed in $D$. If we use the notation $r = |R_D|$ and $s = |S_D|$, then $\mathrm{cr}_D(K_{5,n}) \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ again by (1.1) together with (3.2) force the following relation with respect to edge crossings of the subgraph $H^*$ in $D$:

$$\mathrm{cr}_D(H^*) + 0r + 1s + 2(n - r - s) \leq \mathrm{cr}_D(H^*) + \mathrm{cr}_D(H^*, K_{5,n}) \leq n + \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

that is,

$$\mathrm{cr}_D(H^*) + s + 2(n - r - s) \leq n + \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{3.4}$$

The mentioned inequality (3.4) subsequently enforces $2r + s + 1 \geq \left\lceil \frac{n}{2} \right\rceil + \mathrm{cr}_D(H^*)$. Further, if $\mathrm{cr}_D(H^*) = 0$ and $r = 0$, then $s \geq \left\lceil \frac{n}{2} \right\rceil - 1$. Now, we will deal with possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the drawing $D$ and show that a contradiction with the assumption (3.2) can be obtained in all cases.

*Case 1.* $\mathrm{cr}_D(H^*) = 0$. In this case, we can only suppose the planar drawing of $H^*$ induced by $D$ given in Figure 4(a). Because no face is incident to all five vertices of $H^*$ in $D(H^*)$, there is no possibility to obtain a subdrawing of $H^* \cup T^i$ for a $T^i \in R_D$, that is, $r = 0$. By Lemma 3.1, the edges of $P_n^*$ do not cross any edge of the separating cycle $C_3 = v_1v_3v_5v_1$ in $D$. In the rest of the paper, let us also suppose that all vertices $t_j$ of the path $P_n^*$ are placed in the outer region of the cycle $v_1v_3v_5v_1$. As the set $S_D$ is nonempty using the inequality 3.4, all vertices $t_j$ of subgraphs $T^j \in S_D$ must be placed in the region of $D(H^*)$ with four vertices $v_1$, $v_4$, $v_3$, and $v_5$ of $H^*$ on its boundary. For $T^j \in S_D$, there is only one possible subdrawing of $F^j \setminus v_2$ represented by the subrotation $(1435)$, which yields that there are exactly two ways of obtaining the subdrawing of $H^* \cup T^j$ depending on which of two edges $v_1v_5$ and $v_3v_5$ is crossed by the edge $t_jv_2$. For both cases of $T^j \in S_D$ described by either $(14352)$ or $(14325)$, if all vertices of $P_n^*$ are placed in the same quadrangular region of $D(H^*)$ then it is not difficult to verify in five considered regions of $D(H^* \cup T^j)$ that $\mathrm{cr}_D(H^* \cup T^j, T^k) \geq 4$ is fulfilling for each $T^k$, $k \neq j$. Thus, by fixing the subgraph $H^* \cup T^j$, we have

$$\begin{aligned}
\mathrm{cr}_D(H^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, H^* \cup T^j) + \mathrm{cr}_D(H^* \cup T^j) \\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 \\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2.
\end{aligned}$$

This also contradicts the assumption (3.2). It remains to consider the subcase with an existence of some vertex of $P_n^*$ in the triangular region of $D(H^*)$ with three vertices $v_1$, $v_3$, and $v_4$ of $H^*$ on its boundary. Since some edge of $P_n^*$ is crossed in $D$, there are at least $\lceil \frac{n}{2} \rceil$ subgraphs $T^j$ by which edges of $H^*$ are crossed just once again provided by (3.4) in the form $s + 2(n - r - s) \leq n + \lfloor \frac{n}{2} \rfloor$. Hence, by fixing the subgraph $T^j$, we have

$$\begin{aligned}
\mathrm{cr}_D(H^* + P_n) &\geq \mathrm{cr}_D(H^* + P_{n-1}) + \mathrm{cr}_D(K_{5,n-1}, T^j) + \mathrm{cr}_D(H^*, T^j) \\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor + 2 + 3(s-1) + 1(n-s) + 1 \\
&= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 2s - 1 \\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lceil \frac{n}{2} \right\rceil - 1 \\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2,
\end{aligned}$$

where $\mathrm{cr}_D(T^j, T^k) \geq 3$ is also used for any $T^k \in S_D$, $k \neq j$. Only one interchange of the adjacent elements of (14352) produces the cyclic permutation (14325), and so $\mathrm{cr}_D(T^j, T^k) \geq \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor - 1 = 3$. Note that two different subgraphs from $S_D$ with same rotations produce at least four crossings on their edges.

*Case 2.* $\mathrm{cr}_D(H^*) = 1$. Again using Lemma 3.1, we can consider only the drawing of $H^*$ with the vertex notation in such a way as shown in Figure 4(b). As $r = 0$ and $s \geq 2\lceil \frac{n-1}{2} \rceil \geq 4$ due to Lemma 3.2, there are at least four different subgraphs $T^i$ by which edges of $H^*$ are crossed just once. Thus, we deal with four possible configurations $\mathcal{A}_p$ of $H^* \cup T^i$ belonging to the nonempty set $\mathcal{M}_D$ (they have been already introduced in [29]) and depending on which of four edges of the graph $H^*$ can be crossed by the edge $t_i v_2$ or $t_i v_3$. The lower bounds for number of crossings of two configurations are presented in Table 1 (they were also established in Table 1 of [29]).

For $s \geq 4$ and using the highest values in Table 1, there are at least two different subgraphs $T^k, T^l \in S_D$ with $\mathrm{cr}_D(T^k, T^l) \geq 4$. Thus, by fixing such a considered

**Table 1**

The minimum number of crossings between $T^i$ and $T^j$ such that $\mathrm{conf}(H^* \cup T^i) = \mathcal{A}_p$ and $\mathrm{conf}(H^* \cup T^j) = \mathcal{A}_q$.

| | $\mathcal{A}_1$ | $\mathcal{A}_2$ | $\mathcal{A}_3$ | $\mathcal{A}_4$ |
|---|---|---|---|---|
| $\mathcal{A}_1$ | 4 | 2 | 2 | 2 |
| $\mathcal{A}_2$ | 2 | 4 | 2 | 2 |
| $\mathcal{A}_3$ | 2 | 2 | 4 | 4 |
| $\mathcal{A}_4$ | 2 | 2 | 4 | 4 |

subgraph $T^k$, we have

$$\begin{aligned}\mathrm{cr}_D(H^* + P_n) &\geq \mathrm{cr}_D(H^* + P_{n-1}) + \mathrm{cr}_D(K_{5,n-1}, T^k) + \mathrm{cr}_D(H^*, T^k)\\ &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor\\ &\quad + 2 + 2(s-2) + 4 + 1(n-s) + 1\\ &= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + s + 2\\ &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lceil \frac{n-1}{2} \right\rceil + 2\\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2.\end{aligned}$$

This also confirms a contradiction with the assumption (3.2) in $D$.

*Case 3.* $\mathrm{cr}_D(H^*) \geq 2$. Based on assumptions of Lemma 3.1, let us consider the nonplanar subdrawing of $H^*$ induced by $D$ given in Figure 4(c). For $r \geq 1$, the proof can proceed in the same way as in Lemma 3.2. For $r = 0$ and $s \geq \lceil \frac{n}{2} \rceil + 2 \geq 4$, all vertices $t_i$ of subgraphs $T^i \in S_D$ are placed in the region of $D(H^*)$ with all five vertices of $H^*$ on its boundary. We have only two ways to obtain a subdrawing of $H^* \cup T^i$ depending on which of two edges of the graph $H^*$ can be crossed by the edge $t_iv_3$ or $t_iv_4$. For both these subdrawings, we can also easily verify in ten possible regions of $D(H^* \cup T^i)$ that $\mathrm{cr}_D(H^* \cup T^i, T^j) \geq 4$ holds for each other subgraph $T^j$, $j \neq i$. Hence, all these mentioned subcases again contradict (3.2) in $D$.

We have shown that there is no good drawing $D$ of $H^* + P_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings, and proof of Theorem 3.5 is done. □

## 4. THE CROSSING NUMBERS OF $G^* + C_n$ AND $H^* + C_n$

Let $t_1, t_2, \ldots, t_n, t_1$ be the vertex notation of the $n$-cycle $C_n$ for $n \geq 3$. The join product $G^* + C_n$ consists of one copy of the graph $G^*$, one copy of the cycle $C_n$, and the edges joining each vertex of $G^*$ with each vertex of $C_n$. Let $C_n^*$ denote the cycle as a subgraph of $G^* + C_n$ induced on the vertices of $C_n$ not belonging to the subgraph $G^*$. The subdrawing $D(C_n^*)$ induced by any good drawing $D$ of $G^* + C_n$ represents some drawing of $C_n$. For the vertices $v_1, v_2, \ldots, v_5$ of the graph $G^*$, let $T^{v_i}$ denote the subgraph induced by $n$ edges joining the vertex $v_i$ with $n$ vertices of $C_n^*$. The edges joining the vertices of $G^*$ with the vertices of $C_n^*$ form the complete bipartite graph $K_{5,n}$, and so

$$G^* + C_n = G^* \cup \left(\bigcup_{i=1}^{5} T^{v_i}\right) \cup C_n^*. \tag{4.1}$$

In the proofs of both main theorems of this section, the following three statements related to some restricted subdrawings of $G + C_n$ will be also required.

**Lemma 4.1** ([15, Lemma 2.2])**.** *For $m \geq 2$ and $n \geq 3$, let $D$ be a good drawing of $D_m + C_n$ in which no edge of $C_n^*$ is crossed, and $C_n^*$ does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \dots, m$, two different subgraphs $T^{v_i}$ and $T^{v_j}$ cross each other in $D$ at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

**Corollary 4.2** ([24, Corollary 4])**.** *For $m \geq 2$ and $n \geq 3$, let $D$ be a good drawing of the join product $D_m + C_n$ in which the edges of $C_n^*$ do not cross each other and $C_n^*$ does not separate $p$ vertices $v_1, v_2, \dots, v_p$, $2 \leq p \leq m$. Let $T^{v_1}, T^{v_2}, \dots, T^{v_q}$, $q < p$, be the subgraphs induced on the edges incident with the vertices $v_1, v_2, \dots, v_q$ that do not cross $C_n^*$. If $k$ edges of some subgraph $T^{v_j}$ induced on the edges incident with the vertex $v_j$, $j \in \{q+1, q+2, \dots, p\}$, cross the cycle $C_n^*$, then the subgraph $T^{v_j}$ crosses each of the subgraphs $T^{v_1}, T^{v_2}, \dots, T^{v_q}$ at least $\lfloor \frac{n-k}{2} \rfloor \lfloor \frac{(n-k)-1}{2} \rfloor$ times in $D$.*

**Lemma 4.3** ([24, Lemma 1])**.** *For $m \geq 1$, let $G$ be a graph of order $m$. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of $C_n^*$ do not cross each other.*

Again, using the algorithm on the website http://crossings.uos.de/, we can also determine the crossing numbers of two small graphs $G^* + C_3$ and $G^* + C_4$ in Lemma 4.4.

**Lemma 4.4.** $\mathrm{cr}(G^* + C_3) = 7$ *and* $\mathrm{cr}(G^* + C_4) = 14$.

**Theorem 4.5.** $\mathrm{cr}(G^* + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 4$.

*Proof.* By Lemma 4.4, the result holds for $n = 4$. Into both drawings in Figures 2 and 3, it is possible to add the edge $t_1 t_n$ which forms the cycle $C_n^*$ on vertices of $P_n^*$ with just two additional crossings, i.e., $C_n^*$ is crossed by two edges $v_1 v_2$ and $v_2 v_3$ of the graph $G^*$. Thus, $\mathrm{cr}(G^* + C_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$, and let suppose that there is a good drawing $D$ of $G^* + C_n$ such that

$$\mathrm{cr}_D(G^* + C_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{for some } n \geq 5. \tag{4.2}$$

By Theorem 2.1, at most one edge of the cycle $C_n^*$ can be crossed in $D$, and so edges of $C_n^*$ do not cross each other also by Lemma 4.3. The subdrawing of $C_n^*$ induced by $D$ divides the plane into two regions with at least four vertices of $G^*$ in one of them, and so two possible cases may occur:

*Case 1.* There is no crossing on edges of $C_n^*$. Since at least four vertices of $G^*$ are placed in one region of $D(C_n^*)$, any two such different considered subgraphs $T^{v_i}$ and $T^{v_j}$ cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 4.1. Hence, there are at least

$\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings in $D$ which confirms a contradiction with the assumption (4.2).

*Case 2.* There is exactly one crossing on edges of $C_n^*$, which yields that at least four vertices of $G^*$ are placed in one region of $D(C_n^*)$. The graph $G^*$ with no bridge cannot force only one crossing on edges of $C_n^*$. If $\mathrm{cr}_D(T^{v_5}, C_n^*) = 1$, the same idea as in Case 1 also contradicts the assumption (4.2). Now, assume $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ for only one $i \in \{1, 2, 3, 4\}$. For easier reading, let $r = |R_D|$. This, by Lemma 4.1 and Corollary 4.2 for $p = 4$, $q = 3$ and $k = 1$, enforces at least

$$\binom{3}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 1 + 1 + n - r \tag{4.3}$$

crossings in $D$, because $\mathrm{cr}_D(T^{v_i}, T^{v_5}) \geq 1$ and $\mathrm{cr}_D(G^*, T^j) \geq 1$ for any subgraph $T^j \notin R_D$. The number of crossings obtained in (4.3) can confirm a contradiction in $D$ for all $n \geq 6$ or $r \leq 3$. In the next, we assume that $n = 5$ and $r = 5$. Klešč *et al.* [25] proved that if two different subgraphs $T^i$ and $T^j$ do not cross $G^*$ in $D$, then $T^i$ and $T^j$ cross each other at least twice. In Table 2, there are all necessary numbers of crossings between two subgraphs $T^i$ and $T^j$ with configurations $\mathcal{X}_k$ and $\mathcal{X}_l$ of the subgraphs $F^i = G^* \cup T^i$ and $F^j = G^* \cup T^j$, respectively.

**Table 2**

The minimum number of crossings between $T^i$ and $T^j$ such that $\mathrm{conf}(F^i) = \mathcal{X}_k$ and $\mathrm{conf}(F^j) = \mathcal{X}_l$, where $\mathcal{X} = \mathcal{A}$ for the planar subdrawing of $G^*$ in Figure 1(a) and $\mathcal{X} = \mathcal{B}$ for the nonplanar subdrawing of $G^*$ in Figure 1(c) induced by $D$, respectively.

| | $\mathcal{X}_1$ | $\mathcal{X}_2$ | $\mathcal{X}_3$ | $\mathcal{X}_4$ |
|---|---|---|---|---|
| $\mathcal{X}_1$ | 4 | 2 | 3 | 3 |
| $\mathcal{X}_2$ | 2 | 4 | 3 | 3 |
| $\mathcal{X}_3$ | 3 | 3 | 4 | 2 |
| $\mathcal{X}_4$ | 3 | 3 | 2 | 4 |

At least two different subgraphs $F^i$ and $F^j$ have the same configuration $\mathcal{X}_k$ with $\mathrm{cr}_D(T^i, T^j) \geq 4$, and therefore we obtain at least $\binom{5}{2}2 + 2 = 22$ crossings in $D$, which yields a contradiction with (4.2). Finally, let $n = 5$ and $r = 4$. In the rest of the paper, let edges of $G^*$ be crossed by the subgraph $T^5$. Moreover, $T^5$ must cross edges of $G^*$ just once, otherwise, we receive 2 crossings instead of $n - r$ in (4.3). In this case, we have at least 16 crossings on edges of $\bigcup_{i=1}^4 T^i$ again using minimal values in Table 2, at least 4 crossings between $\bigcup_{i=1}^4 T^i$ and $T^5$ due to at most three vertices of $G^*$ on boundary in all regions of $D(F^i)$, one crossing on edges of $C_n^*$, and one crossing offers $T^5 \in S_D$ on edges of $G^*$. So, we obtain also at least 22 crossings contradicting the assumption (4.2).

We have shown that there is no good drawing $D$ of the graph $G^* + C_n$ with fewer than $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings, and this completes the proof. □

Now, let us turn to the crossing numbers of $H^* + C_n$. Given the use of arguments similar to those in the first part of the proof of Lemma 3.2, the proofs of Lemmas 4.6

and 4.7 can be omitted if $\mathrm{cr}_D(H^*) + \mathrm{cr}_D(H^* + D_n, C_n^*) \geq 3$ is fulfilling in a good drawing $D$ of $H^* + C_n$ with the nonempty set $R_D$.

**Lemma 4.6.** *For $n \geq 5$, let $D$ be a good drawing of $H^* + C_n$ with the subdrawing of $H^*$ induced by $D$ given in Figure 4(b). If $|R_D| \geq 1$ with at least two crossings on edges of the cycle $C_n^*$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in $D$.*

**Lemma 4.7.** *For $n \geq 3$, let $D$ be a good drawing of $H^* + C_n$ with the subdrawing of $H^*$ induced by $D$ given in Figure 4(c). If $|R_D| \geq 1$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in $D$.*

Note that the set $R_D$ must be empty if there is a separating cycle $C_3$ of $H^*$ in the subdrawing $D(H^*)$ induced by some drawing $D$ of $H^* + C_n$. The crossing numbers of $H^* + C_3$ and $H^* + C_4$ in Lemma 4.8 are also given using the algorithm on the website http://crossings.uos.de/.

**Lemma 4.8.** $\mathrm{cr}(H^* + C_3) = 12$ *and* $\mathrm{cr}(H^* + C_4) = 18$.

**Theorem 4.9.** $\mathrm{cr}(H^* + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ *for* $n \geq 3$.

*Proof.* By Lemma 4.8, the result is true for $n = 3$ and $n = 4$. In the following, let $n \geq 5$. Into the drawing in Figure 5, it is possible to add the edge $t_1t_n$ which forms the cycle $C_n^*$ on vertices of $P_n^*$ with just two additional crossings, i.e., $C_n^*$ is crossed by two edges $v_1v_3$ and $v_3v_4$ of the graph $G^*$. Thus, $\mathrm{cr}(H^* + C_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$, and let suppose that there is a good drawing $D$ of $H^* + C_n$ such that

$$\mathrm{cr}_D(H^* + C_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 3 \quad \text{for some } n \geq 5. \tag{4.4}$$

By Theorem 3.3, at most three edges of the cycle $C_n^*$ can be crossed in $D$, and we can also suppose that edges of $C_n^*$ do not cross each other using Lemma 4.3. The subdrawing of $C_n^*$ induced by $D$ divides the plane into two regions with at least four vertices of $H^*$ in one of them, and so four possible cases may occur:

*Case 1.* There is no crossing on edges of $C_n^*$, that is, all vertices of $H^*$ are placed in one region of $D(C_n^*)$. For $i, j \in \{1, 2, 3, 4, 5\}$, any two different considered subgraphs $T^{v_i}$ and $T^{v_j}$ cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 4.1. Hence, there are at least $\binom{5}{2}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in $D$ which confirms a contradiction with the assumption (4.4).

*Case 2.* There is exactly one crossing on edges of $C_n^*$. All five vertices of the graph $H^*$ must be placed in one region of $D(C_n^*)$ because $H^*$ contains no bridge. This enforces that there is exactly one vertex $v_i$, $i \in \{1, 2, 3, 4, 5\}$, such that $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$. By Lemma 4.1, Corollary 4.2 for $p = 5$, $q = 4$ and $k = 1$, we have at least $\binom{4}{2}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor$ crossings in $D$, but the obtained number of crossings also contradicts (4.4) in $D$.

In the following, the edges of $C_n^*$ are crossed at least twice in $D$. Let us first show that $R_D = \emptyset$ in the considered drawing $D$. If the subdrawing $D(H^*)$ contains a separating cycle $C_3$, then there is no possibility to obtain a subdrawing of $H^* \cup T^i$ for any $T^i \in R_D$. Lemma 4.6 or 4.7 contradicts the assumption (4.4) for the subdrawing of $H^*$ in $D$ given in Figure 4(b) or (c) if $|R_D| \geq 1$, respectively. Now, we can assume that there are at least $n$ crossings between $H^*$ and $\bigcup_{i=1}^n T^i$, because $\mathrm{cr}_D(H^*, T^i) \geq 1$ for all $i = 1, 2, \ldots, n$.

*Case 3.* There are two crossings on edges of $C_n^*$. We discuss three subcases:

(a) Let $\mathrm{cr}_D(H^*, C_n^*) = 2$ and $C_n^*$ be crossed twice by only one edge of the graph $H^*$. All vertices of $H^*$ are placed in one region of $D(C_n^*)$ and the same idea as in Case 1 contradicts the assumption (4.4).

(b) Let either $\mathrm{cr}_D(H^*, C_n^*) = 2$ and $C_n^*$ be crossed by two different edges of the graph $H^*$ or $\mathrm{cr}_D(T^{v_i}, C_n^*) = 2$ for only one $v_i$, $i \in \{1, 2, 3, 4, 5\}$. Since at least four vertices of $H^*$ are placed in one region of $D(C_n^*)$, we have at least $\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 4$ crossings in $D$ according to Lemma 4.1 and $R_D = \emptyset$.

(c) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$ for two different subgraphs $T^{v_i}$, $T^{v_j}$, where $i, j \in \{1, 2, 3, 4, 5\}$. Again all five vertices of $H^*$ are placed in one region of $D(C_n^*)$. By Lemma 4.1, Corollary 4.2 for $p = 5$, $q = 3$ and $k = 1$, and $R_D = \emptyset$, we obtain at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (3+3)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + n$ crossings contradicting (4.4) in $D$.

*Case 4.* There are three crossings on edges of $C_n^*$. We discuss five following subcases:

(a) Let $\mathrm{cr}_D(H^*, C_n^*) = 2$ and $C_n^*$ be crossed twice by only one edge of the graph $H^*$, and let also there be exactly one $T^{v_i}$, $i \in \{1, 2, 3, 4, 5\}$, such that $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$. All vertices of $H^*$ are placed in one region of $D(C_n^*)$ and the same idea as in Case 2 contradicts the assumption (4.4).

(b) Let $\mathrm{cr}_D(H^*, C_n^*) = 2$ and $C_n^*$ be crossed by two different edges of the graph $H^*$, and let also there be exactly one $T^{v_i}$, $i \in \{1, 2, 3, 4, 5\}$, with $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$. Four vertices of $H^*$ are placed in one region and one vertex, say $v_5$, is placed in other region of $D(C_n^*)$. If $\mathrm{cr}_D(C_n^*, T^{v_5}) = 1$, then the same idea as in Case 3(b) again confirms a contradiction with (4.4). If $\mathrm{cr}_D(C_n^*, T^{v_i}) = 1$ for $i \neq 5$, then there are at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + n + 3 + 1$ crossings in $D$ provided by Lemma 4.1, Corollary 4.2 for $p = 4$, $q = 3$ and $k = 1$, and $R_D = \emptyset$, also due to three crossings on edges of $C_n^*$ and $\mathrm{cr}_D(T^{v_i}, T^{v_5}) \geq 1$. The obtained number of crossings also contradicts (4.4) in $D$.

(c) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 3$ for only one $T^{v_i}$, $i \in \{1, 2, 3, 4, 5\}$. All vertices of $H^*$ are placed in one region of $D(C_n^*)$ and the similar idea as in Case 3(b) contradicts the assumption (4.4).

(d) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_j}, C_n^*) = 2$ for two distinct $i, j \in \{1, 2, 3, 4, 5\}$. By Lemma 4.1, Corollary 4.2 for $p = 5$, $q = 3$ and $k = 1$, $k = 2$, and $R_D = \emptyset$, we have at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 3\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + n$ crossings contradicting (4.4) in $D$ provided by again all five vertices of $H^*$ are placed in one region of $D(C_n^*)$.

(e) Let $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$, $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$, and $\mathrm{cr}_D(T^{v_l}, C_n^*) = 1$ for three distinct $i, j, l \in \{1, 2, 3, 4, 5\}$. For such a index pair $i, j$, the subgraph $T^{v_i} \cup T^{v_j} \cup C_n^*$ is isomorphic to the graph $D_2 + C_n$. Consider $n - 2$ vertices of the cycle $C_n^*$ incident with edges of $T^{v_i}$ and $T^{v_j}$ which do not cross $C_n^*$. Let us delete all edges of $T^{v_i}$ and $T^{v_j}$ which are not incident with these $n - 2$ vertices. The resulting subgraph is homeomorphic to the graph $D_2 + C_{n-2}$ and, in its subdrawing $D^{'}$ induced by $D$, we obtain $\mathrm{cr}_{D'}(T^{v_i}, T^{v_j}) \geq \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ thanks to Lemma 4.1. Clearly, the same holds for both remaining index pairs $i, l$ and $j, l$. Thus, we have at least $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (2 + 2 + 2)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 3\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + n + 3$ crossings in $D$ again using Lemma 4.1, Corollary 4.2 for $p = 5$, $q = 2$ and $k = 1$, and $R_D = \emptyset$, also due to three crossings on edges of $C_n^*$.

We have shown, in all cases, that there is no good drawing $D$ of the graph $H^* + C_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ crossings. □

## 5. SOME CONSEQUENCES OF THE MAIN RESULTS

In Figure 6, let $G_1$ be the graph obtained from $G^*$ by adding the edge $v_2v_5$ into the drawing in Figure 1(a). Since we can add this edge to the graph $G^*$ without additional crossings in Figures 2 and 3, the drawing of $G_1 + P_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings is obtained. By adding the considered edge $v_2v_5$, it is also possible to add the edge $t_1t_n$ that creates $C_n^*$ on vertices of $P_n^*$ with just two additional crossings. The same holds for the graph $H_1$ if we add the edge $v_2v_5$ into the drawing in Figure 4(a), and therefore, the next results are obvious.

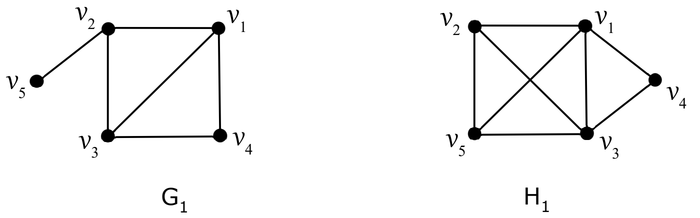


**Fig. 6.** Two graphs $G_1$ and $H_1$ by adding one edge to the graphs $G^*$ and $H^*$

**Corollary 5.1.** $\mathrm{cr}(G_1 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 2$.

**Corollary 5.2.** $\mathrm{cr}(G_1 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 3$.

Note that $\mathrm{cr}(G_1 + C_3) = 8$ is one more than $\mathrm{cr}(G^* + C_3) = 7$ which is caused by adding the edge $v_2v_5$ in Figure 3. Of course, such a result is confirmed again using the algorithm on the website http://crossings.uos.de/.

**Corollary 5.3.** $\mathrm{cr}(H_1 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 2$.

**Corollary 5.4.** $\mathrm{cr}(H_1 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ *for* $n \geq 3$.

## 6. CONCLUSIONS

We suppose that similar forms of discussions can be used to estimate unknown values of the crossing numbers of two remaining connected graphs of order five with a much larger number of edges in join products with paths and cycles on $n$ vertices. Especially for the complete graph $K_5$ and the graph $K_5 \setminus e$ obtained by removing one edge from $K_5$.

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