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THE EXTENSIVE 1-MEDIAN PROBLEM WITH RADIUS ON NETWORKS

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Abstract. The median location problem concerns finding locations of one or several new facilities that minimize the overall weighted distances from the existing to the new facilities. We address the problem of locating one new facility with a radius r on networks. Furthermore, the radius r is flexible and the objective function is the conic combination of the traditional 1-median function and the value r. We call this problem an extensive 1-median problem with radius on networks. To solve the problem, we first induce the so-called finite dominating set, that contains all points on the underlying network and radius values which are candidate for the optimal solution of the problem. This helps to develop a combinatorial algorithm that solves the problem on a general network $G = (V, E)$ in $O(|E||V|^3)$ time. We also consider the underlying problem with improved algorithm on trees. Based the convexity of the objective function with variable radius, we develop a linear time algorithm to find an extensive 1-median with radius on the underlying tree.

Keywords: extensive facility, median problem, tree, convex.

Mathematics Subject Classification: 90B10, 90C35.

1. INTRODUCTION

1.1. LITERATURE REVIEW

Recently, location theory has become an interesting topic in operations research with both theoretical and practical contributions. In a location problem, one finds optimal locations of one or several new facilities to optimize an objective function. The most popular objectives are the median and the center functions with respect to the sum and the max of the weighted distances from the existing facilities to the new ones. For some references on the classical location problem, one refers to the books of Drezner and Hamacher [4], Love *et al.* [12], and Eiselt *et al.* [5] or the seminal papers of Kariv and Hakimi [9, 10] and references therein. Also, some special cases with efficient solution approaches were also studied, e.g., the 1-median problem on trees is solvable in linear time algorithm (see [6, 7]), the 1-center problem on the plane and on networks can be solved in linear time (see [3, 13]), the p-median on trees can be solved in $O(pn^2)$ time (see [20]), the p-center problem on trees can be also solved in polynomial time (see [8]).

A facility in classical location problems is often represented by a point on the plane or on network. However, when the facilities are too large in size with its environment, one considers the *extensive* location problem, where the facilities are objects with shape. For the motivation and solution methods of locating a line/hyperplane in space that minimizes the median/center objective functions, we can refer to the book of Schöbel [18] and references therein. This problem can be applied in constructing an optimal urban railway. The first who concerned the extensive location problem on a network was Morgan and Slater [15]. They defined a core of a tree as the path connecting two different leaves and aimed to find a core with minimum overall distances from vertices of the tree to it. They proposed a linear time algorithm to solve the problem. Then the minisum core problem, which finds a path connecting two points on a tree with specified length to minimize the weighted sum function, was considered by Minieka and Patel [14]. The mentioned problem with pos/neg vertex weights was also investigated and solved by Zaferanieh and Fathali [22]. The problem of locating a subtree on networks was intensively investigated. Kim *et al.* [11] showed the NP-hardness of the problem on networks and solved the confined problem on tree networks in an improved time complexity. Shioura and Shigeno [19] modeled the center subtree problem by a bottleneck knapsack problem and solved the problem in linear time. It is well-known that centdian function is a convex combination of the median and center functions. Then, based on the nestedness property, Tamir *et al.* [21] solved the centdian subtree problem on a tree in $O(n \log n)$ time. Also, Bhattacharya *et al.* [2] solved the problem of locating a center path/tree-shaped facility on a tree by a parametric-pruning method in $O(n \log n)$ time. For survey on the extensive location problems with models and solution methods, one can see Puerto *et al.* [17]. Recently, Berger *et al.* [1] investigated the single facility median location problem in the space with customers having the same radius. They developed efficient $O(n \log^{\mathbf{c}} n)$ algorithm for finding the optimal location of the problem under l_1- and l_∞-norm, where $\mathbf{c}$ is the dimension of the environment. The result on general block norm was also discussed. In this paper, we consider on another hand the extensive 1-median problem on networks where the server has a radius. This concept of extensive server is applicable in locating a radio broadcaster, a base transceiver station, a building with cylinder-shaped, etc.

1.2. PROBLEM SETTING

Let us revisit basic concepts concerning the median location problem on a network. Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. We assign a positive length ℓ_e to e for each edge $e \in E$ and associate a positive weight w_v to v for each vertex $v \in V$. The distance between two vertices in G is the length of the shortest path connecting them. By definition, a point on G is either a vertex or lies on the interior of an edge. In the later situation, a point x on an edge $e = (u, v)$ is identified by a parameter $\lambda \in (0, \ell_e)$ with $d(u, x) = \lambda$ and $d(v, x) = \ell_e - \lambda$. Let $A(G)$ stand for the set of all points in the graph G. The distance $d(a, b)$ between two points a and b

is defined similarly to the distance of two vertices. The classical 1-median problem asks to find a point x on G that minimizes the objective function $\sum_{v\in V} w_v d(x,v)$. An optimal solution to the 1-median problem is called a 1-median of the network. The 1-median problem on networks has been intensively studied with efficient algorithmic approaches; see [6, 7, 10] to mention a few.

If the facility is too big in relation to its environment to be modeled as a point, the extensive location is taken into consideration. Let us now define an extensive facility centered at x with radius r as $B(x,r)$. The mentioned facility presents a station located at x and cover all the points in G with distance to x being less than or equal to r. We can mathematically write

$$B(x,r) := \{x' \in A(G) : d(x,x') \leq r\}.$$

The distance from a vertex $v \in V$ to the extensive facility $B(x,r)$ is defined as

$$d(v,B(x,r)) := \min_{x'\in B(x,v)} d(v,x') = \max\{0, d(v,x) - r\}.$$

Adjusting the value of radius r yields a corresponding cost $\Lambda(r)$ which is a non-decreasing function. Moreover, we take into account the cost with respect to the median objective function, say $\Lambda'(\sum_{v\in V} w_v d(v,B(x,r)))$, which is also nondecreasing in the value of 1-median function. The extensive 1-median problem with radius on G is to find an extensive facility $B(x,r)$, i.e., a point x on $A(G)$ and a radius $r \geq 0$, that minimizes the combination of the two mentioned costs, that is

$$F(x,r) := \Lambda(r) + \Lambda'\Big(\sum_{v\in V} w_v d(v,B(x,r))\Big).$$

In the scope of this paper, we regard the cost functions $\Lambda(r)$ and $\Lambda'(\sum_{v\in V} w_v d(v,B(x,r)))$ as linear functions, i.e.,

$$\Lambda(r) := \alpha r \quad \text{and} \quad \Lambda'\left(\sum_{v\in V} w_v d(v,B(x,r))\right) := \beta \sum_{v\in V} w_v d(v,B(x,r))$$

for non-negative α and β. Then we rewrite

$$F(x,r) := \alpha r + \beta \sum_{v\in V} w_v d(v,B(x,r)). \tag{1.1}$$

If $\alpha = 0$, we actually know that there is no cost for setting the radius of the extensive facility. This situation leads to a trivial solution with the radius large enough to cover all vertices of the graph and the corresponding objective is zero. If $\beta = 0$, the 1-median value does not contribute to the objective. Therefore, any point on the network (with $r = 0$) is an optimal solution to the problem.

Let a graph G in Figure 1 be given such that all edge lengths are 2 and all weights are 1. Let $\alpha = \beta = 1$. Consider an extensive facility $B(v_1, 1)$, we can compute $F(v_1, 1) = 4$.

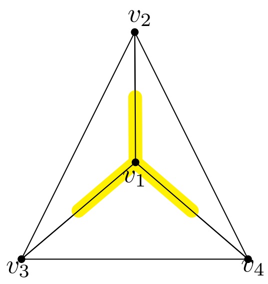

Fig. 1. An instance of the extensive 1-median problem graph G with radius

The rest of this paper is organized as follows. In Section 2, we study the property of the problem on general graphs. a finite dominating set result is constructed. Based on a brute force computation, we solve the problem on graphs in $O(|E||V|^3)$ time and on trees in $O(|V|^3)$ time. In Section 3, we discuss the property of the objective function on a tree graph and then develop a linear time algorithm to find an extensive 1-median with radius on the underlying tree.

2. ON GENERAL GRAPHS

Let a graph $G = (V, E)$ and an extensive facility $B(x, r)$ be given. We denote by

$$\mathcal{O}(x, r) := \{v \in V \mid d(x, v) > r\}$$

the set of all vertices that are outside of $B(x, r)$. Furthermore, let

$$\mathcal{V}(x, r) := \{v \in V : d(x, v) = r\}$$

and

$$\mathcal{I}(x, r) = V \setminus (\mathcal{O}(x, r) \cup \mathcal{V}(x, r))$$

be the set of vertices in the border and inside of $B(x, r)$, respectively. Then we know that

$$d(v, B(x, r)) = \begin{cases} d(v, x) - r, & \text{if } v \in \mathcal{O}(x, r). \\ 0, & \text{otherwise.} \end{cases}$$

We abbreviate the weight of a set $S \subset V$ by $W(S) := \sum_{v \in S} w_v$. In the following, let us investigate a property concerning an extensive 1-median $B(x, r)$ of G with relation to the set $\mathcal{V}(x, r)$.

Proposition 2.1. *There exists an extensive 1-median $B(x, r)$ on the graph G such that one of the following conditions are satisfied.*

(i) $|\mathcal{V}(x, r)| \geq 1$ *if x is a vertex.*
(ii) $|\mathcal{V}(x, r)| \geq 2$ *if x is in the interior of an edge.*

Proof. Part (i). We take a vertex x with $|\mathcal{V}(x,r)| = 0$. For $r' := r + \varepsilon$ with

$$\varepsilon < \min_{v \in \mathcal{O}(x,r)} \{d(v,x) - r\},$$

elementary computation gives

$$F(x,r') - F(x,r) = (\alpha - \beta W(\mathcal{O}(x,r)))\,\varepsilon$$

as $\mathcal{O}(x,r') = \mathcal{O}(x,r)$. Similarly, for $r'' := r - \delta$ with

$$\delta < \min_{v \in \mathcal{I}(x,r)} \{r - d(v,x)\},$$

it yields

$$F(x,r'') - F(x,r) = -\,(\alpha - \beta W(\mathcal{O}(x,r)))\,\delta$$

as $\mathcal{O}(x,r'') = \mathcal{O}(x,r)$. As either $(\alpha - \beta W(\mathcal{O}(x,r)))$ or $-\,(\alpha - \beta W(\mathcal{O}(x,r)))$ is non-positive, we can improve the objective by adjusting the radius r until $|\mathcal{V}(x,v)| \geq 1$. Hence, the mentioned property for the vertex x holds.

Part (ii). If x is not a vertex on G, we analyse the following cases.

Case 1. $|\mathcal{V}(x,r)| = 1$. We can assume that x is an interior point in $e = (u,v)$ such that $d(u,x) = \lambda < d(u,v)$. We denote by

$$A(\star,x) := \{v' \in V : d(v',x) = d(v',\star) + d(\star,x)\}$$

for $\star \in \{u,v\}$ and $\Gamma(x) = A(u,x) \cap A(v,x)$.

We consider $x' \in (u,v)$ such that $d(u,x') = \lambda - \varepsilon$ with

$$\varepsilon < \min_{v' \in A(u,x) \cap \mathcal{O}(x,r)} \{d(x,v') - r, \lambda\}$$

and $r' := r + \varepsilon$. For $v' \in A(u,x) \cap \mathcal{O}(x,r)$, one obtains

$$d(v', B(x',r')) = d(v',x') - r' = (d(v',x) - \varepsilon) - (r + \varepsilon) = d(v', B(x,r)) - 2\varepsilon.$$

For $v' \in (A(v,x) \backslash \Gamma(x)) \cap \mathcal{O}(x,r)$, one obtains

$$d(v', B(x',r')) = d(v',x') - r' = (d(v',x) + \varepsilon) - (r + \varepsilon) = d(v', B(x,r)).$$

Therefore,

$$F(x',r') - F(x,r) = [\alpha - 2\beta W(A(u,x) \cap \mathcal{O}(x,r))]\varepsilon.$$

We consider $x'' \in (u,v)$ such that $d(u,x'') = \lambda + \delta$ with

$$\min_{v' \in A(v,x) \cap \mathcal{L}(x,r)} \{d(x,v') - r, \lambda\}$$

and $r'' := r - \delta$. For $v' \in (A(u,x) \backslash \Gamma(x)) \cap \mathcal{L}(x,r)$, one obtains

$$d(v', B(x'',r'')) = d(v',x'') - r'' = (d(v',x) + \delta) - (r - \delta) = d(v', B(x,r)) + 2\delta.$$

For $v' \in A(v,x) \cap \mathcal{O}(x,r)$, one obtains

$$d(v', B(x'', r'')) = d(v', x'') - r' = (d(v', x) - \varepsilon) - (r - \varepsilon) = d(v', B(x, r)).$$

Therefore,

$$F(x'', r'') - F(x, r) = -[\alpha - 2\beta W(A(u, x) \cap \mathcal{O}(x, r) \backslash \Gamma(x))]\delta.$$

If $\alpha - 2\beta W(A(u,x) \cup \mathcal{O}(x,r)) < 0$, we know $F(x', r') < F(x, r)$. Furthermore, as x' is in an interval from u to x of the edge e, we further imply that

$$A(u, x) \cap \mathcal{O}(x, r) \subset A(u, x') \cap \mathcal{O}(x, r)$$

and thus we can improve the objective function until x' is a vertex of $|\mathcal{V}(x', r')| \geq 2$. Otherwise, if $\alpha - 2\beta W(A(u,x) \cap \mathcal{O}(x,r)) \geq 0$, we know that $-[\alpha - 2\beta W(A(u,x) \cap \mathcal{O}(x,r) \backslash \Gamma(x))] \leq 0$ and hence $F(x'', r'') \geq F(x, r)$. Furthermore, we know that for x'' in the interval from x to v of edge e, one obtains

$$A(u, x'') \cap \mathcal{O}(x'', r'') \backslash \Gamma(x'') \subset A(u, x) \cap \mathcal{O}(x, r) \backslash \Gamma(x).$$

Therefore, we can improve the objective function until $x'' \equiv v$ or $|\mathcal{V}(x'', r'')| \geq 2$.

Case 2. $|\mathcal{V}(x, r)| = 0$.

By either reducing or increasing the radius r of the extensive facility $B(x, r)$, we can reduce this case to Case 1 or (i) and also obtain the result. □

If $r = 0$ is fixed, we get the classical 1-median problem on graphs. The finite dominating set result (see see [10]) states, that there exists 1-median which is indeed a vertex. This is indeed a special case of Proposition 2.1 with $|\mathcal{V}(x, 0)| = 1$ for x being vertices. By Proposition 2.1, we now construct a dominating set of the extensive 1-median problem with radius based on the pairs (x, r), where x is the center of the extensive facility and r is its radius. Let us define the set

$$\mathcal{P}_1 := \{(v, d(v, v')) : v, v' \in V\}$$

associated with a vertex v and the distance from v to another vertex v'. For each pair (x, r) in $\mathcal{P}_1$, we know that the extensive facility $B(x, r)$ satisfies condition (i) of Proposition 2.1. For an edge $e = (u, v)$, we also denote the set

$$\mathcal{EQ}(e) := \{x \in A(G) : d(x, u) = d(x, v)\}.$$

For the sake of tractability, we just take the boundary points of $\mathcal{EQ}(e)$ in the case $\mathcal{EQ}(e)$ includes continuum points. For example, in Figure 2 we take two points v_1 and v_2 in the set of equilibrium points with respect to v_3 and v_4. As the objective function is piecewise linear according to radius r (see the analysis in Proposition 2.1), the exclusion of continuum points does not change the optimal solution. Then $\mathcal{Q} = \bigcup_{e \in E} \mathcal{EQ}(e)$ is the set of of all equilibrium points in G (see [16]). We set

$$\mathcal{P}_2 := \{(x, d(x, u)) : x \in \mathcal{Q}(e) \text{ for } e = (u, v) \text{ and } e \in E\}.$$

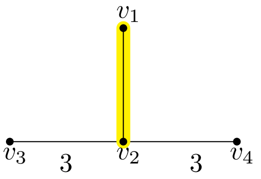

Fig. 2. All points in the edges (v_1, v_2) are equilibrium points of v_3 and v_4

Then any pair (x, r) in $\mathcal{P}_2$ satisfy the criterion (ii) of Proposition 2.1. Hence, we finally induce $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$, that is the set contains all candidate solutions due to Proposition 2.1.

We now discuss the complexity to find the set $\mathcal{P}$. First, the set $\mathcal{P}_1$ can be found via identifying the distances between all pairs of vertices in G in $O(|V|^3)$ time by Floyd-Warshall algorithm. Applying the approach of of Nickel [16], one finds the set $\mathcal{EQ}(e)$ in $O(|V|^2)$. Therefore, the set $\mathcal{P}_2$ can be found in $O(|E||V|^2)$ time.

In order to solve the extensive 1-median problem on graphs with radius, we apply a brute force algorithm to compute the objective values at all pairs in $\mathcal{P}$. For the set $\mathcal{P}_1$, we know that the complexity is $O(|V|^3)$ as there are $O(|V|^2)$ elements in $\mathcal{P}_1$. Also, as there are $O(|V|^2)$ many equilibrium points in each edge e, computing all objective values of elements in $\mathcal{P}_2$ costs $O(|E||V|^3)$ time. These complexity analysis leads to the following result.

Theorem 2.2. *The extensive 1-median problem with radius on a general graph $G = (V, E)$ can be solved in $O(|E||V|^3)$ time.*

Example 2.3. We consider an instance of a graph $G = (V, E)$ as in Figure 1 where $\alpha = \beta = 1$, all edge lengths are 2 and all vertex weights are 1. The objective values at the facilities $B(v_i, 0)$ ($B(v_i, 2)$) for $i = 1, 2, 3, 4$ are 6 (2); at $B(x, 1)$ ($B(x, 3)$) for x being the midpoints of v_i and v_j for $i \neq j$ and $i, j = 1, 2, 3, 4$ are 5 (3). Hence, the extensive 1-median with radius on G is $B(v_i, 2)$ for $i = 1, 2, 3, 4$.

Let us discuss how to find the an extensive 1-median on a tree $T = (V, E)$ with radius. For the tree structure, we can simply find the set $\mathcal{P}_2$ by

$$\mathcal{P}_2 := \left\{ \left(m_{uv}, \frac{d(u,v)}{2} \right) : u, v \in V, u \neq v \right\},$$

where m_{uv} is the midpoint of the unique path connecting u and v in the tree T. The set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ can be found in quadratic time by a naive algorithm. Also, we know that there are $O(|V|^2)$ many elements in $\mathcal{P}$ and computing the objective value at an element in $\mathcal{P}$ costs linear time. Therefore, we can find an extensive 1-median on a tree by a brute force algorithm in $O(|V|^3)$ time. We consider in the next section some structural properties of the corresponding problem on trees which help to improve the complexity significantly.

3. ON TREE GRAPHS

We first fix the radius r and consider the property of the center x of $B(x,r)$ to minimize (1.1) for x in $A(T)$, i.e., we aim to solve the problem

$$\min_{x\in A(T)} w_v d(x, B(x,r)). \tag{3.1}$$

We call the minimizer $B(x,r)$ of (3.1) a r-1-median of T. If $r = 0$, the problem reduces to minimize $\sum_{v\in V} w_v d(v,x)$ for $x \in A(T)$. Hence, the 0-1-median is indeed the classical 1-median problem on T and is solvable in linear time; see Goldman [6] and Hua [7]. Denote by $\mathcal{T}(x)$ the set of all subtrees induced by deleting x and all edges that are incident to or contain x. Assume that $\mathcal{T}(x) = \{T_1, T_2, \ldots, T_k\}$ and denote the total weight of vertices in a subset $S \subset V$ by $W(S) := \sum_{v\in S} w_v$, we revisit the classical 1-median criterion as follows.

Theorem 3.1 (see [6, 7])**.** *The point x is a 1-median of T if and only if*

$$W(T_i) \le \frac{\sum_{v\in V} w_v}{2}$$

for all $i = 1, \ldots, k$.

Now let us consider the case $r > 0$. We denote by

$$\begin{aligned}\mathcal{L}(T_i, r) &:= \{v \in V(T_i) \,|\, d(x,v) > r\},\\ \mathcal{M}(T_i, r) &:= \{v \in V(T_i) \,|\, d(x,v) \ge r\},\\ \mathcal{C}(T_i, r) &:= \{v \in V(T_i) \,|\, d(x,v) < r\}\end{aligned}$$

for a subtree $T_i \in \mathcal{T}(x)$. We derive the following optimality criterion.

Theorem 3.2 (Optimality criterion)**.** *The facility $B(x,r)$ is a r-1-median of T if and only if*

$$W(\mathcal{L}(T_i, r)) \le \frac{\sum_{j=1}^{k} W(\mathcal{M}(T_j, r))}{2}$$

for $i = 1, \ldots, k$.

Proof. As $d(v,x)$ is convex is a convex function along each simple path of the tree (see Goldman [6]), the distance functions between vertices and the extensive facility $d(v, B(x,r)) = \max\{0, d(v,x) - r\}$ are convex for all $v \in V$. This implies the convexity of the objective function $\sum_{i=1}^{n} w_v d(v, B(x,r))$ along each path of the tree. Hence, the extensive facility $B(x,r)$ is an r-1-median of T if it is a local minimizer. Take a vertex v_i in T_i that is incident to v and a point x' in the interior of x and v' with

$$d(x,x') = \epsilon < \min_{v\in V}\{d(x,v_i), \max\{0, d(x,v) - r\}\}.$$

By elementary computation, we get

$$F(x',r) - F(x,r) = \left(\sum_{j\in\{1,\ldots,k\}\setminus\{i\}} W(\mathcal{M}(T_j,r)) - W(\mathcal{L}(T_i,r))\right)\epsilon.$$

The facility $B(x,r)$ is an r-1-median if $F(x',r) \geq F(x,r)$. This fact implies the optimality criterion in the proposition. $\square$

For $r = 0$, the optimality condition in Theorem 3.2 coincides with the result of Goldman [6]. Thus, Theorem 3.2 is an extension of the 1-median criterion. Also, we derive the so-called nestedness property for the extensive 1-median facility on a tree.

Corollary 3.3 (Nestedness property). *Any extensive 1-median facility $B(x,r)$ must contain a 1-median of the tree T.*

Proof. Let $B(x,r)$ be an extensive facility such that there is no 1-median v^* in $B(x,r)$, i.e., $d(v^*,x) > r$. In the path $P(x,v^*)$, we take a vertex v adjacent to v^*. We take a subtree T_i in $\mathcal{T}(v)$ such that T_i contains the 1-medians of T. Then

$$W(T_i) > \frac{\sum_{v\in V} w_v}{2}$$

as v is not a 1-median of T. We now consider the subtrees $T_1', T_2', \ldots, T_k'$ in $\mathcal{T}(x)$ with $T_i \subset T_i'$. Then we get

$$W(\mathcal{L}(T_i',r)) > \frac{\sum_{j=1}^{k} W(\mathcal{M}(T_j',r))}{2}.$$

Hence, $B(x,r)$ is not an r-1-median of T. Let $B(x',r)$ be an r-1-median of T, then

$$\sum_{v\in V} w_v d(v, B(x',r)) < \sum_{v\in V} w_v d(v, B(x,r)).$$

Then we get $F(x',r) < F(x,r)$. The corollary is proved. $\square$

Next we aim to solve the extensive 1-median problem on the tree T with radius, say

$$\min_{\substack{r>0\\ x\in A(T)}} F(x,r) := \alpha r + \beta \sum_{v\in V} w_v d(x, B(x,r)) \tag{3.2}$$

Next we reformulate (3.2) as a uni-variate optimization problem. We first set

$$f(r) := \alpha r + \beta \min_{x\in A(T)} \sum_{v\in V} w_v d(v, B(x,r)).$$

Then it is obvious that

$$\min_{r>0, x\in A(T)} F(x,r) = \min_{r>0} f(r).$$

We get the following result.

Proposition 3.4. *The function $f(r)$ is a piecewise linear convex function for $r > 0$.*

Proof. For simplicity, we denote the set of all vertices with distances being larger than r by $\mathcal{O}$ instead of $\mathcal{O}(x,r)$. For the linearity of $f(\cdot)$, we write

$$f(r) = \alpha r + \beta \sum_{v\in\mathcal{O}} w_v(d(v,x^r) - r) = (\alpha - \beta W(\mathcal{O}))r + \sum_{v\in\mathcal{O}} w_v(d(v,x^r),$$

where

$$x^r := \arg\min_{x\in A(T)} \sum_{v\in V} w_v d(x, B(x,r)).$$

This presentation of $f(r)$ shows that this function is piecewise linear with the slope at r being $\alpha - \beta W(\mathcal{O})$. We next show the convexity of $f(r)$ by proving that the slope $\alpha - \beta W(\mathcal{O})$ is nondecreasing on r.

We also shortly denote $\mathcal{L}_i$ and $\mathcal{M}_i$ instead of $\mathcal{L}(T_i,r)$ and $\mathcal{M}(T_i,r)$ for the sake of simplicity. Furthermore, let

$$\mathcal{L}_{i_0} := \max_{i=1,\dots,\deg(x^r)} W(\mathcal{L}_i).$$

By Theorem 3.2, we know that

$$W(\mathcal{L}_{i_0}) \le \frac{\sum_{i=1}^{\deg(x^r)} W(\mathcal{M}_i)}{2}$$

or equivalently $W(\mathcal{L}_{i_0}) \le \sum_{i\neq i_0} W(\mathcal{M}_i)$. We consider the following cases.
Case 1. $W(\mathcal{L}_{i_0}) \le \sum_{i\neq i_0} W(\mathcal{L}_i)$; see Figure 3 where all vertex weights are 1.

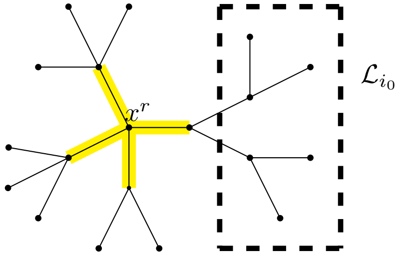

Fig. 3. An instance of the tree T such that Case 1 holds

Set $\varepsilon := \min_{v\in\mathcal{O}}\{d(v,x^r) - r\}$, then the point x^r satisfies

$$x^r \equiv x^{r'} = \arg\min_{x\in A(T)} \sum_{v\in V} w_v d(v, B(x,r'))$$

for $r' \in [r, r+\varepsilon]$ as the optimality criterion in Theorem 3.2 holds. Denote by $\mathcal{O}' = \mathcal{O}'(x^r,r')$, we can compute

$$f(r') = (\alpha - \beta W(\mathcal{O}'))r' + \sum_{v\in\mathcal{O}'} w_v d(v,x^r).$$

Here, we know that $W(\mathcal{O}') = W(\mathcal{O})$ for $r' \in [r, r+\varepsilon)$ and $W(\mathcal{O}') < W(\mathcal{O})$ for $r' = r + \varepsilon$. Hence, the slope is nondecreasing.

Case 2. $W(\mathcal{L}_{i_0}) > \sum_{i \neq i_0} W(\mathcal{L}_i)$ (see Figure 4).

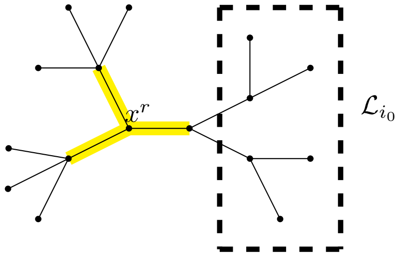

Fig. 4. An instance of the tree T such that Case 2 holds

Let

$$v_{i_0} := \arg\min\{d(x^r, v) : v \in T_{i_0}\}.$$

Set

$$\delta := \min_{v \in \mathcal{L}_{i_0}} \{d(v, x) - r, d(x, v_{i_0})\},$$

then for $r'' \in [r, r+\delta]$, the point $x^{r''}$ such that $x^{r''}$ in the interior of x and v_{i_0} with $d(x^{r''}, x^r) = \frac{r''-r}{2}$ satisfies the optimality condition in Theorem 3.2 or

$$x^{r''} = \arg\min_{x \in A(T)} \sum_{v \in V} w_v d(v, B(x, r')).$$

Set $\mathcal{O}'' = \mathcal{O}(x^{r''}, r'') \subset \mathcal{O}$, we get $W(\mathcal{O}'') \leq W(\mathcal{O})$. Hence, the slope of $f(r'')$ is larger than the slope of $f(r)$.

From the previous case analysis, the slope of $f(r)$ is nondecreasing and hence it is a convex function. □

By Proposition 3.4, we can check if a facility $B(x^r, r)$ is an extensive 1-median with radius on T by computing the corresponding slope of $f(r)$. Precisely, if the slope changes from negative to non-negative, the new facility is an extensive 1-median with radius on T. Now we develop a combinatorial algorithm for the corresponding problem. By Corollary 3.3, we know that there is an extensive facility on tree with radius that contains a 1-median, thus we start with the facility centered at a 1-median of the tree and radius $r := 0$. Then, we increase the radius r to the value such that the slope of $f(r)$ increases (see the two cases in Proposition 3.4) and the current facility $B(x, r)$ is always an r-1-median. We stop whenever the slope is non-negative.

Algorithm 1 is correct due to the convexity of the objective function $f(r)$. We now analyze the complexity of the algorithm. In the initialization step, the classical 1-median on the tree can be found in linear time. Moreover, we can find the set $\mathcal{L}_i$, $\mathcal{M}_i$ and $\mathcal{O}$ in linear time. In each iteration, we update the slope and $\mathcal{L}_{i_0}$. The total number of vertices added to the border set $\mathcal{V}(x, r+\epsilon)$ or $\mathcal{V}_{i_0}(x, r+\epsilon)$ is $O(|V|)$. Therefore, the total complexity of the algorithm is linear time.

Algorithm 1 Solves the extensive 1-median problem on trees with radius.

Input: An instance of a tree T and two positive numbers α, β.
Let x be a classical 1-median of T and set $r := 0$.
Set $\mathcal{L}_i := \{v \in V(T_i) | d(x,v) > r\}, \mathcal{M}_i := \{v \in V(T_i) | d(x,v) \geq r\}, \mathcal{C}_i := \{v \in V(T_i) | d(x,v) < r\}$ for $i = 1, \ldots, |\mathcal{T}(x)|$.
Let $\mathcal{O} := \bigcup_{i=1}^{|\mathcal{T}(x)|} \mathcal{L}_i$, $W(\mathcal{L}_{i_0}) := \max_{i=1,\ldots,|\mathcal{T}(x)|} W(\mathcal{L}_i)$ and set *slope* $:= \alpha - \beta W(\mathcal{L})$.
while *slope* < 0 **do**
  if $W(\mathcal{L}_{i_0}) > \sum_{i \neq i_0} W(\mathcal{L}_i)$ **then**
    Set $\varepsilon := \min_{v \in \mathcal{O}} \{d(v,x) - r\}$ and $\mathcal{V}(x, r+\varepsilon) := \{v \in V : d(v,x) = r + \varepsilon\}$.
    Let $r := r + \varepsilon$ and update *slope* $:=$ *slope* $+ \beta W(\mathcal{V}(x, r+\varepsilon))$.
    Update $\mathcal{L}_i := \mathcal{L}_i \backslash (\mathcal{V}(x, r+\varepsilon) \cap V(T_i))$ for $i = 1, \ldots, |\mathcal{T}(x)|$ and choose $W(\mathcal{L}_{i_0}) := \max_{i=1,\ldots,|\mathcal{T}(x)|} W(\mathcal{L}_i)$.
  end if
  if $W(\mathcal{L}_{i_0}) \leq \sum_{i \neq i_0} W(\mathcal{L}_i)$ **then**
    Take $v_{i_0} := \arg\min\{d(x,v) : v \in T_{i_0}\}$.
    Set $\delta := \min_{v \in \mathcal{L}_{i_0}} \{d(v,x) - r, d(x, v_{i_0})\}$ and $\mathcal{V}_{i_0}(x, r+\delta) := \{v \in V(T_{i_0}) : d(v,x) = r + \delta\}$.
    Take a point x' in (x, v_{i_0}) such that $d(x,x') = \delta$ and set $x := x'$.
    Let $r := r + \delta$ and update *slope* $:=$ *slope* $+ \beta W(\mathcal{V}_{i_0}(x, r+\delta))$.
    Update $\mathcal{L}_{i_0} := \mathcal{L}_{i_0} \backslash \mathcal{V}_{i_0}(x, r+\delta)$ and choose $W(\mathcal{L}_{i_0}) := \max_{i=1,\ldots,|\mathcal{T}(x)|} W(\mathcal{L}_i)$.
  end if
end while
Output: An extensive 1-median $B(x,r)$ with radius on T.

Theorem 3.5. *The extensive 1-median problem on with radius trees can be solved in linear time.*

Example 3.6. Let an instance of the tree $T = (V,E)$ be given as in Figue 5 with all vertex weights are 1. Also, let $\alpha = 10$ and $\beta = 1$ as a part of the input.

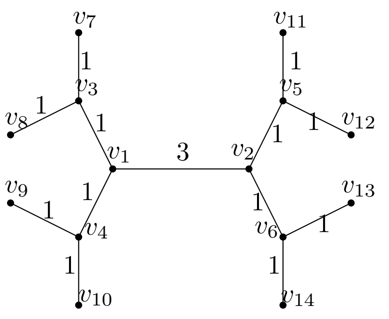

Fig. 5. An instance of the tree $T = (V,E)$ in Example 3.6

Initialization: Take a 1-median $x \equiv v_1$, then we can compute *slope* $:= \alpha - \beta W(\mathcal{O}) = -3 < 0$. We also get the set $\mathcal{T}(x) = \{T_1, T_2, T_3\}$ with T_1 induced by

$\{v_2, v_5, v_6, v_{11}, v_{12}, v_{13}, v_{14}\}$ satisfying $W(\mathcal{L}_1) > W(\mathcal{L}_2)+W(\mathcal{L}_3)$. The slope is $slope := \alpha - \beta W(\mathcal{O}) = -3 < 0$.

Iteration 1. Update x as the midpoint of (v_1, v_2) and $r = 1.5$. The slope is updated as $slope := -2$. Then $\mathcal{T}(x) = \{T_1, T_2\}$ with T_1 induced by $\{v_2, v_5, v_6, v_{11}, v_{12}, v_{13}, v_{14}\}$ satisfying $W(\mathcal{L}_1) = W(\mathcal{L}_2)$.

Iteration 2. Update $r = 2.5$. Then the slope is updated as $slope := 2$. We stop the algorithm.

The extensive 1-median with radius on T is $B(x, 2.5)$ with x being the midpoint of (v_1, v_2).

4. CONCLUSIONS

We addressed the problem on locating an extensive facility centered with radius on networks to minimize the conic combination of the radius value and the 1-median objective with respect to the facility. For the problem on general graphs, we derive the so-called finite dominating set. Then a brute force algorithm, that computes all objective values at all elements in the finite dominating set, is applied to to solve the problem on a graph $G = (V, E)$ in $O(|E||V|^3)$. Finally, we developed an improved algorithm to solve the problem on trees in linear time based on the piecewise linear convexity of the objective function. Further research on multi-facility median location with radius on networks is a promising topic. There is an open question if there is the so-called finite dominating set result for the multi-facility problem on general graphs and polynomial algorithm on tree graphs as it is important to reduce the search space for an optimal solution.

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