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ANISOTROPIC p-LAPLACE EQUATIONS ON LONG CYLINDRICAL DOMAIN

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Abstract. The main aim of this article is to study the Poisson type problem for anisotropic p-Laplace type equation on long cylindrical domains. The rate of convergence is shown to be exponential, thereby improving earlier known results for similar type of operators. The Poincaré inequality for a pseudo p-Laplace operator on an infinite strip-like domain is also studied and the best constant, like in many other situations in literature for other operators, is shown to be the same with the best Poincaré constant of an analogous problem set on a lower dimension.

Keywords: pseudo p-Laplace equation, cylindrical domains, asymptotic analysis.

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1. INTRODUCTION

Let $\Omega_\ell := \ell\omega_1 \times \omega_2$ be a cylindrical domain of length $\ell > 0$, where $\omega_1 \subset \mathbb{R}^{n-m}$ and $\omega_2 \subset \mathbb{R}^m$ are open sets. It is also assumed that $0 \in \omega_1$. Let us denote a generic point in $\mathbb{R}^n$ by $x = (X_1, X_2)$ with $X_1 \in \mathbb{R}^{n-m}$ and $X_2 \in \mathbb{R}^m$, respectively. The set ω_2 will be referred to as a cross section of the cylindrical domains Ω_ℓ. For $p_i, p \in (1,\infty)$, $i \in \{1, 2, \ldots, n\}$, consider the following generalisations of the Laplace equation:

$$\begin{cases} -\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(|(u_\ell)_{x_i}|^{p_i-2}\dfrac{\partial u_\ell}{\partial x_i}\right) = f(X_2) \quad \text{in } \Omega_\ell, \\ u_\ell = 0 \quad \text{on } \partial\Omega_\ell, \end{cases} \tag{1.1}$$

where $f : \omega_2 \to \mathbb{R}$ is in $C(\bar{\omega}_2)$. Let $p^+ = \max\{p_1, \ldots, p_n\} < n$. A typical space for considering a solution of (1.1) would be the anistropic Sobolev space

$$W(\Omega_\ell) := \left\{ u \in L^{p^+}(\Omega_\ell) \mid u_{x_i} \in L^{p_i}(\Omega_\ell) \right\}.$$

For detail reading on anisotropic Sobolev spaces we refer to [25, 29]. The existence of a solution for the above (and more general) type of a problem can be found in [30]. u_ℓ in (1.1) is also associated with an appropriate minimisation problem. The operator involved in (1.1) is called an anisotropic p-Laplacian. For the choice of $p_i = p$ for all i, the operator is called a pseudo p-Laplacian. For $p = 2$, a pseudo p-Laplacian is a Laplacian. Problems involving pseudo p-Laplace operators are widely studied in literature as it is able to model several real life problems with a great deal of accuracy. For example, problems in image processing and computer vision [33] are modeled by pseudo p-Laplace operators. The pseudo p-Laplace operator is also known as an "orthotropic p-Laplacian operator" as it is invariant with respect to the dihedral group of $N = 2$ (see [4]). This operator also naturally appears from optimal transport problems [6].

Consider an analogous equation on the cross section ω_2 defined as follows:

$$\begin{cases} -\sum_{i=n-m+1}^{n} \dfrac{\partial}{\partial x_i}\left(|W_{x_i}|^{p_i-2}\dfrac{\partial W}{\partial x_i}\right) = f(X_2) \quad \text{in } \omega_2, \\ W = 0 \quad \text{on } \partial\omega_2. \end{cases} \tag{1.2}$$

It will be helpful for our analysis. Define

$$D(u)(x) = \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i},$$

$$D_{X_2}(u)(x) = \sum_{i=n-m+1}^{n} |u_{x_i}(x)|^{p_i},$$

$$D_{X_1}(u)(x) = \sum_{i=1}^{n-m} |u_{x_i}(x)|^{p_i}.$$

Assumption (H): Assume for each $i \in \{1, \ldots, n-m\}$ there exists $j \in \{n-m+1, \ldots, n\}$ such that $p_i = p_j$.

Notice that whenever $m = 1$, Assumption (H) implies the operator to be a pseudo p-Laplace operator. But whenever the cross section ω_2 is more than one dimensional, than the operator is a strictly variable exponent pseudo p-Laplacian operator. The first aim of this article is to study the asymptotic behavior of u_ℓ as ℓ tends to infinity, and in this regard we will prove the following theorem.

Theorem 1.1. *Assume* (H) *holds, then for* $p_i \geq 2, \alpha \in (0, 1)$ *and for some constant* $C > 0$ *independent of* ℓ,

$$\int_{\Omega_{\alpha\ell}} D(u_\ell - W)dx \leq Ce^{-\alpha\ell}.$$

It is understood that W *is extended as a function of* X_2 *in the whole* $\Omega_{\alpha\ell}$.

For the case $p_i = 2$ for all i, a variable exponent (anisotropic) pseudo p-Laplace operator is a Laplace operator and the above theorem is studied in [12, 34]. The polynomial rate of convergence was first obtained in [12], whereas sharp exponential rate of convergence (on the right-hand side) was obtained in [34]. When $p_i = p$ the asymptotic behavior of the corresponding u_ℓ was studied in [9]. In [17], a similar problem was studied for purely variational problems and it has to be noticed that u_ℓ in (1.1) can also be obtained for the choice of

$$G(X) = \sum_{i=1}^{n} \frac{|x_i|^{p_i}}{p_i}$$

in [17]. Though it is not clear to us at this point if the above G satisfies uniform "convexity of power-q type" assumption of [17], but nevertheless even if it satisfies, only the polynomial rate of convergence can be obtained as an application of their main theorem Theorem 1.1, whereas we prove the exponential rate of convergence. This can be considered as the main novelty of this article. A similar problem for a fractional Laplacian operator, hyperbolic operators, parabolic operators, large solutions, purely variational problems and Allen–Cahn type equations are carried out in [2, 3, 7, 14, 17, 20] and [18], respectively.

The next goal of this article is to study (1.1) with $p_i = p > 1$ and we will stay with this assumption for the rest of the our results. The unique function u_ℓ in (1.1) has also the following variational characterisation:

$$J_\ell(u_\ell) = \inf_{u \in W_0^{1,p}(\Omega_\ell)} J_\ell(u),$$

where J_ℓ is the energy functional associated to the problem (1.1) and is defined as

$$J_\ell(u) = \sum_{i=1}^{n} \frac{1}{p} \int_{\Omega_\ell} |u_{x_i}|^p dx - \int_{\Omega_\ell} f u dx.$$

Also it is well known that

$$J_{\omega_2}(W) = \inf_{u \in W_0^{1,p}(\omega_2)} J_{\omega_2}(u)$$

where

$$J_{\omega_2}(u) = \sum_{i=n-m+1}^{n} \frac{1}{p} \int_{\omega_2} |u_{x_i}|^p dX_2 - \int_{\omega_2} f u dX_2$$

and where W is as in (1.2). Our next theorem in this direction is the following:

Theorem 1.2 (Convergence of the energy)**.** *If $p \geq 2$, then for some constant $C > 0$ we have*

$$J_{\omega_2}(W) + \frac{P(\omega_1)\mu_{n-m}(\ell\omega_1)}{\ell^p} \int_{\Omega_\ell} |u_\ell|^p dx \leq \frac{J_\ell(u_\ell)}{\mu_{n-m}(\Omega_\ell)} \leq J_{\omega_2}(W) + \frac{C}{\ell}.$$

The above theorem can be thought as a generalisation of Theorem 4.2 of [17] again for the choice of G made in there. The extra term

$$\frac{P(\omega_1)\mu_{n-m}(\ell\omega_1)}{\ell^p} \int_{\Omega_\ell} |u_\ell|^p dx$$

in the lower bound is the new contribution. Also our technique is new when compared to those used in [17], though we present both the techniques in the proof.

Next aim of this article is to study the Poincaré type inequality, related to a pseudo p-Laplace operator, on an infinite strip type domains $\Omega_\infty := \mathbb{R}^{n-m} \times \omega_2$ (such a name is used because one gets Ω_∞ by putting $\ell = \infty$ in Ω_ℓ.) Though Ω_∞ is not a bounded domain, but it is well known (any standard proof of Poincaré inequality will show this) that the Poincaré inequality holds for such domains. To be more precise, by the Poincaré inequality we mean the following inequality: For $u \in W_0^{1,p}(\mathbb{R}^{n-m} \times \omega_2)$ and a constant $C(\Omega_\infty) > 0$ (independent of u)

$$0 < C(\Omega_\infty) := \frac{\int_{\Omega_\infty} |\nabla u|^p dx}{\int_{\Omega_\infty} |u|^p dx}.$$

$C(\Omega_\infty)$ is understood to be the best possible constant in the above inequality. In [26], it was shown (after taking the matrix $A = I_{n\times n}$) that $C(\Omega_\infty) = C(\omega_2)$, where

$$C(\omega_2) := \inf_{u \in W_0^1(\omega_2)} \frac{\int_{\omega_2} |\nabla_{X_2} u|^p dX_2}{\int_{\omega_2} |u|^p dX_2}.$$

Similarly, the result for the case $p = 2$ but with a general matrix A, is done in [13] by using an approximation argument on the matrix A. Such types of results are also established on fractional Sobolev spaces and more generally on Orlicz fractional Sobolev spaces in [22] and [1], respectively. A generalisation of the work [13] for a generalised p-Laplace operator is carried out in [26].

Now using the fact that all norms in the Euclidean space are equivalent, it can be easily shown that for $\ell \in (0, \infty]$ we have

$$0 < P(\Omega_\ell) := \inf_{u \in W_0^1(\Omega_\ell)} \frac{\sum_{i=1}^n \int_{\Omega_\ell} |u_{x_i}|^p dx}{\int_{\Omega_\ell} |u|^p dx}$$

and

$$0 < P(\omega_2) := \inf_{u \in W_0^1(\omega_2)} \frac{\sum_{i=n-m+1}^n \int_{\omega_2} |u_{x_i}|^p dX_2}{\int_{\omega_2} |u|^p dX_2}.$$

The following theorem is our second main result in this direction.

Theorem 1.3. *If $p \in [1, \infty)$, then $P(\Omega_\infty) = P(\omega_2)$. Moreover, for ℓ sufficiently large enough and for some constant $C, D > 0$, one has*

$$P(\omega_2) + \frac{D}{\ell^p} \le P(\Omega_\ell) \le P(\omega_2) + \frac{C}{\ell^p}.$$

The above theorem says that the rate of convergence is ℓ^{-p}, and it is sharp. To the best of our knowledge, such a matching rate (an optimal rate) of convergence is not obtained in any other work in literature, except for the linear case of $p = 2$.

There is now a lot of literature available for problems on long cylindrical domains. In the paper [23] (see also [5]) it was proved, for a semilinear superlinear equation (i.e., with a homogeneous perturbation of exponent $m > 1$), that the decay to zero of the solution, for a general unbounded domain Ω, when $|x| \to \infty$, is of order of the inverse of a polynomial in $|x|$, but when the domain Ω is a cylinder then the decay is negatively exponential. So, the shape of the domain changes the type of decay to zero at the infinity of the solution.

For other related work on semilinear equations, polynominal rate of convergence is obtained in [23] and [5]. We refer to [3, 8, 10, 11, 16, 21, 24, 28, 31, 32] and the references therein for more detailed survey in this direction and related areas.

Application from the numerical point of view: The following feature is common in all the three theorems above: The solution (or parameter) on Ω_ℓ which is set on $\mathbb{R}^n$ finds its limiting connection on the cross section ω_2 of the cylinder which is a lower dimensional set (in $\mathbb{R}^{n-m}$). Clearly this has implications from a numerical point of view, as it saves a lot of computational cost (that arises due to the curse of dimensions) to work on a lower dimensional problem. For a direct application, the interested reader may look into [19].

This article is organised in the following way. In the next section, we introduce some function space, preliminaries and some estimates in the form of lemmas that are required. In the last section we present the proofs of our main theorems.

2. PRELIMINARIES

Through out this article $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ will denote a generic point. $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ will denote its Euclidean norm. $|x|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ will denote the ℓ_p norm of the point x. The following inequality (equivalence of all finite dimensional norms) will be used in several place, with making any references: for some constants $c_1, c_2 > 0$,

$$c_1|x| \le |x|_p \le c_2|x|, \quad x \in \mathbb{R}^n.$$

For us $W^{1,p}(\Omega), W_0^{1,p}(\Omega)$ will denote usual Sobolev spaces (see [27]). The space

$$V(\Omega_\ell) := \{\phi \in W(\Omega_\ell) \colon \phi = 0 \text{ on } \ \ell\omega_1 \times \partial\omega_2\}$$

equipped with the W norm. $\nabla, \nabla_{X_1}, \nabla_{X_2}$ will denote gradients in $\mathbb{R}^n, \mathbb{R}^{n-m}$ and $\mathbb{R}^m$, respectively. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^k$ will be denoted by $\mu_k(E)$. Throughout this article the value of the constants will be denoted by a generic number $C > 0$ and may change from line to line. In this article we are not worried about the existence of u_ℓ in (1.1), rather we assume existence, which we believe that it follows from a usual variational technique, and continue in establishing their asypmtotic behavior as ℓ tends to ∞. We say that u_ℓ is a weak solution of the problem (1.1), that is, if $u_\ell \in W(\Omega_\ell)$,

$$\sum_{i=1}^n \int\limits_{\Omega_\ell} |(u_\ell)_{x_i}|^{p_i-2}(u_\ell)_{x_i}(\phi_\ell)_{x_i} = \int\limits_{\Omega_\ell} f\phi, \quad \forall \phi \in W(\Omega_\ell).$$

Lemma 2.1. *There exists a constant $C > 0$ (independent of ℓ) and for some positive integer δ, we have*

$$\int_{\Omega_\ell} |D(u_\ell - W)| dx \le C\ell^\delta.$$

Proof. Using $v = u_\ell$ in (3.5) which if the equation that u_ℓ satisfies weakly, one obtains

$$\int_{\Omega_\ell} |(u_\ell)_{x_i}|^{p_i} dx \le \sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{p_i} dx = \int_{\Omega_\ell} f u_\ell dx, \quad \forall i.$$

Using the Hölder's inequality, we have

$$\int_{\Omega_\ell} |(u_\ell)_{x_i}|^p dx = \int_{\Omega_\ell} f u_\ell dx \le \left(\int_{\Omega_\ell} u_\ell^{p_i} dx \right)^{1/p_i} \left(\int_{\Omega_\ell} f^{q_i} dx \right)^{1/q_i},$$

where $1/p_i + 1/q_i = 1$. Now using the Poincaré inequality and Lemma 2.2, we have

$$\int_{\Omega_\ell} | \ (u_\ell)_{x_i}|^{p_i} dx \le C\ell^\delta,$$

where the constant C depends on $f, n, m, \Omega_1, \Omega_2, p_i$ and δ is a positive integer that depends on n and m only. Now finally,

$$\int_{\Omega_\ell} D(u_\ell - W) dx \le C \sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{p_i} + |W_{x_i}|^{p_i} dx \le C\ell^\delta,$$

possibly for a different δ than before. This finishes the proof of the lemma. □

Lemma 2.2. *Let $P(\ell\omega_1)$, $P(\omega_1)$ denote the best Poincaré constant of the domains $\ell\omega_1$ and ω_1, respectively, then*

$$P(\ell\omega_1) = \ell^{-p} P(\omega_1).$$

Proof. The proof follows via a standard scaling argument. □

We need the following version of the inequality for $n = 1$, but nevertheless we state it more generally.

Lemma 2.3 (An inequality)**.** *If $p \ge 2$, then there exist a constant $C_p > 0$ such that*

$$\sum_{i=1}^{n} \left(|x_i|^{p-2} x_i - |y_i|^{p-2} y_i \right) (x_i - y_i) \ge C_p |x - y|^p,$$

for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Proof. We use the one dimension version of the following well known inequality [15]:

$$\left\langle (|x|^{p-2}x - |y|^{p-2}y), (x-y) \right\rangle \geq D_p |x-y|^p,$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. Just for an idea, the above inequality can be proved using fundamental theorem of calculus, after writing

$$|x|^{p-2}x - |y|^{p-2}y = \int_0^1 g'(\xi) d\xi$$

where $g(\xi) = |(1-\xi)x + y\xi|^{p-2}\left((1-\xi)x + \xi y\right)$ and then making appropriate estimates for $p \geq 2$. Using this inequality for $n = 1$, we get that for each $i = 1, \ldots, n$,

$$\left(|x_i|^{p-2}x_i - |y_i|^{p-2}y_i\right)(x_i - y_i) \geq D_p |x_i - y_i|^p.$$

Therefore summing up over each i gives

$$\sum_{i=1}^{n} \left(|x_i|^{p-2}x_i - |y_i|^{p-2}y_i\right)(x_i - y_i) \geq D_p \sum_{i=1}^{n} |x_i - y_i|^p = D_p |x-y|_p^p \geq \tilde{D}_p |x-y|^p.$$

This finishes the proof of the lemma. □

Lemma 2.4 (Uniform Poincaré inequality)**.** *There exists a constant* $C > 0$ *(independent of* ℓ*) such that for each* $i = \{1, \ldots, n\}$ *and* $\ell' + 1 \leq \ell$*,*

$$\int_{\Omega_{\ell'+1} \setminus \Omega_{\ell'}} D(\phi) dx \geq C \int_{\Omega_{\ell'+1} \setminus \Omega_{\ell'}} |\phi|^{p_i} dx, \text{ for all } \phi \in V(\Omega_\ell).$$

Proof. It is sufficient to prove the inequality for $\phi \in C_c^\infty(\Omega_\ell)$, as it is a dense subspace of $V(\Omega_\ell)$. First let us deal with the case when $i = n-m+1, \ldots, n$. Notice that, since ω_2 is bounded subset of $\mathbb{R}^{n-m}$, we can use the usual Poincaré inequality to obtain

$$\int_{\omega_2} |\phi_{x_i}|^{p_i} dX_2 \geq C \int_{\omega_2} \phi^{p_i} dX_2.$$

The required inequality then using the estimate $|D(\phi)| \geq |\phi_{x_i}|^{p_i}$, and then integrating both sides over the rest of the variables.

Now let us consider the case when $i \in \{1, \ldots, n-m\}$. Fix an i. Let us consider $i = 1$ without any loss of generality. Now using our assumption that there exists $j \in \{n-m+1, \ldots, n\}$ such that $p_j = p_1$. We have apply the Poincaré inequality in the x_j direction to get

$$\int_{\omega_2} |\phi|^{p_1} dX_2 \leq C \int_{\omega_2} |\phi_{x_j}|^{p_1} dX_2 = \int_{\omega_2} |\phi_{x_j}|^{p_j} dX_2 \leq \int_{\omega_2} D(\phi) dX_2.$$

The result then again follows after integrating over rest of the variables. □

Lemma 2.5. *The function* $V(X_1, X_2) := W(X_2) \in W(\Omega_\ell)$, *where* W *is as in* (1.2) *satisfies the following equation weakly, for each* $\ell > 0$*:*

$$\begin{cases} -\sum_{i=1}^n \dfrac{\partial}{\partial x_i}\left(|(V_\ell)_{x_i}|^{p_i-2}\dfrac{\partial V_\ell}{\partial x_i}\right) = f(X_2) \quad in \;\; \Omega_\ell, \\ V_\ell = 0 \quad\; on \quad \ell\omega_1 \times \partial\omega_2, \\ V_\ell = W \quad on \quad \partial\ell\omega_1 \times \omega_2. \end{cases}$$

Proof. By a weak solution above we mean that for any $v \in W_0^{1,p}(\Omega_\ell)$, the following inequality is satisfied:

$$\sum_{i=1}^n \int_{\Omega_\ell} |(V_\ell)_{x_i}|^{p_i-2}\frac{\partial V_\ell}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \int_{\Omega_\ell} f(X_2)v dx.$$

Using Fubini's theorem we get

$$\begin{aligned} &\sum_{i=1}^n \int_{\Omega_\ell} |(V_\ell)_{x_i}|^{p_i-2}\frac{\partial V_\ell}{\partial x_i}\frac{\partial v}{\partial x_i}dx \\ &= \int_{\ell\omega_1}\left(\sum_{i=n-m+1}^n \int_{\omega_2} |(W)_{x_i}(\_, X_2)|^{p_i-2}\frac{\partial W}{\partial x_i}(\_, X_2)\frac{\partial v}{\partial x_i}(\_, X_2)dX_2\right)dX_1. \end{aligned}$$

Now using the weak formulation for the equation (1.2) and Fubini's theorem again, we get

$$\begin{aligned} &\sum_{i=1}^n \int_{\Omega_\ell} |(V_\ell)_{x_i}|^{p_i-2}\frac{\partial V_\ell}{\partial x_i}\frac{\partial v}{\partial x_i}dx \\ &= \int_{\ell\omega_1}\left(\int_{\omega_2} f(X_2)v(\_, X_2)dX_2\right)dX_1 = \int_{\Omega_\ell} f(X_2)v dx. \end{aligned}$$

This finishes the proof of the lemma. □

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.3. Let $\phi \in W_0^1(\Omega_\ell)$, where $\ell \in (0, \infty]$ (notice that the case $\ell = \infty$ is also included). Clearly,

$$\sum_{i=1}^n \int_{\Omega_\ell} |\phi_{x_i}|^p dx = \sum_{i=1}^{n-m} \int_{\Omega_\ell} |\phi_{x_i}|^p dx + \sum_{i=n-m+1}^n \int_{\Omega_\ell} |\phi_{x_i}|^p dx.$$

Using Fubini's Theorem and Lemma 2.2 on the first part, we get

$$\sum_{i=1}^{n-m}\int\limits_{\Omega_\ell}|\phi_{x_i}|^p dx = \sum_{i=1}^{n-m}\int\limits_{\omega_2}\left(\int\limits_{\ell\omega_1}|\phi_{x_i}(X_1,X_2)|^p dX_1\right)dX_2$$
$$\geq \ell^{-p}P(\omega_1)\int\limits_{\Omega_\ell}|\phi|^p dx.$$

Using Fubini's Theorem and the definition of $P(\omega_2)$, we get

$$\sum_{i=1}^{n-m}\int\limits_{\Omega_\ell}|\phi_{x_i}|^p dx = \sum_{i=n-m+1}^{n}\int\limits_{\ell\omega_1}\left(\int\limits_{\omega_2}|\phi_{x_i}(X_1,X_2)|^p dX_2\right)dX_1$$
$$\geq P(\omega_2)\sum_{i=n-m+1}^{n}\int\limits_{\ell\omega_1}\left(\int\limits_{\omega_2}|\phi(X_1,X_2)|^p dX_2\right)dX_1$$
$$= P(\omega_2)\sum_{i=n-m+1}^{n}\int\limits_{\Omega_\ell}|\phi|^p dx.$$

Now since the space $C_c^\infty(\Omega_\ell)$ is dense in $W_0^{1,p}(\Omega_\ell)$, it implies, after taking infimum over the previous inequality, that $P(\omega_2)\leq P(\Omega_\ell)$.

For the other part of the inequality, first consider the following sequence of functions $\phi_\ell:\ell\omega_1\to\mathbb{R}$, with the property that $\phi_\ell\in W_0^{1,p}(\ell\omega_1)$, $0\leq\phi\leq 1$, $\phi_\ell=1$ on $\frac{\ell}{2}\omega_1$ and

$$|\nabla_{X_1}\phi_\ell|\leq\frac{C}{\ell},\quad \forall x\in\ell\omega_1. \tag{3.1}$$

Finally, consider the function $\tilde{W}\phi_\ell\in W_0^{1,p}(\Omega_\ell)$, where $\tilde{W}\in W_0^{1,p}(\omega_2)$ satisfies

$$P(\omega_2)\int\limits_{\omega_2}|\tilde{W}|^p dX_2 = \sum_{i=n-m+1}^{n}\left(\int\limits_{\omega_2}|\tilde{W}_{x_i}|^p dX_2\right). \tag{3.2}$$

Since $W_0^{1,p}(\Omega_\ell)\subset W_0^{1,p}(\Omega_\infty)$ (trivially extending each function with the value 0 outside Ω_ℓ), using the definition of $P(\Omega_\ell)$, we get

$$P(\Omega_\ell)\leq\frac{\sum_{i=1}^{n}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx},\quad \forall\ell>0. \tag{3.3}$$

Simplifying the right-hand side of the above expression, we get

$$\begin{aligned}&\frac{\sum_{i=1}^{n}\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx}\\&=\frac{\sum_{i=1}^{n-m}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)+\sum_{i=n-m+1}^{n}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx}\\&=\frac{\sum_{i=1}^{n-m}\left(\int_{\Omega_\ell}|\tilde{W}|^p|(\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}|^p|\phi_\ell|^p dx}+\frac{\sum_{i=n-m+1}^{n}\left(\int_{\Omega_\ell}|\tilde{W}_{x_i}|^p|\phi_\ell|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}|^p|\phi_\ell|^p dx}.\end{aligned}$$

Using Fubini's theorem and (3.2), we get

$$\begin{aligned}&\frac{\sum_{i=1}^{n}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx}\\&=\frac{\sum_{i=1}^{n-m}\left(\int_{\ell\omega_1}|(\phi_\ell)_{x_i}|^p dX_1\right)}{\int_{\ell\omega_1}|\phi_\ell|^p dX_1}+\frac{\sum_{i=n-m+1}^{n}\left(\int_{\omega_2}|\tilde{W}_{x_i}|^p dX_2\right)}{\int_{\omega_2}|\tilde{W}|^p dX_2}\\&=P(\omega_2)+\frac{\sum_{i=1}^{n-m}\left(\int_{\ell\omega_1}|(\phi_\ell)_{x_i}|^p dX_1\right)}{\int_{\ell\omega_1}|\phi_\ell|^p dX_1}.\end{aligned}$$

Now using the inequality $\left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}}\leq C\left(\sum_{i=1}^{k}|x_i|^2\right)^{\frac{1}{p}}$, where $k\in\mathbb{N}$, we obtain

$$\frac{\sum_{i=1}^{n}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx}\leq P(\omega_2)+C\frac{\int_{\ell\omega_1}|\nabla_{X_1}\phi_\ell|^p dX_1}{\int_{\ell\omega_1}|\phi_\ell|^p dX_1}.$$

Now we use (3.1), to get the following estimates:

$$\frac{\sum_{i=1}^{n}\left(\int_{\Omega_\ell}|(\tilde{W}\phi_\ell)_{x_i}|^p dx\right)}{\int_{\Omega_\ell}|\tilde{W}\phi_\ell|^p dx}\leq P(\omega_2)+C\frac{\ell^{-p}\mu_{n-m}(\ell\omega_1\setminus\frac{\ell}{2}\omega_1)}{\mu_{n-m}(\frac{\ell}{2}\omega_1)}\leq P(\omega_2)+\frac{\tilde{C}}{\ell^p},\quad(3.4)$$

where μ_{n-m} denotes the $n-m$ dimensional Lebesgue measure of a measurable set in $\mathbb{R}^{n-m}$. The upper bound follows after combining (3.3) and (3.4).

Now for the first part of the theorem, it is already proved that $P(\omega_1)\leq P(\Omega_\infty)$ in the beginning of the proof. The other part of the inequality follows after observing that $P(\Omega_\infty)\leq P(\Omega_\ell)$, for each $\ell>0$, and then letting ℓ tend to ∞ together with the second part of the proof. □

Now we turn to the proof of our first theorem.

Proof of Theorem 1.1. As u_ℓ satisfies (1.1) weakly, this means that for any $v\in W_0^{1,p}(\Omega_\ell)$ one has

$$\sum_{i=1}^{n}\int_{\Omega_\ell}|(u_\ell)_{x_i}|^{p_i-2}\frac{\partial u_\ell}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\int_{\Omega_\ell}f(X_2)v\,dx.\quad(3.5)$$

This together with Lemma 2.5 gives for all $v \in W_0^{1,p}(\Omega_\ell)$,

$$\sum_{i=1}^{n} \int_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - |W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right) \frac{\partial v}{\partial x_i} dx = 0.$$

For $\ell' \in (0, \ell-1)$, let $\rho_{\ell'}$ be a function, whose precise properties will be specified later, such that $v(= v_\ell) := \rho_\ell(u_\ell - W) \in W_0^{1,p}(\Omega_\ell)$. Substituting this v in the previous equation yields

$$\sum_{i=1}^{n} \int_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - | W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right) \left\{ (u_\ell - W) \frac{\partial \rho_\ell}{\partial x_i} + \rho_\ell \frac{\partial}{\partial x_i}(u_\ell - W) \right\} = 0.$$

Hence we have

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - | W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right) \left\{ \rho_\ell \frac{\partial}{\partial x_i}(u_\ell - W) \right\} dx \\ &= -\sum_{i=1}^{n} \int_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - | W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right) \left\{ (u_\ell - W) \frac{\partial \rho_\ell}{\partial x_i} \right\} dx. \end{aligned}$$

Next using the inequality in Lemma 2.3, we get

$$\begin{aligned} &C_p \int_{\Omega_\ell} \rho_\ell D(u_\ell - W) dx \\ &\leq -\sum_{i=1}^{n} \int_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - |W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right) \left\{ (u_\ell - W) \frac{\partial \rho_\ell}{\partial x_i} \right\} dx \\ &\leq \sum_{i=1}^{n} \int_{\Omega_\ell} \left| |(u_\ell)_{x_i}|^{p_i-2} \frac{\partial u_\ell}{\partial x_i} - |W_{x_i}|^{p_i-2} \frac{\partial W}{\partial x_i} \right| \left| u_\ell - W \right| \left| \frac{\partial \rho_\ell}{\partial x_i} \right| dx. \end{aligned}$$

Now we make the choice of $\rho_{\ell'}$ in the following way:

$$\rho_{\ell'} = \rho_{\ell'}(X_1), \quad 0 \leq \rho_{\ell'} \leq 1, \quad \rho_{\ell'} = 1 \text{ on } \Omega_{\ell'}, \quad \rho_{\ell'} = 0 \text{ on } \Omega_{\ell'+1}^c \text{ and } |\nabla_{X_1} \rho_{\ell'}| \leq 1.$$

We get

$$\begin{aligned} &C_p \int_{\Omega_\ell} \rho_\ell D(u_\ell - W) dx \\ &\leq \sum_{i=1}^{n} \int_{\Omega_{\ell'+1} \setminus \Omega_{\ell'}} \left| |(u_\ell)_{x_i}|^{p_i-2} (u_\ell)_{x_i} - |W_{x_i}|^{p_i-2} W_{x_i} \right| \left| u_\ell - W \right| \left| \frac{\partial \rho_\ell}{\partial x_i} \right| dx. \end{aligned}$$

Now $|(\rho_\ell)_{x_i}| \leq 1$ and $\rho_\ell = 1$ over $\Omega_{\ell'}$, so

$$C_p \int\limits_{\Omega_{\ell'}} D(u_\ell - W)dx \leq \sum_{i=1}^{n-m} \int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |(u_\ell - W)_{x_i}|^{p_i-1}|u_\ell - W|dx.$$

Using the Hölder inequality and the Poincaré inequality, we have

$$C_p \int\limits_{\Omega_{\ell'}} D(u_\ell - W)dx$$
$$\leq \sum_{i-1}^{n-m} \left(\int\limits_{\Omega_{\ell'+1\setminus\Omega_{\ell'}}} |(u_\ell - W)_{x_i}|^{p_i} dx \right)^{\frac{p_i-1}{p_i}} \left(\int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |u_\ell - W|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

Now using the uniform Poincaré inequality (Lemma 2.4), $|(u_\ell - W)_{x_i}|^{p_i}| \leq D(u_\ell - W)$ on the right-hand side of the above expression we get for some constant $C > 0$,

$$\int\limits_{\Omega_{\ell'}} D(u_\ell - W)dx \leq C \int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} D(u_\ell - W)dx.$$

This is nothing but

$$\int\limits_{\Omega_{\ell'}} D(u_\ell - W)dx \leq \frac{C}{C+1} \int\limits_{\Omega_{\ell'+1}} D(u_\ell - W)dx.$$

Now iterating the above inequality after choosing $\ell' = \frac{\ell}{2}, \frac{\ell}{2}+1, \frac{\ell}{2}+2, \dots, \frac{\ell}{2}+[\frac{\ell}{2}]$, where $[\frac{\ell}{2}]$ denotes the greatest integer less than or equal to $\frac{\ell}{2}$, we obtain

$$\begin{aligned}\int\limits_{\Omega_{\ell/2}} D(u_\ell - W)dx &\leq \left(\frac{C}{C+1}\right)^{[\ell/2]} \int\limits_{\Omega_{\ell/2+[\ell/2]}} D(u_\ell - W)dx \\ &\leq \left(\frac{C}{C+1}\right)^{[\ell/2]} \int\limits_{\Omega_\ell} D(u_\ell - W)dx.\end{aligned}$$

Rewriting the above equation in a different way we have

$$\int\limits_{\Omega_{\ell/2}} D(u_\ell - W)dx \leq e^{\log(\frac{C}{C+1})\ell/2} \int\limits_{\Omega_\ell} D(u_\ell - W)dx.$$

The proof the theorem for $\alpha = 1/2$ finally follows after using Lemma 2.1 and observing that

$$\log\left(\frac{C}{C+1}\right) = \beta < 0.$$

For a general $\alpha \in (0,1)$ one has to choose $\ell' = \alpha\ell, \alpha\ell+1, \dots, \alpha\ell + [\ell - \alpha\ell]$ to get the general result. □

Clearly, Theorem 1.1 implies that up to a subsequence u_ℓ converges to W in $L^p(\Omega_{\alpha\ell})$ and hence pointwise as well. One may ask if the above convergence also takes place on the entire Ω_ℓ. We believe that the answer is negative, as W does not satisfy zero boundary conditions on the lateral part of the boundary. In this aspect a more relevant question that one can ask is as to whether the convergence happens in $\Omega_{\ell-1}$, but this is also unclear from our approach.

Now we present the proof of Theorem 1.2

Proof of Theorem 1.2. A weaker estimate for the lower bound: Consider the sequences of test functions $\psi_\ell \in W_0^{1,p}(\Omega_\ell)$ defined as

$$\psi_\ell(X_2) := \frac{1}{\mu_{n-m}(\ell\omega_1)} \int_{\ell\omega_1} u_\ell(\_, X_2)dX_1.$$

Since we have

$$J_{\omega_2}(W) = \inf_{u \in W_0^{1,p}(\omega_2)} J_{\omega_2}(u),$$

this implies that for $\ell > 0$,

$$\begin{aligned} J_{\omega_2}(W) \leq J_{\omega_2}(\psi_\ell) &= \frac{1}{p}\left(\sum_{i=n-m+1}^{n} \int_{\omega_2} |(\psi_\ell)_{x_i}|^p dX_2\right) - \int_{\omega_2} f\psi_\ell dX_2 \\ &\leq \frac{1}{p}\left(\sum_{i=n-m+1}^{n} \int_{\omega_2} \left| \frac{1}{\mu_{n-m}(\ell\omega_1)} \int_{\ell\omega_1} (u_\ell)_{x_i}(\_, X_2)dX_1 \right|^p dX_2\right) \\ &\quad - \int_{\omega_2} \frac{f(X_2)}{\mu_{n-m}(\ell\omega_1)} \left(\int_{\ell\omega_1} u_\ell(\_, X_2)dX_1\right) dX_2. \end{aligned}$$

Now using Jensen's inequality for the integrals, one has

$$\begin{aligned} \mu_{n-m}(\ell\omega_1)J_{\omega_2}(W) &\leq \frac{1}{p}\left(\sum_{i=n-m+1}^{n} \int_{\omega_2}\int_{\ell\omega_1} |(u_\ell)_{x_i}(\_, X_2)|^p \, dX_1 dX_2\right) \\ &\quad - \int_{\omega_2}\int_{\ell\omega_1} f u_\ell(X_1, X_2)dX_1 dX_2 \\ &\leq \frac{1}{p}\left(\sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}(\_, X_2)|^p \, dX_1 dX_2\right) \\ &\quad - \int_{\Omega_\ell} f u_\ell(X_1, X_2)dX_1 dX_2 = J_\ell(\Omega_\ell). \end{aligned}$$

A sharper estimate for the lower bound: By definition,

$$J_\ell(\Omega_\ell) = \frac{1}{p}\sum_{i=1}^{n-m}\int\limits_{\Omega_\ell}|(u_\ell)_{x_i}|^p + \left(\frac{1}{p}\sum_{i=n-m+1}^{n}\int\limits_{\Omega_\ell}|(u_\ell)_{x_i}|^p - \int\limits_{\Omega_\ell} f u_\ell\right).$$

Using Lemma 2.2 on the first integrand, it is easy to obtain that

$$\sum_{i=1}^{n-m}\int\limits_{\Omega_\ell}|(u_\ell)_{x_i}|^p \geq P(\omega_1)\ell^{-p}\int\limits_{\Omega_\ell} u_\ell^p,$$

whereas using Fubini's theorem and the definition of $J_{\omega_2}(W)$ on the second term, we have

$$\frac{1}{p}\sum_{i=n-m+1}^{n}\int\limits_{\Omega_\ell}|(u_\ell)_{x_i}|^p - \int\limits_{\Omega_\ell} f u_\ell \geq \mu_{n-m}(\ell\omega_1)J_{\omega_2}(W).$$

Combining the above inequalities the stronger estimate follows.

For the second one, first we consider a Lipschitz continuous cutoff function $\rho_\ell = \rho_\ell(X_1)$, $0 \leq \rho_\ell \leq 1, |\nabla_{X_1}\rho_\ell| \leq C$. We also further assume that $\rho_\ell = 1$ on $(\ell-1)\omega_1$ and $\rho_\ell = 0$ on $\partial(\ell\omega_1)$. Since $\rho_\ell(X_1)W(X_2) \in W_0^{1,p}(\Omega_\ell)$, we have $J_\ell(u_\ell) \leq J_\ell(\rho_\ell W)$.

Now estimating the right-hand side gives

$$\begin{aligned}
J_\ell(\rho_\ell W) &= J_{\ell-1}(W) + \frac{1}{p}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|(\rho_\ell W)x_i|^p dx\right) - \int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|\rho_\ell W|^p dx \qquad (3.6)\\
&= J_\ell(W) + \left\{\frac{1}{p}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|(\rho_\ell W)x_i|^p dx\right) - \int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|\rho_\ell W|^p dx\right\}\\
&\quad - \left\{\frac{1}{p}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|(W)x_i|^p dx\right) - \int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|W|^p dx\right\}\\
&= \mu_{n-m}(\ell\omega_1)J_{\omega_2}(W) + \mathcal{A}_\ell - \mathcal{B}_\ell.
\end{aligned}$$

First notice that

$$\mathcal{B}_\ell = \mu_{n-m}(\ell\omega_1 \setminus (\ell-1)\omega_1)J(W). \qquad (3.7)$$

Then we estimate the term $\mathcal{A}_\ell$.

$$\begin{aligned}
\mathcal{A}_\ell &\leq \frac{1}{p}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|(\rho_\ell W)x_i|^p dx\right)\\
&\leq \frac{C_p}{p}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|\rho_\ell W_{x_i}|^p + |W(\rho_\ell)_{x_i}|^p dx\right).
\end{aligned}$$

Using the properties of ρ_ℓ, we can further estimate, and get for some other constant $D_{W,p} > 0$,

$$\begin{aligned}\mathcal{A}_\ell &\leq \frac{C_p}{p}\left(\sum_{i=1}^{n}\int_{\Omega_\ell\setminus\Omega_{\ell-1}} |W_{x_i}|^p + |W|^p dx\right) \\ &= \frac{D_{W,p}}{p}\mu_{n-m}(\ell\omega_1 \setminus (\ell-1)\omega_1) = C\ell^{n-m-1}.\end{aligned} \tag{3.8}$$

Finally combining, (3.6), (3.7) and (3.8), we get

$$\frac{J_\ell(u_\ell)}{\mu_{n-m}(\ell\omega_1)} \leq J_{\omega_2}(W) + \frac{C}{\ell}.$$

This finishes the proof of the theorem. □

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