Opuscula Mathematica • Vol. 25 • No. 1 • 2005

Stanisław Białas

A SUFFICIENT CONDITION FOR SCHUR STABILITY OF THE CONVEX COMBINATION OF THE POLYNOMIALS

Abstract. In this paper is given a simple sufficient condition for Schur stability of the convex combination of the real polynomials.

Keywords: convex sets of polynomials, stability of polynomial sets, Schur stability.

Mathematics Subject Classification: 93D09, 15A63.

1. INTRODUCTION

The stability analysis of many control systems are concerned with the zero locations of their characteristic polynomials. If the coefficients of a polynomial are known exactly, then its zero locations can be determined readily by the well-known methods. However, in practice the system model is used at best an approximation of the real process. Thus, one of the real problems of the stability analysis is to determine the stability of family of polynomials or matrices.

One of such family of polynomials is a convex combination of the real polynomials.

Ackermann (1988) has given the necessary and sufficient conditions for Schur stability of the convex combination of polynomials but these conditions are not practical in calculation.

In the present paper, we given the simple sufficient condition for the Schur stability of the convex combination of the polynomials.

In this note we consider only the real polynomials.

The real polynomial

$$g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - x_1)(x - x_2)\cdots(x - x_n) \qquad (1)$$

is called Schur stable if $|x_i| < 1$ $(i = 1, 2, \ldots, n)$.

A set of polynomials W is called Schur stable if each polynomial $f(x) \in W$ is Schur stable. In this paper, we will write "stable" instead of "Schur stable".

Let

$$S_1(g) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_3 & a_2 \\ 0 & a_n & a_{n-1} & \dots & a_4 & a_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_n \end{bmatrix},$$

$$S_2(g) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & a_2 & \dots & a_{n-3} & a_{n-2} \end{bmatrix},$$

$$S(g) = S_1(g) - S_2(g).$$

It is easy to see that $S_1(g)$, $S_2(g)$, $S(g) \in R^{(n-1)\times(n-1)}$. The matrix $S(g)$ plays a fundamental role of Schur stability of polynomials, see [2].

For simplicity, we consider monic polynomials in the main result below.

Consider the real polynomials

$$f_j(x) = x^n + a_{n-1}^{(j)} x^{n-1} + \dots + a_1^{(j)} x + a_0^{(j)} \tag{2}$$

for $j = 1, 2, \dots, m$.

We apply the following notations:

$$P_n = \left\{ f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \colon a_i \in R \ (i = 0, 1, \dots, n-1) \right\},$$
$$L_n = \{ f(x) = (x - x_1)(x - x_2)\cdots(x - x_n) \in P_n \colon |x_i| < 1 \ (i = 1, 2, \dots, n) \},$$
$$V_m = \{ (\alpha_1, \alpha_2, \dots, \alpha_m) \in R^m \colon \alpha_j \geq 0 \ (j = 1, 2, \dots, m), \ \alpha_1 + \alpha_2 + \dots + \alpha_m = 1 \},$$
$$C(f_1, f_2, \dots, f_m) = \{ (\alpha_1 f_1(x) + \alpha_2 f_2(x) + \dots + \alpha_m f_m(x) \colon (\alpha_1, \alpha_2, \dots, \alpha_m) \in V_m \},$$
$$C(f_i, f_j) = \{ (\alpha_i f_i(x) + \alpha_j f_j(x) \colon (\alpha_i, \alpha_j) \in V_2 \}.$$

Let B^T denote the transpose of the matrix B, $\lambda_i(A)$-eigenvalue of the matrix $A \in R^{n\times n}$.

It is seen that $C(f_1, f_2, \dots, f_m)$ denotes the convex combination of the polynomials $f_1(x), f_2(x), \dots, f_m(x)$. It is known that from the stability of the polynomials $f_i(x)$, $f_j(x)$ does not imply that $C(f_i, f_j) \subset L_n$.

In this paper, we present the simple sufficient condition for Schur stable of the set $C(f_1, f_2, \dots, f_m)$. We assume that the polynomials $f_j(x) \in L_n$ $(j = 1, 2, \dots, m)$. Hence, there exists the inverse matrix $S^{-1}(f_j)$ $(j = 1, 2, \dots, m)$.

Let

$$\lambda_k(S^{-1}(f_j)S(f_i)) \quad (k = 1, 2, \dots, n-1;\ i, j = 1, 2, \dots, m;\ j < i)$$

denote the eigenvalues of the matrix $S^{-1}(f_j)S(f_i)$.

If the matrix $Q = Q^T \in R^{n\times n}$ and $\lambda_i(Q) > 0$ $(i = 1, 2, \dots, n)$, then we write $Q > 0$.

The following theorems are true:

Theorem 1.1. [1] *If the polynomials* (2) *are Schur stable, then the convex combination* $C(f_1, f_2, \dots, f_m) \subset L_n$ *if and only if* $\lambda_k(S^{-1}(f_j)S(f_i)) \notin (-\infty, 0\rangle$ *for* $k = 1, 2, \dots, n-1$*;* $i, j = 1, 2, \dots, m$*;* $j < i$*.*

Theorem 1.2. [3] *If the matrix* $A \in R^{n\times n}$ *and there exists the matrix* $P \in R^{n\times n}$ *such that*

$$P^T + P > 0,$$
$$A^T P^T + PA > 0,$$

then $\lambda_i(A) \notin (-\infty, 0\rangle$ $(i = 1, 2, \dots, n)$*.*

2. MAIN RESULT

Now, we will prove the simple sufficient condition for Schur stability of the convex combination of the polynomials (2).

Theorem 2.1. *If the real polynomials* (2) *are Schur stable and there exists the matrix* $P_{ij} \in R^{(n-1)\times(n-1)}$ *such that*

$$S^T(f_i)P_{ij}^T + P_{ij}S(f_i) > 0 \tag{3}$$

$$S^T(f_j)P_{ij}^T + P_{ij}S(f_j) > 0 \tag{4}$$

for $i, j = 1, 2, \dots, m$*;* $j < i$*, then the convex combination* $C(f_1, f_2, \dots, f_m) \subset L_n$*.*

Proof. From the assumption $f_i(x), f_j(x) \in L_n$ it follows that there exist the inverse matrices $S^{-1}(f_i)$, $S^{-1}(f_j)$. For the matrix $S^{-1}(f_i)S(f_j)$ we have

$$\left[S^{-1}(f_i)S(f_j)\right]^T [P_{ij}S(f_i)]^T + [P_{ij}S(f_i)]\left[S^{-1}(f_i)S(f_j)\right] = S^T(f_j)P_{ij}^T + P_{ij}S(f_j) > 0.$$

From the last relation and the assumption (3), by applying Theorem 1.2, we have

$$\lambda_k(S^{-1}(f_i)S(f_j)) \notin (-\infty, 0\rangle \quad (k = 1, 2, \dots, n-1)$$

for $i, j = 1, 2, \dots, m$; $j < i$. Hence and from Theorem 1.1, it follows that the convex combination $C(f_1, f_2, \dots, f_m) \subset L_n$. This completes the proof of Theorem 2.1. □

From Theorem 2.1, it follows

Corollary 2.1. *If the polynomials* (2) *are Schur stable and*

$$S^T(f_i) + S(f_i) > 0 \quad (i = 1, 2, \dots, m),$$

then the convex combination $C(f_1, f_2, \dots, f_m) \subset L_n$.

The inverse thesis is not true.

REFERENCES

[1] Ackermann J. E., Barmish B. R.: *Robust Schur stability of a polytope of polynomials*. IEEE Trans. Automat. Contr., 33, No. 10 (1988), 984–986.

[2] Jury E. I.: *Inners and Stability of Dynamic Systems*. New York, Wiley 1974.

[3] Neumann M.: *Weak stability for matrices*. Linear and Multilinear Algebra **7** (1979), 257–262.

Stanisław Białas
sbialas@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

Received: February 20, 2004.