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Petru A. Cojuhari, Alexander M. Gomilko

THE ABEL SUMMATION OF THE KONTOROVICH–LEBEDEV INTEGRAL REPRESENTATION

Abstract. A new result on the summation of the Kontorovich–Lebedev integral representation in the sense of Abel mean is given.

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1. INTRODUCTION

In this note a result concerning the representation a function by the Kontorovich–Lebedev transform is given. The Kontorovich–Lebedev transform is relative to the class of integral transformations of the Bessel type in which the integration is taken with respect to the index of the Bessel's functions. For the first time a such transformation, later called in the name of the authors, was considered by M.I. Kontorovich and N.N. Lebedev in the work [4]. It was found that transformations of this type are useful in the study of various problems of the mathematical physics as, for instance, are static boundary value problems on wedge-shaped domains (in this respect, we note the works [7, 9] and see also [3]). The Kontorovich–Lebedev transform $F(\tau)$ of $f(x)$ is a function of the real positive variable τ defined by

$$F(\tau) = \int_0^\infty f(x) K_{i\tau}(x)dx, \quad \tau > 0, \tag{1.1}$$

where $K_{i\tau}(x)$ is the cylindrical Macdonald function. In [4], it is shown that under certain restrictions the transform given by (1.1) can be inverted by the following formula

$$f(x)=\frac{2}{\pi^2 x}\int\limits_0^\infty K_{i\tau}(x)\tau\,\mathrm{sh}\,\pi\tau F(\tau)d\tau,\quad x>0. \tag{1.2}$$

The conditions that was imposed on the function f is that f is of the class C^1 and that the functions $xf(x)$ and $x^2f(x)$ are summable over $0<x<\infty$. Note that the inverse transformation (1.2) is also valid under less restrictive conditions (see [5, 6] and [7]) or for other classes of functions [10]. In line with [10], for each function $M(s)$ with property that $M(\frac{1}{2}+it)\in L_1(-\infty,\infty)$ and that

$$\lim_{|\sigma|\to\infty}\int\limits_{\frac{1}{2}+i\sigma}^{\frac{1}{2}+i(\sigma+1)}\left|\frac{1}{2}+it\right|^{\frac{1}{4}}\left|M\left(\frac{1}{2}+it\right)\right|dt=0. \tag{1.3}$$

the inverse Mellin transform

$$f(x)=\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}M(s)x^{-s}ds,\ x>0, \tag{1.4}$$

determines a function for which the Kontorovich–Lebedev representation (1.1) and (1.2) are realized.

In this paper we propose a new method of the summation (more exactly, the Abel summation) of the Kontorovich–Lebedev integral representation. In the framework of our approach the condition (1.3) is not necessary to be fulfilled. Namely, defining

$$f_\epsilon(x)=\frac{2}{\pi^2}\int\limits_0^\infty K_{i\tau}(x)\tau\,\mathrm{sh}\left((\pi-\epsilon)\tau\right)F(\tau)d\tau,\quad x>0,\ \epsilon\in(0,\pi), \tag{1.5}$$

for any two functions f and F related by the formula (1.1), it can be proved that if, in addition, the function $f(x)$ admits a representation of the from (1.4), where $M\in L_1\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right)$, then $f_\epsilon(x)$ uniformly converges to $f(x)$ over each finite interval $[a,b]$ from the positive semi-axis.

2. THE MAIN RESULT

Before formulating our main result it will be convenient to recall a few notations and facts from the theory of the gamma-functions and cylindrical functions. Concerning the gamma-functions $\Gamma(z)$ we restrain that it is a meromorphic function on the whole complex plane and has only simple poles at the points $z=-k$, $k=0,1,...,$ and

$$\mathrm{Re}\,s_{z=-k}\Gamma(z)=(-1)^k\frac{1}{k}\quad(k=0,1,\ldots).$$

In the sequel the following estimate

$$\left|\Gamma\left(\sigma+\frac{1}{2}-i\tau\right)\right| \le c\left(|\tau|+1\right)^{\sigma}\exp\left(-\pi|\tau|/2\right), \tag{2.1}$$

held uniformly over the domain $\{|\sigma| \le \sigma_1 < \infty,\ |\tau| \ge 1\}$, will be useful.

Further, we note the following integral representation of the modified Bessel function $I_\nu(z)$ for $\operatorname{Re}\nu > -\frac{1}{2}$:

$$I_\nu(z) = \frac{z^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}\left(1+x^2\right)^{\nu-\frac{1}{2}}\operatorname{ch}(zx)dx,\ \ |\arg z| < \pi, \tag{2.2}$$

and also note that the Macdonald function $K_{i\tau}(z)$ $(\tau \ne 0)$ can be expressed by the function $I_\nu(z)$ as follows

$$K_{i\tau}(z) = \frac{\pi}{2i\operatorname{sh}(\pi\tau)}\left(-I_{i\tau}(z)+I_{-i\tau}(z)\right). \tag{2.3}$$

It follows from (2.1) and (2.2) that for $x \in R$ and $y \in [0,\sigma]$, $0 < \sigma < \infty$,

$$|I_{-i\tau}(z)| \le c\left(1+|x|^{-\frac{1}{2}}\right)^{-\frac{1}{2}-y}\exp(\pi|x|/2), \quad \tau = x+iy, \tag{2.4}$$

and, also, estimations

$$k\left|\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s-k}{2}\right)I_k(r)\right| \le \frac{c}{(k-1)!}\left(\frac{r}{2}\right)^k\operatorname{ch}r,$$

$$|(s+2k)\Gamma(s+k)I_{s+2k}(r)| \le \frac{c}{s+2k-1}\left(\frac{r}{2}\right)^{\frac{1}{2}+2k}\left(1+r^2\operatorname{ch}^2 r\right), \tag{2.5}$$

$$|s\Gamma(s)I_s(r)| \le c\left(\frac{r}{2}\right)^{\frac{1}{2}}\left(1+r^2\operatorname{ch}^2 r\right),$$

which hold for $\operatorname{Re}s = \frac{1}{2}$, $r > 0$ and $k = 1, 2, \ldots$ with a constant $c > 0$ independent of r, s and k.

Next, we introduce the following function

$$R_\epsilon(s,\tau,r) = \tau\frac{\operatorname{sh}(\pi-\epsilon)}{\operatorname{sh}\pi\tau}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)I_{-i\tau}(r), \tag{2.6}$$

where $s = \frac{1}{2}+it$, $r > 0$, and $t, \tau \in R$. In the subsequent consideration the following auxiliary assertion will be useful.

Lemma 2.1. *Let $R_\epsilon(s,\tau,r)$ be given by* (2.6), *and let*

$$a_{k,\epsilon}(s) = \frac{1}{\pi}k\sin(\epsilon\pi k)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s-k}{2}\right) \quad (k = 0, 1, \ldots)$$

and

$$b_{k,\epsilon}(s) = (-1)^k \frac{2(s+2k)\Gamma(s+k)}{k!\sin\pi s} \sin((\pi-\epsilon)s - 2\epsilon k) \quad (k = 0, 1, \ldots).$$

Then, the following expansion

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} R_\epsilon(s,\tau,r)d\tau = \sum_{k=0}^{\infty} \left(a_{k,\epsilon}(s)I_k(r) + b_{k,\epsilon}(s)I_{s+2k}(r)\right). \tag{2.7}$$

holds.

Proof. First of all we remark that $R_\epsilon(s,\tau,z)$ for a fixed $s = \frac{1}{2} + it$ and z is a meromorphic function with respect to the variable τ. The poles of this function belong to the upper half-plane $\operatorname{Im}\tau > 0$, namely they are $\tau_k^1 = ik$ $(k = 1, 2, \ldots)$ (i.e. the zeros of the function $\operatorname{sh}\pi\tau$) and $\tau_k^2 = i(s+2k) = t + i\left(\frac{1}{2} + 2k\right)$ $(k = 0, 1, \ldots)$ (i.e. the corresponding poles of the gamma-function $\Gamma((s+i\tau)/2)$).

Furthermore, we observe that the estimation (2.1) implies the following one

$$\left|\Gamma\left(\frac{s+iz}{2}\right)\Gamma\left(\frac{s-iz}{2}\right)\right| \le c(\delta,t)\,|x|^{-\frac{1}{2}} \exp\left(\frac{-\pi\,|x|}{2}\right), \quad s = \frac{1}{2} + it,$$

where $z = x + iy$, $|x| \ge 1$, $0 \le y \le \delta$ and $||x| - |y|| \ge 1$. Taking into account these facts, by the relation (2.2), one obtains

$$\int_{q-i\delta}^{q+i\delta} R_\epsilon(s,\tau,r)d\tau \le c(\delta,t,z) \int_0^\delta |q|^{-y}\, dy \to 0, \quad q \to \pm\infty. \tag{2.8}$$

Let $\delta_0 \in \left(0, \frac{1}{2}\right)$ be chosen so that for the sequence $\delta_n = \delta_0 + 2n$ $(n = 1, 2, \ldots)$ the following inequalities

$$\left|\operatorname{Im}\tau_k^2 - \delta_n\right| \ge c > 0, \quad k, n = 1, 2, \ldots.$$

are satisfied. Then, using the relation $\Gamma(t+1) = t\Gamma(t)$, it can be shown that there exists a constant $c > 0$ such that for each $x \in \mathbb{R}$ and $n = 1, 2, \ldots$, the following estimate is true

$$|R_\epsilon(s, x+i\delta_n, r)| \le c\,|x+i\delta_n| \left(\frac{r}{2}\right)^{\delta_n} \delta_n^{-1} \left|\Gamma\left(-ix + \frac{1}{2}\right)\right|^{-1} \times$$
$$\times \left|\Gamma\left(\frac{1-2\delta+2i(t-x)}{4}\right)\Gamma\left(\frac{1+2\delta+2i(t-x)}{4}\right)\right| \times$$
$$\times \prod_{k=1}^{n} \left|\frac{-3+2\delta_k+2i(t-x)}{1-2\delta_k+2i(t+x)}\right| \cdot \prod_{k=1}^{2n} \left|-\frac{1}{2} + \delta_0 + k - ix\right|^{-1}.$$

Now, by straightforward evaluation, it can be deduced

$$|R_\epsilon(s, x + i\delta_n, r)| \leq \frac{c(r,t)}{(2n-1)!} \left(\sqrt{2}r\right)^{\delta_n} \delta_n^{-1} (1 + |x|)^{-\frac{3}{2}},$$

and consequently

$$\lim_{n\to\infty} \int\limits_{-\infty+i\delta_n}^{+\infty+i\delta_n} R_\epsilon(s,\tau,r)d\tau = 0. \tag{2.9}$$

Next, consider the contours

$$\gamma_n = \{z \in \mathbb{C} \colon z = \tau + i\delta_n, \tau \in \mathbb{R}\}, \quad n = 1, 2, \ldots$$

In view of (2.8) by virtue of the Cauchy theorem, it follows

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} R_\epsilon(s,\tau,r)d\tau =$$

$$= \sum_{k=1}^{2n} \operatorname{Re} s_{z=\tau_k^1} R_\epsilon(s,z,r) + \sum_{k=1}^{2n} \operatorname{Re} s_{z=\tau_k^2} R_\epsilon(s,z,r) + \int\limits_{\gamma_n} R_\epsilon(s,z,r)dz.$$

This result combined with (2.9) implies the desired conclusion, i.e. the expansion (2.7). □

Henceforth, f and F denote as above functions of the real positive variable related by the formula (1.1). The main objective of this section is to establish the following result.

Theorem 2.1. *Let the function f admit a representation of the form* (1.4)*, where $M \in L_1\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$, and let f_ϵ be given by* (1.5)*. Then f_ϵ uniformly converges to f over each finite integral from the positive semi-axis.*

Proof. First of all, note that our hypotheses together with the fact that $r^{-\frac{1}{2}}K_{i\tau}(r)$ is a summable function on the positive semi-axis indicate that a result from [8], namely the result given by Theorem 42 [8], can be applied. Due to that the transform $F(\tau)$ can be represented as follows

$$F(\tau) = \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^{2-s}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right) M(1-s)ds, \quad \tau > 0,$$

and then, by employing (2.1), (2.3) and (2.4), the function $f_\epsilon(r)$, $r > 0$ defined by (1.5) in turn can be writhen as

$$f_\epsilon(r) = -\frac{1}{8\pi^2 r} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^s M(1-s) R_\epsilon(s,r)ds,$$

where

$$R_\epsilon(s,r) = \int_{-\infty}^{+\infty} R_\epsilon(s,\tau,r)d\tau \quad (r > 0)$$

and $R_\epsilon(s.\tau, r)$ is defined as above by (2.6). On the other hand, if the change of variable $s \longrightarrow 1 - s$ is made, the representation (1.4) for functions $f(r)$ becomes

$$f(r) = \frac{1}{2\pi i r} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(1-s) r^s ds, \quad r > 0.$$

Upon eliminating r^s by using the expansion

$$r^s = \sum_{k=0}^{\infty} (-1)^k \frac{(s+2k)2^s}{k!} \Gamma(s+k) I_{s+2k}(r), \quad s \neq 0, -1, -2, \dots,$$

taking into account the estimations (2.5), and by invoking Lemma 2.1, we obtain

$$|f_\epsilon(r) - f(r)| \leq c(a,b) \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^s M(1-s) \left(\sum_{k=0}^{\infty} \left(1 - \frac{\sin\left((\pi-\varepsilon)s - 2\varepsilon k\right)}{\sin \pi s} \right) \right) \times$$
$$\times (-1)^k \frac{s+2k}{k!} \Gamma(s+k) I_{s+2k}(r) - \frac{1}{2\pi} \sin\left(\pi\varepsilon k\right) \Gamma\left(\frac{s+k}{2}\right) \Gamma\left(\frac{s-k}{2}\right) I_k(r) \Big) ds, \tag{2.10}$$

for $0 < a \leq r \leq b < \infty$ and with a constant $c(a,b) > 0$.

Since

$$|\sin(\pi\varepsilon k)| \leq \pi\varepsilon k, \qquad \left| \frac{\sin(\pi s) - \sin\left((\pi-\varepsilon)s - 2\varepsilon k\right)}{(s+2k-1)\sin \pi s} \right| \leq c\varepsilon,$$

it follows from (2.10) that

$$\sup_{a \leq r \leq b} |f_\epsilon(r) - f(r)| \leq c(a,b) \int_{\frac{1}{2}-i\varepsilon}^{\frac{1}{2}+i\varepsilon} |M(1-s)| \left(\left| 1 - \frac{\sin(\pi-\varepsilon)s}{\sin \pi s} \right| + \varepsilon \right) |ds| \, .$$

But

$$\left| 1 - \frac{\sin(\pi-\varepsilon)s}{\sin \pi s} \right| \leq c\varepsilon \, |s| \, , \quad \left| 1 - \frac{\sin(\pi-\varepsilon)s}{\sin \pi s} \right| \leq c$$

for $\operatorname{Re} s = \frac{1}{2}$, and so the assertion of Theorem follows. □

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Petru A. Cojuhari
cojuhari@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

Alexander M. Gomilko
gomilko@ukrpost.net

National University of Trade and Economics
Department of Higher and Applied Mathematics
Kiev, Ukraine

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