Opuscula Mathematica • Vol. 25 • No. 1 • 2005

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RECOVERING A PART OF POTENTIAL BY PARTIAL INFORMATION ON SPECTRA OF BOUNDARY PROBLEMS

Abstract. Under additional conditions uniqueness of the solution is proved for the following problem. Given 1) the spectrum of the Dirichlet problem for the Sturm–Liouville equation on $[0, a]$ with real potential $q(x) \in L_2(0, a)$, 2) a certain part of the spectrum of the Dirichlet problem for the same equation on $[\frac{a}{3}, a]$ and 3) the potential on $[0, \frac{a}{3}]$. The aim is to find the potential on $[\frac{a}{3}, a]$.

Keywords: sine-type function, Lagrange interpolation series, Dirichlet boundary value problem, Dirichlet–Neumann boundary value problem.

Mathematics Subject Classification: 34B24, 34A55, 34B10, 73K03.

1. INTRODUCTION

The first result on uniqiness of the potential of the Sturm–Liouville equation producing the prescribed spectrum of a corresponding boundary problem was obtained in [1]. In this paper it was shown that if the spectrum of the problem with the Neumann boundary conditions coincides with the set $\{k^2\}$ where $k \in \{0\} \cup \mathbb{N}$, then $q(x) \overset{a.e.}{=} 0$. In [2] it was proved that in most cases two spectra of corresponding boundary problems uniquely determine the potential. Leaving aside the history of other aspects of Sturm–Liouville inverse theory (see [3]–[5]) we should mention that important step was done in [6] where it was shown that the spectrum of one boundary problem on $[0, a]$ and the potential no $[0, \frac{a}{2}]$ uniquely determine the potential on $[\frac{a}{2}, a]$. It was shown in [7], that a half of the spectrum of a boundary problem (for example Dirichlet boundary problem) on $[0, a]$ and the potential on $[0, \frac{3}{4}a]$ uniquely determine the potential on $[\frac{3}{4}a, a]$. In [8] the authors showed that the spectrum of a boundary problem (for example Dirichlet problem), a half of the spectrum of another boundary problem (for example, Dirichlet–Neumann one) and the potential on $[0, \frac{a}{4}]$ uniquely determine the potential on $[\frac{a}{4}, a]$.

Three spectral problems were considered in [9], [10], where it was proved that the spectra of three Dirichlet boundary problems on the intervals $[0, a]$, $[0, \frac{a}{2}]$ and $[\frac{a}{2}, a]$ generated by the same potential uniquely determine the potential if these three spectra do not intersect.

In the present paper the potential is supposed to be known on the interval $[0, \frac{a}{3}]$ as well as the spectrum of the Dirichlet problem on the wlole interval $[0, a]$ and a certain part of the spectrum of the Dirichlet problem on $[\frac{a}{3}, a]$. It is proven that under some additional conditions these data uniquely determine the potential on $[0, a]$.

2. MAIN RESULT

Let us consider the following Sturm–Liouville problems with the Dirichlet boundary conditions and common real potential $q(x) \in L_2(0, a)$.

$$y'' + \lambda^2 y - q(x)y = 0, \tag{1}$$

$$y(0) = y(a) = 0, \tag{2}$$

$$y'' + \lambda^2 y - q(x)y = 0,$$

$$y(0) = y\left(\frac{a}{3}\right) = 0, \tag{3}$$

$$y'' + \lambda^2 y - q(x)y = 0,$$

$$y(0) = y'\left(\frac{a}{3}\right) = 0, \tag{4}$$

$$y'' + \lambda^2 y - q(x)y = 0,$$

$$y\left(\frac{a}{3}\right) = y(a) = 0. \tag{5}$$

We denote by $\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$ the spectrum of problem (1), (2), by $\{\nu_k\}_{-\infty,k\neq0}^{\infty}$ the spectrum of (1), (3), by $\{\mu_k\}_{-\infty,k\neq0}^{\infty}$ the spectrum of (1), (4) and by $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ the spectrum of (1), (5). For the sake of simplicity we assume $q(x)$ to be positive almost everywhere on $[0, a]$. Then the four above mentioned spectra are real. It is well known that the eigenvalues of these spectra are simple. We enumerate them such that $\lambda_{-k} = -\lambda_k$, $\lambda_{k+1} > \lambda_k$ for all $k \in \mathbb{N}$ and so on for each sequence of eigenvalues.

In the sequel we suppose the following condition to be satisfied:

Condition 2.1. $\{\lambda_k\}_{-\infty,k\neq0}^{\infty} \cap \{\nu_k\}_{-\infty,k\neq0}^{\infty} = \emptyset$ and $\{\lambda_k\}_{-\infty,k\neq0}^{\infty} \cap \{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty} = \emptyset$.

Let us denote by $s(\lambda, x)$ the solution of equation (1) which satisfies the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$, by $s_1(\lambda, x)$ the solution of (1) which satisfies the conditions

$$s_1\left(\lambda, \frac{a}{3}\right) = s_1'\left(\lambda, \frac{a}{3}\right) - 1 = 0 \tag{6}$$

and by $c_1(\lambda, x)$ the solution of (1) which satisfies the conditions

$$c_1\left(\lambda, \frac{a}{3}\right) - 1 = c_1'\left(\lambda, \frac{a}{3}\right) = 0. \tag{7}$$

It is easy to check up that

$$s(\lambda, a) = s'\left(\lambda, \frac{a}{3}\right) s_1(\lambda, a) + s\left(\lambda, \frac{a}{3}\right) c_1(\lambda, a). \tag{8}$$

Relation (8) implies that if $\nu_k^{(1)} = \nu_p$ for some k and p then $\nu_p = \lambda_s$ for some s. That means that Condition 1 implies $\{\nu_k\}_{-\infty,k\neq 0}^{\infty} \cap \{\nu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty} = \emptyset$.

The spectrum $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ possesses the following asymptotics (see [11])

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi k}{a} + \frac{B_0}{k} + \frac{\alpha_k}{k}, \tag{9}$$

where

$$B_0 = \frac{1}{2\pi}\int_0^a q(x)dx, \quad \{\alpha_k\}_{-\infty,k\neq 0}^{\infty} \in l_2.$$

Applying the same results of [11] to the subintervals $\left[0, \frac{a}{3}\right]$ and $\left[\frac{a}{3}, a\right]$ we obtain

$$\nu_k \underset{k\to\infty}{=} \frac{3\pi k}{a} + \frac{B}{k} + \frac{\beta_k}{k}, \tag{10}$$

and

$$\nu_k^{(1)} \underset{k\to\infty}{=} \frac{3\pi k}{2a} + \frac{B_1}{k} + \frac{\beta_k^{(1)}}{k}, \tag{11}$$

where

$$B = \frac{1}{2\pi}\int_0^{\frac{a}{3}} q(x)dx, \quad \{\beta_k\}_{-\infty,k\neq 0}^{\infty} \in l_2,$$

$$B_1 = \frac{1}{2\pi}\int_{\frac{a}{3}}^{a} q(x)dx, \quad \{\beta_k^{(1)}\}_{-\infty,k\neq 0}^{\infty} \in l_2.$$

We call *fitting* any subsequence $\left\{\nu_{k_p}^{(1)}\right\}_{-\infty,p\neq 0}^{\infty}$ such that $\nu_{k_{-p}}^{(1)} = -\nu_{k_{-p}}^{(1)}$,

$$\nu_{k_p}^{(1)} \underset{p\to\infty}{=} \frac{3\pi}{2a}(2p-1) + \frac{B_1}{2p-1} + \frac{\beta_p}{p}, \tag{12}$$

where $\{\beta_p\}_{-\infty,k\neq 0}^{\infty} \in l_2$ and $\left\{\nu_{k_p}^{(1)}\right\}_{-\infty,p\neq 0}^{\infty} \cap \{\nu_k\}_{-\infty,k\neq 0}^{\infty} = \emptyset$.

Theorem 2.1. *Let the following data be given:*

1) *the spectrum* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$*;*
2) *the real potential* $q(x) \in L_2\left(0, \frac{a}{3}\right)$ *on the interval* $\left[0, \frac{a}{3}\right]$ *(almost everywhere);*
3) *any fitting subsequence* $\left\{\nu_{k_p}^{(1)}\right\}_{-\infty,p\neq 0}^{\infty}$.

Then these data uniquely determine the potential $q(x)$ *almost everywhere on* $[0, a]$.

Proof. Knowing $q(x)$ on $\left[0, \frac{a}{3}\right]$ we can find $s(\lambda, x)$ and $s'(\lambda, x)$ on $\left[0, \frac{a}{3}\right]$ solving equation (1) with the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$. Therefore, we can find $s\left(\lambda, \frac{a}{3}\right)$ and $s'\left(\lambda, \frac{a}{3}\right)$. Then we find the set $\{\nu_k\}_{-\infty,p\neq 0}^{\infty}$ of zeros of $s\left(\lambda, \frac{a}{3}\right)$ and the set of values $s'\left(\nu_k, \frac{a}{3}\right)$. □

Let us consider the union $\{\zeta_k\}_{-\infty}^{\infty} = \{\nu_k\}_{-\infty,k\neq 0}^{\infty} \cup \left\{\nu_{k_p}^{(1)}\right\}_{-\infty,p\neq 0}^{\infty} \cup \{\zeta_0\}$. Here we have set $\zeta_0 = 0$ and changed the enumeration to have $\zeta_{-k} = -\zeta_k$ and $\zeta_k < \zeta_{k+1}$ for all k. Due to (10) and (12) we obtain

$$\zeta_k \underset{k\to\infty}{=} \frac{3\pi}{2a}k + O(\frac{1}{k}). \tag{13}$$

The following definition is due to [14]:

Definition 2.1. *An entire function* $\omega(\lambda)$ *of exponential type* $\sigma > 0$ *is said to be a function of sine-type if:*

1) *all the zeros of* $\omega(\lambda)$ *lie in a strip* $|Im\lambda| < h < \infty$*;*
2) *for some* h_1 *and all* $\lambda \in \{\lambda\colon Im\lambda = h_1\}$ *the following inequalities hold:* $0 < m \leq$ $\leq |\omega(\lambda)| \leq M < \infty$*;*
3) *the type of* $\omega(\lambda)$ *in the lower half-plane coincides with that in the upper half-plane.*

Thus, using Corollary after Lemma 4 in [12] we conclude that $\{\zeta_k\}_{-\infty,k\neq 0}^{\infty}$ is the set of zeros of a sine-type function. This function can be given as

$$\varphi(\lambda) \underset{n\to\infty}{=} \prod_{-n,k\neq 0}^{n} C\left(1 - \frac{\lambda}{\zeta_k}\right). \tag{14}$$

Now our aim is to construct $s_1(\lambda, a)$. We know the part $\left\{\nu_p^{(1)}\right\}_{-\infty,p\neq 0}^{\infty}$ of the set of zeros of this function. From (8) we obtain

$$s_1(\nu_k, a) = \frac{s(\nu_k, a)}{s'\left(\nu_k, \frac{a}{3}\right)} \tag{15}$$

We have already shown how to find all the values $s'\left(\nu_k, \frac{a}{3}\right)$. Now we find (see [11])

$$s(\lambda, a) = a\prod_{k=1}^{\infty}\left(\frac{a}{\pi k}\right)^2\left(\lambda_k^2 - \lambda^2\right).$$

Using this we find $s_1(\nu_k, a)$ for all k via (15). Also we know from [11] that

$$s_1(\lambda, a) = \frac{\sin \frac{2}{3}\lambda a}{\lambda} - \frac{\pi B_1}{\lambda^2} \cos \frac{2}{3}\lambda a + \frac{\psi(\lambda)}{\lambda^2}, \tag{16}$$

where $\psi(\lambda) \in \mathcal{L}_{\frac{2}{3}a}$ ($\mathcal{L}_a$ stands for the set of entire functions of exponential type $\le a$ which belong to $L_2(-\infty, \infty)$ for real λ). Using (15) and (16) we obtain

$$\psi(\nu_k) = \nu_k^2 \left(\frac{s(\nu_k, a)}{s'\left(\nu_k, \frac{a}{3}\right)} - \frac{1}{\nu_k} \sin \frac{2}{3}\nu_k a + \frac{\pi B_1}{\nu_k^2} \cos \frac{2}{3}\nu_k a \right).$$

According to Lemma 1.4.3 in [11] $\{\psi(\nu_k)\}_{-\infty, k\neq 0}^{\infty} \in l_2$.

Let us associate $a_p = 0$ with every $\zeta_k = \nu_{k_p}^{(1)}$ and with ζ_0. Each ζ_k which does not coinside with any of $\nu_{k_p}^{(1)}$ coinside with one of ν_k. Let it be ν_{k_1}, then we associate $a_{k_1} = \psi(\nu_{k_1})$ with this ζ_k. Thus we obtain the sequence $\{a_k\}_{-\infty}^{\infty} \in l_2$. From the other hand $\{\zeta_k\}$ is the set of zeros fo a sine-type function $\varphi(\lambda)$ defined by (14). Solving the interpolation problem we obtain

$$\psi(\lambda) = \varphi(\lambda) \sum_{-\infty}^{\infty} \frac{a_k}{\varphi'(\lambda)|_{\lambda=\zeta_k} (\lambda - \zeta_k)}. \tag{17}$$

This Lagrange series converges uniformly on any compact of complex plane and in the norm of $L_2(-\infty, \infty)$ for real λ (see [13]). Here $\psi(\lambda) \in \mathcal{L}_{\frac{2}{3}a}$. Substituting obtained $\psi(\lambda)$ into (16) we find $s_1(\lambda, a)$. Using (8) we find $c_1(\lambda, a)$:

$$c_1(\lambda, a) = \frac{s(\lambda, a) - s'\left(\lambda, \frac{a}{3}\right) s_1(\lambda, a)}{s\left(\lambda, \frac{a}{3}\right)}.$$

Now knowing $s_1(\lambda, a)$ and $c_1(\lambda, a)$ we construct the potential $q(x)$ via the procedure presented in [11] which we describe below.

First of all we introduce the function

$$e(\lambda) = e^{-i\lambda a} \left(c(\lambda, a) + i\lambda s(\lambda, a)\right)$$

which is so-called Jost function of the prolonged Sturm–Liouville problem on the semiaxis

$$\begin{aligned} y'' + \lambda^2 y - \tilde{q}(x) y = 0, \\ y_j(\lambda, 0) = 0, \end{aligned}$$

where

$$\tilde{q}(x) \overset{a.e.}{=} \begin{cases} q(x), & if\ x \in [0, \frac{2}{3}a] \\ 0, & if\ x \in (\frac{2}{3}a, \infty). \end{cases}$$

For the sake of simplicity let $\mu_1^2 > 0$ (otherwise we may shift). Then the Jost function $e(\lambda)$ has no zeros in the closed lower half-plane. Introduce so-called "S-matrix"

$$S(\lambda) = \frac{e(\lambda)}{e(-\lambda)}$$

and the function

$$F(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} (1 - S(\lambda))\, e^{i\lambda x} d\lambda.$$

The Marchenko integral equation

$$K(x,t) + F(x+t) + \int\limits_{x}^{\infty} K(x,s)F(s+t)ds = 0$$

possesses unique solution $K_j(x,t)$, and the potential

$$\tilde{q}(x) = -2\frac{dK(x,x)}{dx} \tag{18}$$

is real and belongs to $L_2(0,\infty)$ and $\tilde{q}_j(x) = 0$ for $x \in (\frac{2}{3}a, \infty)$. The shift $q(x + \frac{a}{3})$ of the projection $g(x)$ of $\tilde{q}(x)$ onto the interval $[0, \frac{2}{3}a]$ gives the unknown part of the potential we are looking for. Now the uniqueness of the procedure of recovering follows from the uniqueness of the recovering procedure in [11]. Theorem is proved.

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Received: September 22, 2004.