

Stanisław Formella

**A NOTE ON GEODESIC
AND ALMOST GEODESIC MAPPINGS
OF HOMOGENEOUS RIEMANNIAN MANIFOLDS**

Abstract. Let M be a differentiable manifold and denote by ∇ and $\tilde{\nabla}$ two linear connections on M . ∇ and $\tilde{\nabla}$ are said to be geodesically equivalent if and only if they have the same geodesics. A Riemannian manifold (M, g) is a naturally reductive homogeneous manifold if and only if ∇ and $\tilde{\nabla} = \nabla - T$ are geodesically equivalent, where T is a homogeneous structure on (M, g) ([7]). In the present paper we prove that if it is possible to map geodesically a homogeneous Riemannian manifold (M, g) onto $(M, \tilde{\nabla})$, then the map is affine. If a naturally reductive manifold (M, g) admits a nontrivial geodesic mapping onto a Riemannian manifold (\bar{M}, \bar{g}) then both manifolds are of constant curvature. We also give some results for almost geodesic mappings $(M, g) \rightarrow (M, \tilde{\nabla})$.

Keywords: homogeneous Riemannian manifold, geodesic, almost geodesic, geodesic mapping, almost geodesic mapping..

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1. INTRODUCTION

Let (M, g) be an n -dimensional Riemannian manifold of class C^∞ . Let $\mathfrak{F}(M)$ be the ring of differentiable functions and $\mathfrak{X}(M)$ the $\mathfrak{F}(M)$ -module of differentiable vector fields on M . A complete and simply connected manifold (M, g) is homogeneous if there exists a transitive and effective group G of isometries of M . Ambrose and Singer proved (see [7]) that a complete and simply connected Riemannian manifold (M, g) is homogeneous if and only if there exists a tensor field T of type (1, 2) such that:

$$\begin{aligned} \text{(i)} \quad & g(T_X Y, Z) + g(Y, T_X Z) = 0, \\ \text{(ii)} \quad & (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \\ \text{(iii)} \quad & (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \end{aligned} \tag{1.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Here ∇ and R denote the Levi-Civita connection and the Riemannian tensor field, respectively. A tensor field T satisfying the conditions (1.1) on M is called a homogeneous structure on (M, g) . It is easy to see that the conditions (1.1) are equivalent to

$$\left(\tilde{\nabla}_X g\right)(Y, Z) = 0, \quad \left(\tilde{\nabla}_X R\right)(Y, Z) = 0, \quad \left(\tilde{\nabla}_X T\right)(Y, Z) = 0 \quad (1.2)$$

where $\tilde{\nabla}$ is the connection determined by

$$\tilde{\nabla}_X Y = \nabla_X Y - T(X, Y). \quad (1.3)$$

where $T(X, Y) = T_X Y$, $X, Y, Z \in \mathfrak{X}(M)$.

In [7] F. Tricerri and L. Vanhecke studied the decomposition of the space of all the tensors T satisfying the conditions (1.1) into the irreducible components under the action of orthogonal group. As is well-known, a geodesic in a Riemannian manifold M is a curve of $c : I \rightarrow M$ whose tangent vector field \dot{c} is parallel along c (I is an open interval in the real line R^1). A curve c is almost geodesic in a Riemannian manifold M if there exists a 2-dimensional distribution E^2 complanar along c , to which the tangent vector \dot{c} of this curve belongs at every point. Let $(\bar{M}, \bar{\nabla})$ be a differentiable manifold with a linear symmetric connection $\bar{\nabla}$. A mapping $f : (M, g) \rightarrow (\bar{M}, \bar{\nabla})$ is called geodesic or projective if f carries geodesics in M to geodesics in \bar{M} . The mapping f is an almost geodesic mapping if, as a result of f , every geodesic in the manifold M passes into an almost geodesic curve in the manifold \bar{M} . If \bar{M} coincides with M and f is a diffeomorphism, f is called a geodesic or an almost geodesic transformation of M .

It is well known, that the identity transformation is geodesic if and only the connection deformation tensor $P(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$ has the form ([1, 2, 3, 5])

$$P(X, Y) = \psi(X)Y + \psi(Y)X, \quad (1.4)$$

where ψ is a certain 1-form and ∇ denotes the Levi-Civita connection of (M, g) , $X, Y \in \mathfrak{X}(M)$.

In this case, $\bar{\nabla}$ and ∇ are said to be geodesically (or projectively) equivalent or geodesically (projectively) related. Two such connections define the same system of geodesics. Obviously \sim is an equivalence relation and an equivalence class $[\nabla]$ containing ∇ is called a projective structure on M .

Sinyukow [5] defined three kinds of almost geodesic mappings, namely π_1 , π_2 and π_3 which are characterized, respectively, by the conditions:

$$\pi_1: \sum_{X, Y, Z} [(\nabla_X P)(Y, Z) + P(P(X, Y), Z) - a(X, Y)Z - P(X, Y)b(Z)] = 0 \quad (1.5)$$

(\mathfrak{S} is cyclic sum),

$$\begin{aligned} \pi_2: P(X, Y) &= \psi(X)Y + \psi(Y)X + F(X)\varphi(Y) + F(Y)\varphi(X), \\ &(\nabla_X F)(Y) + (\nabla_Y F)(X) + F(F(X))\varphi(Y) + F(F(Y))\varphi(X) = \\ &= \mu(X)F(Y) + \mu(Y)F(X) + \rho(X)Y + \rho(Y)X; \end{aligned} \tag{1.6}$$

$$\begin{aligned} \pi_3: P(X, Y) &= \psi(X)Y + \psi(Y)X + a(X, Y)\nu, \\ \nabla_X \nu &= \theta(X)\nu + \lambda X, \quad \lambda \in \mathfrak{F}(M); \quad X, Y, Z \in \mathfrak{X}(M), \end{aligned} \tag{1.7}$$

where $P(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$ is the connection deformation tensor and $\varphi, \psi, b, \theta, \rho, \nu, a, F$ are tensors of the corresponding types.

In the present paper we shall study a geodesic and an almost geodesic related connections ∇ and $\tilde{\nabla} = \nabla - T$, where T is a homogeneous structure on (M, g) .

2. GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

By [7], Theorem 6.8, a complete and simply connected Riemannian manifold (M, g) is naturally reductive homogeneous manifold if and only if there exists a tensor field T of type (1, 2) satisfying the conditions (1.1) and such that $\tilde{\nabla}$ and ∇ are geodesically equivalent.

Now we shall prove

Lemma 2.1. *If it is possible to map geodesically a homogeneous Riemannian manifold (M, g) onto a manifold $(M, \tilde{\nabla})$, then the map is affine.*

Proof. The connections ∇ and $\tilde{\nabla}$ are geodesically equivalent if and only if the connection deformation D have the form

$$D(X, Y) = -T(X, Y) = \psi(X)Y + \psi(Y)X + S(X, Y) \tag{2.1}$$

where ψ is a 1-form and the tensor field S satisfies

$$S(X, Y) + S(Y, X) = 0.$$

We put

$${}^0P(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X))$$

and

$${}^0P(X, Y, Z) = g({}^0P(X, Y), Z).$$

From (1.1 (i)) we obtain

$$\mathfrak{S}_{X, Y, Z} {}^0P(X, Y, Z) = 0.$$

Hence and from (2.1) we have

$$\psi(X)g(Y, Z) + \psi(Y)g(X, Z) + \psi(Z)g(X, Y) = 0.$$

Therefore $\psi(X) = 0$ for all $X \in \mathfrak{X}(M)$. This completes the proof. □

If $T = 0$, then (1.1) implies that (M, g) is a symmetric manifold. In view of the Sinyukov theorem we obtain: if it is possible to map geodesically a complete and simply connected Riemannian manifold with the homogeneous structure $T = 0$ into a manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant sectional curvature.

Let (M, g) be a connected Riemannian manifold and suppose M admits a non-trivial homogeneous structure T by

$$T(X, Y, Z) + T(Y, X, Z) = 0, \quad (2.2)$$

where $X, Y, Z \in \mathfrak{X}(M)$.

From (1.1) and (2.2) we get easily.

Lemma 2.2. *Let (M, g) be a connected Riemannian manifold with the homogeneous structure of type (2.2). Then Ricci tensor on M satisfies*

$$\mathfrak{S}_{X, Y, Z}(\nabla_X Ric)(Y, Z) = 0. \quad (2.3)$$

Now we shall prove

Theorem 2.1. *If it is possible to map geodesically on (M, g) satisfying (2.3) onto a manifold $(\overline{M}, \overline{g})$, then both manifolds are of constant curvature.*

Proof. As is well-known a manifold (M, g) admits a geodesic mapping if and only if there exists a function $\varphi \in \mathfrak{F}(M)$ and a symmetric non-singular bilinear form a on M satisfying

$$(\nabla_X a)(Y, Z) = (Y\varphi)g(X, Z) + (Z\varphi)g(X, Y) \quad (2.4)$$

for all $X, Y, Z \in \mathfrak{X}(M)$ ([5]).

Let $p \in M$ be such that $d\varphi \neq 0$ and (2.4) hold at p . Choose a local coordinate system (U, x) so that $p \in U$. By $R_{ijk}^l, R_{ik}, g_{ik}, a_{ik}, \varphi_{ik}$ we denote the components of the tensors R, Ric, g, a and the Hessian $H\varphi$ of φ in this coordinate system. Differentiating covariantly (2.4) and applying the Ricci identity we get

$$a_{it}R_{jkl}^t + a_{tj}R_{ikl}^t = \varphi_{li}g_{jk} + \varphi_{lj}g_{ik} - \varphi_{ki}g_{jl} - \varphi_{kj}g_{il}. \quad (2.5)$$

Differentiating covariantly (2.5) with respect to x^m , contracting with g^{lm} and applying the Ricci identity, by (2.4) and (2.3), we obtain

$$4\varphi_t R_{jki}^t = 3R_k^t \varphi_t g_{ji} - 4\varphi_k R_{ji} + 4\psi_i R_{jk} - 3g_{jk} R_i^t \varphi_t + a_j g_{ik} - a_k g_{ji}, \quad (2.6)$$

where $a_i = \nabla_s \varphi_{it} g^{ts}$. Transvecting (2.6) with g^{jk} we get $R_i^t \varphi_t = \rho \varphi_i$, $\rho \in \mathfrak{F}(U)$. Following considerations made in [6] we get

$$a_i^t \varphi_t = \tau \varphi_i, \quad \varphi_i^t \varphi_t = \lambda \varphi_i, \quad \tau, \lambda \in \mathfrak{F}(U),$$

and finally we obtain

$$H_\varphi(X, Y) = \Phi(\varphi)g(X, Y) \quad (2.7)$$

where H_φ is the Hessian of φ and $\Phi \in \mathfrak{F}(M)$.

By [7] if a complete and simply connected manifold with homogeneous structure T admits condition (2.7), then the manifold (M, g) is of constant curvature. This completes the proof. \square

From Lemmas 2.1 i 2.2 and Theorem 2.1 we obtain

Theorem 2.2. *On a homogeneous manifold the geodesic of ∇ and $\tilde{\nabla} = \nabla - T$ are the same if and only if M is naturally reductive. The geodesic mapping $(M, g) \rightarrow (M, \tilde{g})$ is affine. If a naturally reductive manifold (M, g) admits a non-trivial geodesic mapping onto a Riemannian manifold (\bar{M}, \bar{g}) , then both manifolds are of constant curvature.*

3. ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS MANIFOLDS

On the basis [7] the most general form of the structure tensor T is following

$$T(X, Y) = g(X, Y)\Phi - g(\Phi, Y)X + \overset{2}{T}(X, Y) \tag{3.1}$$

where Φ is a given vector field on (M, g) and $\overset{2}{T}$ is a tensor field such that

$$g(\overset{2}{T}(X, Y), Z) + g(Y, \overset{2}{T}(X, Z)) = 0, \tag{3.2}$$

$$\tilde{\nabla} \overset{2}{T} = 0,$$

$$C_{12}(\overset{2}{T}) = \sum_{i=1}^n \overset{2}{T}(X_i, X_i) = 0,$$

where X_i is the base vector of the natural frame.

We put

$$\begin{aligned} \overset{1}{P}(X, Y) &= \frac{1}{2} (\psi(X)Y + \psi(Y)X) - g(X, Y), \\ \overset{1}{S}(X, Y) &= \frac{1}{2} (\psi(X)Y - \psi(Y)X), \\ \overset{2}{P}(X, Y) &= -\frac{1}{2} \left(\overset{2}{T}(X, Y) + \overset{2}{T}(Y, X) \right), \\ \overset{2}{S}(X, Y) &= -\frac{1}{2} \left(\overset{2}{T}(X, Y) - \overset{2}{T}(Y, X) \right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P(X, Y) &= \overset{1}{P}(X, Y) + \overset{2}{P}(X, Y) \\ S(X, Y) &= \overset{1}{S}(X, Y) + \overset{2}{S}(X, Y) \end{aligned}$$

where $\psi(X) = g(X, \Phi)$.

Here P denotes the symmetric part of the tensor field T and S – the skew-symmetric one.

Then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + P(X, Y) + S(X, Y). \quad (3.4)$$

and the connection deformation tensor D have the form

$$D(X, Y) = P(X, Y) + S(X, Y) \quad (3.5)$$

for all $X, Y \in \mathfrak{X}(M)$.

We shall prove

Theorem 3.1. *On the homogeneous Riemannian manifold the connections ∇ and $\tilde{\nabla}$ defined by (3.3) and (3.4) are almost geodesically related if and only if the tensor fields $\overset{2}{P}$ and $\overset{2}{S}$ satisfy the relations*

$$\begin{aligned} \mathfrak{S}_{X,Y,Z} \left[(\nabla_X \overset{2}{P})(Y, Z) + \overset{2}{P}(\overset{2}{P}(X, Y), Z) - \overset{2}{P}(X, Y)b(Z) - \overset{2}{P}(X, \psi)g(Y, Z) + \right. \\ \left. - \overset{2}{P}(\overset{2}{S}(X, Y), Z) - \overset{2}{S}(X, \psi)g(Y, Z) + \right. \\ \left. - h(X, Y)\nabla_Z \psi + k(X, Y, Z)\psi + q(X, Y)Z \right] = 0, \end{aligned} \quad (3.6)$$

where: b, d, h, k, q are tensors of the corresponding types.

Proof. By [5] the mapping $\nabla \rightarrow \tilde{\nabla}$ is almost geodesic if and only if the connection deformation tensor D satisfies the relations

$$(\nabla_\gamma D_{\alpha\beta}^h + D_{\delta\alpha}^h D_{\beta\gamma}^\delta) \lambda^\alpha \lambda^\beta \lambda^\gamma = b D_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + a \lambda^h \quad (3.7)$$

where $\lambda^i = \frac{dc^i}{dt}$ denotes the vector tangent to the geodesic $c(t) = (c^i(t))$. We conclude from (3.3), (3.4), (3.5), (3.7) that (3.6) holds. This proves the theorem. \square

Corollary 3.1. *If $\overset{2}{P} = 0$ and $\overset{2}{S} = 0$ then the almost geodesic mapping is of the kind (1.7).*

Corollary 3.2. *If $\overset{2}{P} = 0$ and $\overset{2}{S} = 0$ then a homogeneous Riemannian manifold is a manifold of constant curvature (see [7]).*

Corollary 3.3. *If $g(X, Y)\nabla_Z \psi + g(X, Y)\overset{2}{P}(Z, \psi) + g(X, Y)\overset{2}{S}(Z, \psi) + k(X, Y, Z)\psi = 0$ then the almost geodesic mapping (3.6) is of the kind (1.5).*

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Stanisław Formella
stanislaw.formella@ps.pl

Technical University
Institute of Mathematics
al. Piastów 48/49, 70-310 Szczecin, Poland

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