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Beata Orchel

2-BIPLACEMENT WITHOUT FIXED POINTS OF (p, q)-BIPARTITE GRAPHS

Abstract. In this paper we consider 2-biplacement without fixed points of paths and (p, q)-bipartite graphs of small size. We give all (p, q)-bipartite graphs G of size q for which the set $\mathcal{S}^*(G)$ of all 2-biplacements of G without fixed points is empty.

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1. TERMINOLOGY

For a bipartite graph $G = (L, R; E)$ with the vertex set $V(G) = L \cup R$ and the egde set $E(G) = E$ we denote by $L = L(G)$ and $R = R(G)$ the left and the right set of bipartition of the vertex set of G, while the cardinality of the egde set by $e(G)$. Note that the graphs $G = (L, R; E)$ and $G' = (R, L; E)$ are different.

We denote by $N(x, G)$ the set of the neighbors of the vertex x in G. The degree $d(x, G)$ of the vertex x in G is the cardinality of the set $N(x, G)$; $\Delta_L(G)(\delta_L(G))$, $\Delta_R(G)(\delta_R(G))$ and $\Delta(G)(\delta(G))$ are the maximum (minimum) of the vertex degree in the set $L(G)$, $R(G)$ and $V(G)$, respectively. A vertex x of G is called a pendent if $d(x, G) = 1$. $K_{p,q}$ stands for the complete bipartite graph with $|L(K_{p,q})| = p$ and $|R(K_{p,q})| = q$. A bipartite graph G is called (p, q)-bipartite if $|L(G)| = p$ and $|R(G)| = q$.

2. EMBEDDING WITHOUT FIXED POINTS OF GENERAL GRAPHS

Let G be a graph of order n. We say that graph G can be embedded in its complement if there exits a permutation f on $V(G)$ such that if an edge xy belongs to $E(G)$, then

$f(x)f(y)$ does not belong to $E(G)$. A permutation f will be called an embedding of G (in its complement $K_n \setminus G$).

The following theorem was proved by D. Burns and S. Schuster. Theorem A is a sufficient condition for a graph to be embeddable (with some exeptional graphs).

Theorem A ([1]). *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \le n - 1$ then either G is embeddable or G is isomorphic to one of the following graphs: $K_{1,n-1}, K_{1,n-4} \cup K_3$, with $n \ge 8, K_1 \cup K_3, K_2 \cup K_3, K_1 \cup 2K_2, K_1 \cup C_4$.*

An embedding f of $V(G)$ such that $f(x) \ne x$ for every x on $V(G)$ is called an embedding without fixed points.

Let $G_1 = K_{1,2} \cup C_3$ and $G_2 = K_{1,3} \cup C_3$. G_1 and G_2 are embeddable in their complements (by Theorem A) but G_1 and G_2 cannot be embedded without fixed points. All other graphs G with n vertices and $n - 1$ edges which are contained in their complements can be embedded without fixed points.

S. Schuster proved the following theorem.

Theorem B ([5]). *Let $G = (V, E)$ be a graph of order n with $|E(G)| \le n - 1$ and such that G is not an exceptional graph of Theorem A and $G \notin \{G_1, G_2\}$. Then there exists a fixed-point-free embedding of G.*

3. 2-BIPLACEMENT WITHOUT FIXED POINTS OF BIPARTITE GRAPHS

If $G = (L, R; E)$ and $H = (L', R'; E')$ are two (p, q)-bipartite graphs then we say that G and H are mutually placeable (into the complete bipartite graph $K_{p,q}$) if there is a bijection $f : L \cup R \to L' \cup R'$ such that $f(L) = L'$ and $f(x)f(y)$ is not an egde of H whenever xy is an egde of G. The function f is called a biplacement of G and H. If $H = G$ then we say that (p, q)-bipartite graph G is 2-biplaceable and the function f is called a 2-biplacement of G. Richard Rado in [4] has proved a theorem in traversal theory, which may be transformed into a necessary and sufficient condition for two bipartite graphs to be mutually placeable. So, even if the mutual placement of bipartite graphs, in the sense of the definition given above, has been introduced in [3] (see also [2] and [6]), it is clear that the problem of mutual placeability of bipartite graphs is at least ninety years old.

In this paper we shall consider 2-biplacement f of (p, q)-bipartite graph G without fixed points i.e. $f(x) \ne x$ for every x in $V(G)$.

We denote by $\mathcal{S}^*(G)$ the set of all 2-biplacements f of G such that $f(x) \ne x$, for every x in $V(G)$. If $f \in \mathcal{S}^*(G)$ then we denote it briefly by f is $w.f.p.$

We shall present theorems which improve it by specyfying the structure of the 2-biplaceable permutation of bipartite graphs.

A. P. Wojda and J. L. Fouquet proved the following theorem, which is a bipartite version of Theorem A.

Theorem C ([3]). *Let G be a (p, q)-bipartite graph such that either $p \ge 3$, $q \ge 3$, and $e(G) \le p + q - 3$ or $p = 2$, $p \le q$ and $e(G) \le p + q - 2$. Then G is 2-biplaceable.*

Let P_n be a path of order n. Then P_n is $(\frac{n}{2}, \frac{n}{2})$-bipartite graph — if n is even and $(\frac{n+1}{2}, \frac{n-1}{2})$-bipartite graph or $(\frac{n-1}{2}, \frac{n+1}{2})$-bipartite graph — if n is odd. A path P_n we shall denote by $P_{\frac{n}{2}, \frac{n}{2}}$ and $P_{\frac{n+1}{2}, \frac{n-1}{2}}$ or $P_{\frac{n-1}{2}, \frac{n+1}{2}}$, respectively.

If $n \leq 6$ then P_n is not 2-biplaceable (into the complete bipartite graph).

For $n = 7$ (then P_7 is $(4, 3)$-bipartite graph or $(3, 4)$-bipartite graph) there exists a 2-biplacement of P_7 but the set of all 2-biplacements of P_7 without fixed points is empty. A path P_n, for $n \geq 8$ is 2-biplaceable (into an appropriate bipartite graph) and there exits a 2-biplacement of P_n without fixed points. More precisely we shall prove the following theorem.

Theorem 1. *If $k \geq 4$ then there exits 2-biplacement w.f.p. of path P_{2k} in $K_{k,k}$ and path P_{2k+1} in $K_{k,k+1}$ (or in $K_{k+1,k}$).*

If $G = (L, R; E)$ is (p, q)-bipartite graph, $2 \leq p \leq q$ and $e(G) = q$ then, by Theorem C, G is 2-biplaceable. We give all (p, q)-bipartite graphs of size q and $p \leq q$ for which $\mathcal{S}^*(G) = \emptyset$.

If $p = 2$ and there are no isolated vertices in $V(G)$ and isolated edges in $E(G)$ then $\mathcal{S}^*(G) \neq \emptyset$. Let $H_1(2, q) = K_{1,1} \cup K_{1,q-1}$.

If $p = 2$ and $\Delta_R(G) > 1$ then the familly $\mathcal{H}_2(2, q)$ is a set of $(2, q)$-bipartite graphs of size q such that if $L(G) = \{a, b\}$ and if l is number of vertices of degree 2 in $R(G)$ then $d(a, G) = l + 1$ or $d(b, G) = l + 1$.

Let $H_1(3, q)$ be $(3, q)$-bipartite graph of size q, which contains $K_{1,1} \cup K_{1,q-1}$ as a subgraph, and let $H_2(3, q)$ be $(3, q)$-bipartite graph of odd size q, which has a subgraph $K_{1,1} \cup K_{2,\frac{q-1}{2}}$.

Observe $\mathcal{S}^*(H_1(3, q)) = \emptyset$ and $\mathcal{S}^*(H_2(3, q)) = \emptyset$.

We can now formulate our main result. Theorem 2 is a counterpart of Theorem B.

Theorem 2. *Let $G = (L, R; E)$ be (p, q)-bipartite graph, $2 \leq p \leq q$ and $e(G) \leq q$. Then either there exits 2-biplacement w.f.p. of G, or*

(i) *($p = 2$ and ($G = H_1(2, q)$ or $G \in \mathcal{H}_2(2, q)$)) or else*

(ii) *($p = 3$ and ($G = H_1(3, q)$ or $G = H_2(3, q)$)).*

4. PROOFS

We start with two easy remarks.

Remark 1. *Let $T = (L, R; E)$ be a (p, q)-tree and let e be an edge in E. If $T \setminus \{e\} = T_1 \cup T_2$ and T_1, T_2 2-biplaceable (into appropriate complete bipartite graphs) and $\mathcal{S}^*(T_1) \neq \emptyset$, $\mathcal{S}^*(T_2) \neq \emptyset$, then T is 2-biplaceable (into $K_{p,q}$) and $\mathcal{S}^*(T) \neq \emptyset$.*

If $y \in V(G)$ then let $U_y = \{x \in V(G)\colon x \in N(y, G)$ and $d(x, G) = 1\}$.

Remark 2. *Let G be a (p, q)-bipartite graph. If there is a vertex y such that $|U_y| \geq 2$ and $G' = G \setminus U_y$ is 2-biplaceable and $\mathcal{S}^*(G') \neq \emptyset$, then G is 2-biplaceable and $\mathcal{S}^*(G) \neq \emptyset$.*

4.1. PROOF OF THEOREM 1.

The proof is by induction on order of the path, say n. First, we assume that $n = 2k$, $k \geq 4$. We shall check the theorem for $k = 4, 5, 6, 7$.

For $k = 4$ we denote $P_8 = P_{4,4}$ and $L(P_{4,4}) = \{1, 3, 5, 7\}$, $R(P_{4,4}) = \{2, 4, 6, 8\}$ and $P_{4,4}\colon 1{-}2{-}3{-}4{-}5{-}6{-}7{-}8$. Let consider the path $P'_{4,4}\colon 7{-}4{-}1{-}6{-}3{-}8{-}5{-}2$ and define $\sigma_{P_{4,4}}$ such that $\sigma(1) = 7$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 6$, $\sigma(5) = 3$, $\sigma(6) = 8$, $\sigma(7) = 5$, $\sigma(8) = 2$. Then we see $\sigma_{P_{4,4}}$ is 2-biplacement w.f.p. of P.

For $k = 5$ we have:

$$P_{5,5}\colon 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10,$$
$$P'_{5,5}\colon 3 - 6 - 1 - 10 - 7 - 4 - 9 - 2 - 5 - 8;$$

for $k = 6$:

$$P_{6,6}\colon 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 - 12,$$
$$P'_{6,6}\colon 7 - 4 - 1 - 10 - 3 - 8 - 11 - 2 - 5 - 12 - 9 - 6;$$

if $k = 7$ we may define:

$$P_{7,7}\colon 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 - 12 - 13 - 14,$$
$$P'_{7,7}\colon 7 - 4 - 1 - 10 - 13 - 8 - 3 - 6 - 11 - 14 - 9 - 2 - 5 - 12.$$

If $k \geq 8$ then there exists $e \in E(P_{2k})$ such that

$$P_{2k} \setminus \{e\} = P_{\frac{k}{2},\frac{k}{2}} \cup P_{\frac{k}{2},\frac{k}{2}}, \quad \text{for } k\text{-even}$$

and

$$P_{2k} \setminus \{e\} = P_{\frac{k-1}{2},\frac{k-1}{2}} \cup P_{\frac{k+1}{2},\frac{k+1}{2}}, \quad \text{for } k\text{-odd.}$$

By the induction hypothesis paths $P_{\frac{k}{2},\frac{k}{2}}$, $P_{\frac{k-1}{2},\frac{k-1}{2}}$ and $P_{\frac{k+1}{2},\frac{k+1}{2}}$ can be 2-biplacaeble w.f.p. It is easy to complete this packing and obtain a 2–biplacement w.p.f. of $P_{k,k}$ by Remark 1.

Let $n = 2k + 1$, $k \geq 4$. For $k \geq 8$ we proceed as above. Now we verify the theorem for $4 \leq k \leq 7$.

For $k = 4$ we denote $L(P_{4,5}) = \{2, 4, 6, 8\}$, $R(P_{4,5}) = \{1, 3, 5, 7, 9\}$:

$$P_{4,5}\colon 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9,$$
$$P'_{4,5}\colon 9 - 4 - 1 - 6 - 3 - 8 - 5 - 2 - 7;$$

for $k = 5$:

$$P_{5,6}\colon 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11,$$
$$P'_{5,6}\colon 3 - 8 - 5 - 10 - 7 - 2 - 9 - 4 - 11 - 6 - 1;$$

for $k = 6$:

$$P_{6,7}\colon 1-2-3-4-5-6-7-8-9-10-11-12-13,$$
$$P'_{6,7}\colon 5-10-7-12-9-2-11-4-13-6-1-8-3;$$

for $k = 7$:

$$P_{7,8}\colon 1-2-3-4-5-6-7-8-9-10-11-12-13-14-15,$$
$$P'_{7,8}\colon 5-10-7-12-9-14-11-2-13-4-15-6-1-8-3. \quad \square$$

4.2. PROOF OF THEOREM 2.

To prove Theorem 2, we shall need some additional definitions and notation.

$$\text{Let} \quad H'_1(p,2) = K_{1,1} \cup K_{p-1,1} \text{ and } p \geq 3.$$

$\mathcal{H}'_2(p,2)$ is a set of graphs $(p,2)$-bipartite G of size p such that, if $R(G) = \{a,b\}$ then $d(a,G) = l+1$ or $d(b,G) = l+\ 1$, l is the number of vertices of degree 2 in the set $L(G)$;

$G = H'_1(p,3) \Leftrightarrow q = 3, K_{1,1} \cup K_{p-1,1}$ is a subgraph of G, $e(G) = p$;

$G = H'_2(p,3) \Leftrightarrow q = 3, K_{1,1} \cup K_{\frac{p-1}{2},2}$ is a subgraph of G, $e(G) = p$ and p is odd.

Observe, that $H'_1(p,2)$, $H'_1(p,3)$, $H'_2(p,3)$ and the family $\mathcal{H}'_2(p,2)$ are obtained from $H_1(2,q)$, $H_1(3,q)$, $H_2(3,q)$ and the family $\mathcal{H}_2(2,q)$, respectively, by exchanging the sides of corresponding graphs.

We shall give only the main idea of the proof, leaving to the reader long but very easy verification of some details.

The proof is by induction on $p+q$. The result is obvious if $p = 2$ and $q \geq 2$. It is easy to check that the theorem is true for $p = q = 3$ and $p = 3$ and $q = 4$.

Now we assume $p \geq 3$, $q \geq 4$ and the theorem is true for (p',q')-bipartite graph fulfilling the assumptions of the theorem and $p'+q' < p+q$. Let $G = (L,R;E)$ be (p,q)-bipartite graph, $p \leq q$ and $e(G) = q$.

We consider the following two cases.

Case 1. There is an isolated vertex in R. Let $y \in R$, $d(y,G) = 0$, $y' \in R$ and $d(y',G) = \Delta_R(G)$.

We can apply induction to the graph $G' = G \setminus \{y,y'\}$ and if there exits $\sigma_{G'}$-2--biplacement w.f.p. of G', then σ_G such that,

$$\sigma_G(v) = \sigma_{G'}(v), \quad \text{for } v \in V(G'),$$
$$\sigma_G(y) = y',$$
$$\sigma_G(y') = y,$$

define 2-biplacement w.f.p. of G.

Suppose that $d(y', G) = 2$ and $\mathcal{S}^*(G') = \emptyset$. Observe that $G' \neq H_1'(p, 2)$ and $G' \notin \mathcal{H}_2'(p, 2)$.

If $G' = H_1(3, q-2)$, then we have to consider a few simple cases.

If $G' = H_2(3, q-2)$ then either $G = H_2(3, q)$ or $\mathcal{S}^*(G) \neq \emptyset$. For $G' = H_1'(p, 3)$ we have $p = 3$ and $q = 5$ and then either $K_{1,1} \cup K_{2,2} \leq G$ and $G = H_2(3, 5)$, or $P_{3,3} \leq G$.

Finally, we observe that if $G' = H_2'(p, 3)$ then $p = 3$ and $G' = H_1(3, 3)$ and the theorem is easy to check.

Case 2. There are no isolated vertices in R. Hence $\delta_R(G) = 1$. The theorem is true for $p = q$ and $qK_{1,1} = G$, $q \geq 3$.

If $p < q$ or $p = q$ and $pK_{1,1}$ is not the subgraph of G then there are vertices y_1, $y_2 \in R$ such that $N(y_1, G) = N(y_2, G)$. Let $G'' = G \setminus \{y_1, y_2\}$. G'' is (p'', q'')-bipartite graph, $p'' = p \geq 3$ and $q'' = q - 2 \geq 2$ and $e(G'') \leq q - 2$. Hence we can apply the induction hypothesis to the graph G''.

Observe that, if $\mathcal{S}^*(G'') \neq \emptyset$ then, by Remark 2, $\mathcal{S}^*(G) \neq \emptyset$.

If G'' is one of the exceptional graphs, then, by $p \geq 3$ and $\delta_R(G'') = 1$, we have to consider only the situation that $G'' = H_1(3, q-2)$. But in this case either $G = H_1(3, q)$, or there exits 2-biplaceable G w.f.p. and the theorem is proved.

REFERENCES

[1] Burns D., Schuster S.: *Embedding $(p, p-1)$ graphs in their complements.* Israel J. Math. **30** (1978), 313–320.

[2] Catlin P. A.: *Subgraphs of graphs. I.* Discrete Math. **10** (1974), 225–233.

[3] Fouquet J.- L., Wojda A. P.: *Mutual placement of bipartite grahps.* Discrete Math. **121** (1993) 85–92.

[4] Rado R.: *A theorem on a general measure functions.* Proc. London Math. Soc. **44 (2)** (1938), 61–91.

[5] Schuster S.: *Fixed-point-free embeddings in graphs in their complements.* Internat. J. Math. Sci. **1** (1978), 335–338.

[6] Wojda A. P., Vanderlind P.: *Packing bipartite graphs.* Discrete Math. **164** (1997), 303–311.

Beata Orchel
`orchel@agh.edu.pl`

AGH University of Science and Technology,
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

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