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ON INTERTWINING AND w-HYPONORMAL OPERATORS

Abstract. Given $A, B \in B(H)$, the algebra of operators on a Hilbert Space H, define $\delta_{A,B}\colon B(H) \to B(H)$ and $\Delta_{A,B}\colon B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - XB$ and $\Delta_{A,B}(X) = AXB - X$. In this note, our task is a twofold one. We show firstly that if A and B^* are contractions with $C.o$ completely non unitary parts such that $X \in \ker \triangle_{A,B}$, then $X \in \ker \triangle_{A^*,B^*}$. Secondly, it is shown that if A and B^* are w-hyponormal operators such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$, where X and Y are quasi-affinities, then A and B are unitarily equivalent normal operators. A w-hyponormal operator compactly quasi-similar to an isometry is unitary is also proved.

Keywords: w-hyponormal operators, contraction operators and quasi-similarity.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

Let H be an infinite dimensional Complex Hilbert space and let $B(H)$ denote the algebra of operators from H to itself (= bounded linear transformations).

Given $A, B \in B(H)$, define $\delta_{A,B}\colon B(H) \to B(H)$ and $\Delta_{A,B}\colon B(H) \to B(H)$ by

$$\delta_{A,B}(X) = AX - XB \quad \text{and} \quad \Delta_{A,B}(X) = AXB - X.$$

The classical Putnam–Fuglede Theorem [21, p. 104] says that if A and B^* are normal operators, then $\ker \delta_{A,B} = \ker \delta_{A^*,B^*}$.

Analoguesly, if A and B^* are normal operators, then $\ker \Delta_{A,B} = \ker \Delta_{A^*,B^*}$.

A number of generalisations of the Putnam–Fuglede Theorem, and its $\Delta_{A,B}$ analogue, are to be found in the extant literature, amongst them generalisations where the normal operators A and B^* are replaced by larger classes than the normal operators. The particular classes which have drawn alot of attention are those consisting of either subnormal or hyponormal or M-hyponormal or dominant or k-quasi-hyponormal operators as well as p-hyponormal operators.

It is well known that $\ker\delta_{A,B} \subset \ker\delta_{A^*,B^*}$ ($\ker\Delta_{A,B} \subset \ker\Delta_{A^*,B^*}$) for A and B^* belonging to many a pair of these classes ([8, 12, 13, 14, 19, 23, 27, 28, 29] and some of the references there) except for when both A and B^* are dominant (see [12, 14, 15]).

In the first part of this note, using the operator equation $\Delta_{A,B}(X)$, we show among other results that Putnam–Fuglede Theorem holds true for contractions A and B^* with $C_{.}o$ completely non unitary and one can easily deduce that a w-hyponormal contraction operator is unitary.

For $p > 0$, recall that ([1, 2, 12, 17]) an operator A is said to be p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where A^* denotes the adjoint of A. A p-hyponormal is called hyponormal if $p = 1$, semi-hyponormal if $p = \frac{1}{2}$. An invertible operator A is called log-hyponormal if $\log(A^*A) \geq \log(AA^*)$.

An operator A is said to be Paranormal if $\|Ax\|^2 \leq \|A^2x\| \|x\|$, for all $x \in H$, k-paranormal if $\|Ax\|^k \leq \|A^kx\| \|x\|^{k-1}$ for all $x \in H$ and k ≥ 2 is some integer and is said to be k-quasi-hyponormal if $A^{*k}(A^*A - AA^*)A^k \geq 0$ for all $x \in H$ and $k \geq 1$. Of cousre it is well known that neither the class of k-quasi-hyponormal operators nor the class of k-paranormal operators contain each other and are therefore independent.

Let $A = U|A|$ be the polar decomposition of A, then following ([1, 2]), we define the first **Aluthge transform** of A by $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ and define the second **Aluthge transform** of A by $\widetilde{\widetilde{A}} = |\widetilde{A}|^{\frac{1}{2}}\widetilde{U}|\widetilde{A}|^{\frac{1}{2}}$, where $\widetilde{A} = \widetilde{U}|\widetilde{A}|$ is the polar decomposition of $\widetilde{A}$.

An operator A is said to be w-hyponormal if

$$|\widetilde{A}| \geq |A| \geq |\widetilde{A}^*|.$$

The classes of log-and w-hyponormal were introduced and their properties studied in [3, 4, 5, 25, 31, 32] and other references there. In particular, it was shown in [3] and [5] that the class of w-hyponormal contains both the log- and p-hyponormal operators.

The class of log-hyponormal operators were independently introduced by Tanahashi in his paper [31]. There, he gave an interesting example ([31, Example 12]) of a log-hyponormal operator which is not p-hyponormal for $p > 0$. Thus the class of p-hyponormal operators are totally independent of the class of log- hyponormal operators.

Since the class of w-hyponormal operators contains both log- and p-hyponormal operators, it therefore provides a unified approach in studying the latter classes. Indeed, Tanahashi's example can be used to show that the class of w-hyponormal operators properly contains the classes of log- and p-hyponormal operators. For if $A \in B(H)$ is the Tanahashi operator ([31, Example 12]), then $A \oplus 0$ defined on $H \oplus H$ is w-hyponormal operator but is neither log- nor p- hyponormal operator. Thus in general, if B is a non invertible p-hyponormal operator, then $A \oplus B$ is w-hyponormal but is neither log-nor p-hyponormal operator.

It is well known that if an operator A is w-hyponormal, then $\widetilde{A}$ is semi-hyponormal and $\widetilde{\widetilde{A}}$ is hyponormal.

Also if an operator A is p-hyponormal, then $\ker A \subset \ker A^*$ and if A is log-hyponormal, then $\ker A = \ker A^*$. However, if A is w-hyponormal, then it is not known whether the kernel condition $\ker A \subset \ker A^*$ holds. Nevertheless, there are several properties that p-hyponormal operators share with w-hyponormal operators A or w-hyponormal operators A with $\ker A \subset \ker A^*$ ([3] and [5]).

Recall that an operator $A \in B(H)$ is said to be dominant if for each $\lambda \in \mathbf{C}$, there exists a positive number M_λ such that

$$(A-\lambda)(A-\lambda)^* \leq M_\lambda (A-\lambda)^*(A-\lambda).$$

If the constants M_λ are bounded by a positive operator M, then A is said to be M-hyponormal.

Clearly the following inclusions hold.

$$\begin{gathered} \textit{Hyponormal} \subset p\textit{-Hyponormal}(0<p<1) \subset w\textit{-Hyponormal} \subset \textit{Paranormal} \\ \subset K\textit{-paranormal}, \\ Hyponormal \subset Log\text{-}hyponormal \subset w\text{-}Hyponormal \subset Paranormal \\ \subset K\text{-}paranormal, \\ Hyponormal \subset k\text{-}quasi - hyponormal, \end{gathered}$$

and

$$\textit{Hyponormal} \subset M\textit{-hyponormal} \subset \textit{Dominant}.$$

An operator $X \in B(H)$ is called a quasi-affinity if X is both injective and has a dense range. Two operators A and B are said to quasi-similar if $\exists$ quasiaffinities X and Y such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$.

The operator A is said to be pure if there exists no non-trivial reducing subspace N of H such that the restriction of A to N ($A\mid_N$) is normal and is completely hyponormal if it is pure.

Recall that every $A \in B(H)$ has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 and A_2 are normal and pure parts respectively. Of course in the sum decomposition, either A_1 or A_2 may be absent.

We say that the contraction $A \in$ to class $C_{.0}$ of contractions (A$\in C_{.0}$) if $A^{*n} \to 0$ strongly as n $\to \infty$. The contraction A is said to be completely non unitary (*c.n.u.*) if there exists no non-trivial reducing subspace U of H such that A restricted to U is unitary. Every contraction A has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 is unitary and A_2 is c.n.u. and of course either A_1 or A_2 may be absent. Clearly a pure contraction is completely non unitary.

Jeon and Duggal [17] have shown among other results that the normal parts of quasi-similar p-hyponormal operators are unitarily equivalent and that a p-hyponormal operator compactly quasi-similar to an isometry is unitary.

Jeon,Tanahashi and Uchiyama [25] proved that similar results of ([17]) hold true for the class of log-hyponormal operators.

In the second part of this paper, we use the second Aluthge transform operator $\widetilde{\widetilde{A}}$ and the kernel condition $\ker A \subset \ker A^*$ as major tools to show that these results ([17] and [25]) still hold true to the more general case of w-hyponormal operators.

2. INTERTWINING OF w-HYPONORMAL OPERATORS

We begin by proving results on contraction operators with $C_{.}o$ completely non unitary parts.

The following result shows that contraction operators A and B^* with $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}$ are unitarily equivalent unitary operators.

Theorem 1. *If the contractions A and $B^* \in B(H)$ have $C_{.}o$ completely non unitary parts and $X \in \ker \Delta_{A,B}$ for some $X \in B(H)$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B and $A |_{\overline{\operatorname{ran} X}}$ and $B |_{\ker^{\perp} X}$ are unitarily equivalent unitary operators.*

Proof. Decompose A and B^* into their unitary and $C_{.}o$ completely non unitary parts, $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $X = [X_{ij}]_{i,j=1}^2$.

Since A_2 and B_2^* both belong to $C_{.}o$ completely non unitary parts,

$$\|X_{12}x\| = \|A_1^n X_{12} B_2^n x\| \leq \|X_{12}\| \, \|B_2^n x\| \to 0 \text{ as } n \to \infty$$

for all $x \in H$. Using a similar arguements to the equations $X_{21}^* \in \ker \Delta_{B_1^*,A_2^*}$ and $X_{22} \in \ker \Delta_{A_2,B_2}$, $X_{22} = X_{21} = 0$.

Consequently applying Putnam–Fuglede Theorem to $X_{11} \in \ker \Delta_{A_1,B_1}$ where A_1 and B_1 are unitary operators, $X_{11} \in \ker \Delta_{A_1^*,B_1^*}$ and the result follows. □

Corollary 2 ([13]). *If A and B^* are contractions with $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}^n$ for some $X \in B(H)$, then the conclusions in Theorem 1 above hold.*

Proof. Let $X \in \ker \Delta_{A,B}^{n-1} = Y$, then clearly $Y \in \ker \Delta_{A,B}$ and by the Theorem, $Y \in \ker \Delta_{A^*,B^*}$. Thus $\overline{\operatorname{ran} Y}$ reduces A, $\ker^{\perp} Y$ reduces B and $A |_{\overline{\operatorname{ran} Y}}$ and $B |_{\ker^{\perp} Y}$ are unitarily equivalent unitary operators.

Let X has a matrix representation as in the proof of the Theorem. Now if $A = C_1 \oplus C_2$ and $B = D_1 \oplus D_2$ with $H = \overline{\operatorname{ran} Y} \oplus (\overline{\operatorname{ran} Y})^{\perp}$ and $H = \ker^{\perp} Y \oplus (\ker^{\perp} Y)^{\perp}$ respectively, then C_1 and D_1 are unitarily equivalent unitary operators and

$$\mathbf{Y} = \mathbf{X} \in \ker \mathbf{\Delta}_{\mathbf{A},\mathbf{B}}^{\mathbf{n-1}} = \begin{bmatrix} X_{11} \in \ker \Delta_{C_1,D_1}^{n-1} & X_{12} \in \ker \Delta_{C_1,D_2}^{n-1} \\ X_{21} \in \ker \Delta_{C_2,D_1}^{n-1} & X_{22} \in \ker \Delta_{C_2,D_2}^{n-1} \end{bmatrix}.$$

Clearly,

$$X_{12} \in \ker \Delta^{n-1}_{C_1,D_2} = X_{21} \in \ker \Delta^{n-1}_{C_2,D_1} = X_{22} \in \ker \Delta^{n-1}_{C_2,D_2} = 0.$$

Now $X_{11} \in \ker \Delta^{n-1}_{C_1,D_1}$ and so is $X_{11} \in \ker \Delta_{C_1,D_1}$. $X_{11} \in \ker \Delta_{C_1,D_1}$ means

$$C_1X_{11}D_1 = X_{11} = C_1X_{11}D_1 - X_{11} = C_1X_{11}D_1 - C_1C_1^*X_{11} = 0.$$

Consequently $(-1)C_1[C_1^*X_{11} - X_{11}D_1] = 0$ and $(-1)C_1(X_{11} \in \ker \delta_{C_1^*,D_1})$.

Similarly

$$X_{11} \in \ker \Delta^2_{C_1,D_1} = (-1)^2 C_1^2 (X_{11} \in \ker \delta^2_{C_1^*,D_1})$$

and in general

$$X_{11} \in \ker \Delta^n_{C_1,D_1} = (-1)^n C_1^n (X_{11} \in \ker \delta^n_{C_1^*,D_1}).$$

Hence by Lemma 2 of [28],

$$\lim_{n\to\infty} \left\|X_{11} \in \ker \Delta^n_{C_1,D_1}\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|X_{11} \in \ker \delta^n_{C_1^*,D_1}\right\|^{\frac{1}{n}} = 0.$$

Thus $X_{11} \in \ker \Delta^n_{C_1,D_1}$ is a zero operator and so $X_{11} \in \ker \Delta^{n-1}_{C_1,D_1}$.

Consequently $X \in \ker \Delta^{n-1}_{A,B}$ and $X \in \ker \Delta_{A,B}$ is a zero operator and again by the Theorem, $X \in \ker \Delta_{A^*,B^*}$ and the result follows.

Corollary 3. *If A is a k-paranormal or dominant or k-quasihyponormal contractions operator and B^* a contraction operator with $C_{.}o$ c.n.u. parts, such that $X \in \ker \Delta_{A,B}$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B and $A\,|_{\overline{\operatorname{ran} X}}$ and $B\,|_{\ker^{\perp} X}$ are unitarily equivalent unitary operators.*

Clearly if in Corollary 3, X is quasiaffinity, then A and B are unitarily equivalent unitary operators.

Similarly if in Theorem 1, the same is true, then we have the following Corollary.

Corollary 4. *If the contractions A and $B^* \in B(H)$ have $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}$ where X is quasiaffinity, then A and B are unitarily equivalent unitary operators.*

We now prove a Putnam–Fuglede Theorem $\Delta_{A,B}(X)$ analogue for w-hyponormal operators.

Theorem 5. *Let A, $B^* \in B(H)$ be w-hyponormal operators with $\ker A(B^*) \subset \ker A^*(B)$. If $X \in \ker \Delta_{A,B}$ for some $X \in B(H)$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B and $A\,|_{\overline{\operatorname{ran} X}}$ and $B\,|_{\ker^{\perp} X}$ are normal operators.*

To prove the theorem, we need auxiliary lemmas.

The following lemma is well known.

Lemma 6. *If* $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$*, then, for all* $X \in \ker \Delta_{A,B}$, $\overline{\operatorname{ran} X}$ *reduces* A , $\ker^{\perp} X$ *reduces* B *and* $A\,|_{\overline{\operatorname{ran} X}}$ *and* $B\,|_{\ker^{\perp} X}$ *are normal operators.*

The next result was proved in [3, Theorem 2.4].

Lemma 7. *If* A *is* w*-hyponormal, then* $\widetilde{A}$ *is semi-hyponormal and* $\widetilde{\widetilde{A}}$ *is hyponormal.*

The following result is Theorem 2.6 of [3].

Lemma 8. *Let* A *be* w*-hyponormal with* $\ker A \subset \ker A^*$*. If* $\widetilde{A}$ *is normal, then* $A = \widetilde{A}$*.*

Proof of Theorem 5. Let $\widetilde{\widetilde{X}} = \left|\widetilde{A}\right|^{\frac{1}{2}} |A|^{\frac{1}{2}}\, X \left|\widetilde{B^*}\right|^{\frac{1}{2}} |B^*|^{\frac{1}{2}}$. Since $X \in \ker \Delta_{A,B}$, $\widetilde{\widetilde{X}} \in \ker \Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}}$, where $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B^*}}$ are hyponormal operators by Lemma 7.

Applying Putnam–Fuglede Theorem for hyponormal operators analogue to $\Delta_{A,B}(X)$ [15, Theorem 2], it follows that $\widetilde{\widetilde{X}} \in \ker \Delta_{\widetilde{\widetilde{A^*}},\widetilde{\widetilde{B^*}}}$. Hence by Lemma 6,

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} \text{ reduces } \widetilde{\widetilde{A}} \text{ and } \ker^{\perp} \widetilde{\widetilde{X}} \text{ reduces } \widetilde{\widetilde{B}} \text{ and } \widetilde{\widetilde{A}}\,|_{\overline{\operatorname{ran} \widetilde{\widetilde{X}}}} \text{ and } \widetilde{\widetilde{B}}\,|_{\ker^{\perp} \widetilde{\widetilde{X}}}$$

are normal operators.

Consequently, $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$ must be normal operators [9] and by Lemma 8, A and B are normal operators. Thus $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$, and the result follows. □

3. w-HYPONORMAL OPERATORS AND QUASI-SIMILARITY

Douglas ([11]) proved that quasi-similar normal operators are unitarily equivalent.This result was extended by Clary ([10]) who proved that quasi-similar hyponormal operators are unitarily equivalent.

In this section, we extend the result of Clary ([10]) to the class of w-hyponormal operators.

The following lemma is due to Williams [34, Lemma 1.1].

Lemma 9. *Let* A *and* B *be normal operators. If there exist injective operators* X *and* Y *such that* $X \in \ker \delta_{A,B}$ *and* $Y \in \ker \delta_{B,A}$*, then* A *and* B *are unitarily equivalent.*

Theorem 10. *Let* A *and* B^* *be* w*-hyponormal operators with* $\ker A \subset \ker A^*$ *and* $\ker B \subset \ker B^*$ *respectively. If there* $\exists$ *quasiaffinities* X *and* Y *such that* $X \in \ker \delta_{A,B}$ *and* $Y \in \ker \delta_{B,A}$*, then* A *and* B *are unitarily equivalent normal operators.*

Proof. First decompose A and B^* into their normal and pure parts by $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $\widetilde{\widetilde{X}} = \left|\widetilde{A}_2\right|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}\, X \left|\widetilde{B_2^*}\right|^{\frac{1}{2}} |B_2^*|^{\frac{1}{2}}$. Since $X \in \ker \delta_{A_2,B_2}$, $\widetilde{\widetilde{X}} \in \ker \delta_{\widetilde{\widetilde{A}}_2,\widetilde{\widetilde{B_2}}}$, where $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2^*}}$ are hyponormal operators by Lemma 7 and $\widetilde{\widetilde{X}}$ is quasi-affinity.

Now by Putnam–Fuglede Theorem for hyponormal operators,

$$\widetilde{\widetilde{X}} \in \ker \delta_{\widetilde{\widetilde{A_2^*}}, \widetilde{\widetilde{B_2^*}}}$$

and

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} \text{ reduces } \widetilde{\widetilde{A}}_2 \text{ and } \ker^{\perp} \widetilde{\widetilde{X}} \text{ reduces } \widetilde{\widetilde{B_2}} \text{ and } \widetilde{\widetilde{A}}_2 \mid_{\overline{\operatorname{ran} \widetilde{\widetilde{X}}}} \text{ and } \widetilde{\widetilde{B_2}} \mid_{\ker^{\perp} \widetilde{\widetilde{X}}}$$

are unitarily equivalent normal operators. Since $\widetilde{\widetilde{X}}$ is quasiaffinity,

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} = H \quad \text{and} \quad \ker^{\perp} \widetilde{\widetilde{X}} = H$$

and $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2}}$ are unitarily equivalent normal operators. In particular $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2}}$ are normal operators and by Lemmas 8 and 9, the result follows. □

From the Theorem, the following corollaries are immediate.

Corollary 11. *If a w-hyponormal operator A with $\ker A \subset \ker A^*$ is quasi-similar to a normal operator B, then A and B are unitarily equivalent normal operators.*

Corollary 12 ([17, Corollary 6] and [25, Corollary 7]). *If a p-hyponormal or log-hyponormal A is quasi-similar to a normal operator B, then A and B are unitarily equivalent normal operators.*

During the early days of operator theory, Berberian S. K. [9] posed a very interesting question on the class of hyponormal operators: "Does there exist a completely hyponormal operator which is not normal?". While studying the concept of hyponormal operators, Ando [7] gave a negative answer to this question. That is to say, that every completely hyponormal operator is normal.

From Theorem 10, it is easy to deduce that a pure w-hyponormal operator is normal, which therefore generalises Ando's result [7].

However in the sequel, we wish to give an alternative proof of this result.

Theorem 13. *If A is w-hyponormal operator, then $\|A^n\| = \|A\|^n$ for all n.*

Proof. A is w-hyponormal implies

$$\|\widetilde{A}\| = \left\| \widetilde{|A|} \right\| \geq \||A|\| = \|A\|.$$

But

$$\|A\| \geq \|\widetilde{A}\| \geq \|\widetilde{\widetilde{A}}\|$$

is always true. Hence $\|A\| = \|\widetilde{A}\|$. Similarly, $\|\widetilde{A}\| = \|\widetilde{\widetilde{A}}\|$. Now since $\widetilde{\widetilde{A}}$ is hyponormal, by [7]

$$\|A\|^n = \|\widetilde{\widetilde{A}}\|^n = \|\widetilde{\widetilde{A^n}}\| = \|A^n\|$$

for all n. □

Corollary 14. *Every non-zero w-hyponormal operator has a non-zero element in its spectrum.*

Corollary 15. *A pure w-hyponormal operator is normal.*

Stampfli and Wadhwa ([30]) proved that if A is dominant and B is a normal operator such that $X \in \ker \delta_{A,B}$ where X has a dense range, then A is normal.

Recently, Duggal and Jeon ([17]) and Jeon,Tanahashi and Uchiyama ([25]) extended this result to a more general case of p-hyponormal and log-hyponormal respectively.

In the sequel, we try to extend the results of ([17]) and ([25]) to the class of w-hyponormal operators.

Theorem 16 (Generalised Putnam–Fuglede). *Let A be w-hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in \ker \delta_{A,B}$, then A is normal.*

Proof. Decompose $A = A_1 \oplus A_2$ into normal and pure parts respectively. Let $A_2 = U_2 |A_2|$, $\widetilde{A}_2 = |A_2|^{\frac{1}{2}} U |A_2|^{\frac{1}{2}}$ and $\widetilde{\widetilde{A}}_2 = \left|\widetilde{A}_2\right|^{\frac{1}{2}} \widetilde{U} \left|\widetilde{A}_2\right|^{\frac{1}{2}}$.

A_2 being pure, it is injective and $|A_2|^{\frac{1}{2}}$ is quasiaffinity. Also since A_1 is normal, $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}}_1 \oplus \widetilde{\widetilde{A}}_2 = A_1 \oplus \widetilde{\widetilde{A}}_2$.

Now if we let T $= \left|\widetilde{A}_2\right|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}$, then by a simple computation, $\widetilde{\widetilde{A}}_2 T = T A_2$ and T is quasiaffinity.

Also if we let $Z = I_H \oplus T$, then clearly Z is also quasiaffinity such that $\widetilde{\widetilde{A}} Z = ZA$, where $\widetilde{\widetilde{A}}$ is a hyponormal operator.

Thus $\widetilde{\widetilde{A}} ZX = ZAX = ZXB$ and by ([30]), $\widetilde{\widetilde{A}}$ is normal. Hence by Lemma 8, we get the result .

Thus from Theorem 16, we immediately recapture Corollary 11 again. However, the following Corollary says more than this.

Corollary 17. *Let A be w-hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in \ker \delta_{A,B}$, then A and B are unitarily equivalent normal operators.*

The following example due to Clary ([10]) shows that it is not possible to replace a normal operator in Corollary 11 with an isometry.

Example 18. *Let U denote the unweighted unilateral shift with multiplicity 1, and let S_n be the unilateral weighted shift with weights $\frac{1}{n}, 1, 1, 1, \ldots$. Let $U_\infty := \sum^\infty \oplus U$ and $S := \sum_{i=1}^\infty \oplus S_i$. Then each S_n is similar to U and so S and U_∞ are quasi-similar by [22, Theorem 2.5]. Clearly U_∞ is an isometry and S is hyponormal. But since S is not bounded from below, U_∞ and S are not similar.*

This therefore gave rise to the following question.

Problem. Is it possible to replace the normality of B in Corollary 11 with an isometry?

However, in affirmative answer to this question, Duggal and Jeon ([17]) recently proved the result for the case of p-hyponormal operators under the condition that either X or Y is compact.

In the sequel, we try to extend this result to a more general case of w-hyponormal operators as follows.

Theorem 19. *Let A be w-hyponormal with* $\ker A \subset \ker A^*$ *and B be an isometry. If there exist quasiaffinities $X, Y \in B(H)$ such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$ where X or Y is compact, then A and B are unitarily equivalent unitary operators.*

Proof. Since $\widetilde{\widetilde{A}}$ is hyponormal, by [36, Theorem 2.1], $\widetilde{\widetilde{A}}$ is subdecomposable and so $\sigma(\widetilde{\widetilde{A}}) \subseteq \sigma(B) \subseteq \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ denotes a closed unit disc.
Now since $\sigma(\widetilde{\widetilde{A}}) = \sigma(A)$ by [3, Corollary 2.3],

$$\|A\| = \|\widetilde{\widetilde{A}}\| = r(\widetilde{\widetilde{A}}) = r(A) \subseteq \sigma(A) \subseteq \overline{\mathbf{D}}$$

and A is a contraction. Applying theorem 2 of [6] to $Y \in \ker \delta_{B,A}$ if Y is compact, B is unitary. Similarly, by the same theorem if X is compact, then $B(YX) = YAX = = (YX)B$ and B is unitary. □

Now applying theorem 1 of [6] to the operator equation $Y \in \ker \delta_{B,A}$, A is unitary and the result follows.

Corollary 20. *If a log or p-hyponormal operator A is quasi-similar to an isometry B, then A and B are unitarily equivalent unitary operators.*

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