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Anna Pudełko

MONOTONE ITERATION FOR INFINITE SYSTEMS OF PARABOLIC EQUATIONS

Abstract. In the paper the Cauchy problem for an infinite system of parabolic type equations is studied. The general operators of the parabolic type of second order with variable coefficients are considered and the system is weakly coupled. Among the obtained results there is a theorem on diffenential inequality as well as the existence and uniqueness theorem in the class of continuous-bounded functions obtained by monotone iterative method.

Keywords: infinite systems, parabolic equations, Cauchy problem, monotone iteration method, differential inequality..

Mathematics Subject Classification: Primary: 35K15, Secondary: 35K55, 35R45.

1. INTRODUCTION

We consider an infinite system of weakly coupled semilinear parabolic equations of reaction-diffusion-convection type of the form

$$\mathcal{F}^i[u^i](t,x) = f^i(t,x,u(t,x)), \quad i \in S, \tag{1}$$

supplemented with the initial condition

$$u(0,x) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m\,. \tag{2}$$

Here the functions f^i and φ stand for the following mappings:

$$f^i\colon \overline{\Omega} \times \mathbb{R} \ni (t,x,s) \to f^i(t,x,s) \in \mathbb{R},\ \ i \in S,$$
$$\varphi\colon S \times \mathbb{R}^m \ni (i,x) \to \varphi^i(x) \in \mathbb{R},$$

respectively.

Here:

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i,$$

$$\mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{m} b^i_j(t,x)\frac{\partial}{\partial x_j} + c^i(t,x),$$

$(t,x) \in \Omega := \{(t,x)\colon t \in (0,T], x \in \mathbb{R}^m\}$, $0 < T < \infty$, S is the set of indices and $u\colon \overline{\Omega} \ni (t,x) \to u(t,x) \in B(S)$, where $u(t,x) : S \ni i \to u^i(t,x) \in \mathbb{R}$

Let $B(S)$ be the space of all mappings $v\colon S \ni i \to v^i \in \mathbb{R}$, such that $\sup\{|v^i| : i \in S\} < \infty$ endowed with the supremum norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\}.$$

Denote by $CB_S(\overline{\Omega})$ the space of mappings $w\colon \overline{\Omega} \ni (t,x) \to w(t,x) \in B(S)$, where $w(t,x)\colon S \ni i \to w^i(t,x) \in \mathbb{R}$ and the functions $w^i = w^i(t,x)$ are continuous and bounded in $\overline{\Omega}$, $\sup\{|w^i(t,x)| : (t,x) \in \overline{\Omega},\ i \in S\} < \infty$. $CB_S(\overline{\Omega})$ is endowed with the following norm

$$\|w\|_0 := \sup\{|w^i(t,x)| : (t,x) \in \overline{\Omega},\ i \in S\}.$$

The space $CB_S(\mathbb{R}^m)$ is understood in an analogous way. For $w \in CB_S(\overline{\Omega})$ and for a fixed t, $t \geq 0$ we define

$$\|w\|_{0,t} := \sup\{|w^i(\tilde{t},x)| : (\tilde{t},x) \in \overline{\Omega},\ \tilde{t} \leq t,\ i \in S\}.$$

The aim of this paper is to prove the existence and the uniqueness of a solution for system (1) supplemented with the initial condition (2) (Theorem, Section 4). To obtain solution of considered problem we apply so-called monotone iterative method (method of subsolutions and supersolutions) (cf. [13]). The first initial-boundary value problem for infinite system of weakly coupled differential-functional equations of parabolic type was dealt with the same monotone iterative technique in [3, 4]. In this approach we have to assume the monotonicity of the reaction functions in the last variable as well as the existence of a pair of lower and upper solutions for the considered problem. The method sub and super solutions, coupled with monotone iterative technique provides an effective and flexible mechanism that ensures theoretical as well as constructive existence results for nonlinear problems(cf. [12]). The lower and upper solutions serve as bounds for solutions which are improved by a monotone iterative process. We use differential inequality to show that sequences obtained by monotone iteration are sub and supersolutions, respectively, as well as to get the uniform convergence of these sequences. We notice that Proposition 2, Section 3, which proof is based on the Gronwall lemma (cf. [19, 5]), is crucial in the proof of Theorem 1.

An infinite system of ordinary differential equations was considered first time by M. Smoluchowski [17] as a model for coagulation of colloids moving according to a Brownian motion. The classical Smoluchowski's coagulation equations have described the binary coagulation of colloidal particles. Certain generalization of Smoluchowski's model which describes the space and time evolution of a system of a large number of clusters moving by spatial diffusion is treated in [2]. The author assumes that the clusters are composed of a number of identical units and are fully identified by their size but the size of the clusters is not limited a priori therefore the infinite system appears.

A system of infinite number of reaction-diffusion equations related to the system of ODE derived by Smoluchowski arise among others in polymer science [11], aerosol physics [7], phase transitions [9], biology and immunology [6, 14] and astrophysics [16].

This paper is organised as follows. In the next section the necessary notations are introduced. We also formulate the assumptions and an auxiliary lemma (cf. [8, 10]) in Section 2. In Section 3 we state and prove the comparison principles. The last section contains a main result of the paper, i.e. the theorem on the existence and the uniqueness of the solution for the problem (1), (2) and its proof.

2. NOTATIONS, DEFINITIONS AND ASSUMPTIONS

Throughout the paper, we use the following notation.

Let R be a positive number. By D_R we denote the part of the domain Ω contained in the cylindric surface (which we will denote by Γ_R) described by the equation $\sum_{j=1}^{n} x_j^2 = R^2$. D_R stands for the base of S_R^0.

We define the Niemycki operator $\mathbf{F} = \{\mathbf{F}^i\}_{i\in S}$ by setting

$$\mathbf{F}^i[\eta](t,x) := f^i(t,x,\eta(t,x)),$$

for each $\eta \in CB_S(\overline{\Omega})$ and $i \in S$. This operator plays an important role in theory of nonlinear equations (cf. [1])

Now we formulate the assumptions sufficient for existing the fundamental solution of the homogeneous system associated with system (1).

We will assume that

(**H**) the coefficients $a^i_{jk}(t,x)$, $b^i_j(t,x)$, $c^i(t,x)$, $i \in S$, $j,k = 1,\dots,m$ are bounded continuous in $\overline{\Omega}$ functions such that $a^i_{jk}(t,x) = a^i_{kj}(t,x)$ and satisfy the following uniform *Hölder conditions* with exponent α $(0 < \alpha \le 1)$ in $\overline{\Omega}$ there exists $H > 0$ such that:

$$\left|a^i_{jk}(t,x) - a^i_{jk}(t,x')\right| \le H\left|x - x'\right|^{\alpha},$$
$$\left|b^i_j(t,x) - b^i_j(t,x')\right| \le H\left|x - x'\right|^{\alpha},$$
$$\left|c^i(t,x) - c^i(t,x')\right| \le H\left|x - x'\right|^{\alpha},$$

for all (t,x), $(t,x') \in \overline{\Omega}$ and $j,k = 1,\dots,m$.

We suppose as well that the operators $\mathcal{F}^i$, $i \in S$ are uniformly parabolic in $\overline{\Omega}$ i.e. there is $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a^i_{jk}(t,x)\xi_j\xi_k \geq \mu \sum_{j=1}^{m} \xi_j^2$$

for all $(t,x) \in \overline{\Omega}$, and $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$.

Lemma 1 (cf. [8, 10]). *Let the operators $\mathcal{F}^i$ ($i \in S$) be uniformly parabolic in $\overline{\Omega}$ with the constant μ. If the coefficients $a^i_{jk}(t,x)$, $b^i_j(t,x)$, $c^i(t,x)$, $i \in S$, $j,k = 1,\ldots,m$ satisfy the condition* (**H**) *in $\overline{\Omega}$, then there exist the fundamental solutions $\Gamma^i(t,x;\tau,\xi)$ of the equations $\mathcal{F}^i[u^i](t,x) = 0$, $i \in S$ and the following inequalities hold*

$$\left|\Gamma^i(t,x;\tau,\xi)\right| \leq C(t-\tau)^{-\frac{m}{2}} \exp\left(-\frac{\mu^* \left|x-\xi\right|^2}{4(t-\tau)}\right), \quad i \in S$$

for any $\mu^* < \mu$ where μ^* depends on μ and H whereas C depends on μ, α, T and the character of continuity $a^i_{jk}(t,x)$ in t.

Now we reinforce assumptions relating to the principal coefficients. Let the coefficients $a^i_{jk}(t,x)$ satisfy the uniform *Hölder condition* in $\overline{\Omega}$ with exponent α with respect to t and x in the sense of the parabolic distance, i.e. there exists $H > 0$ such that

$$(\mathbf{H_a}) \qquad \left|a^i_{jk}(t,x) - a^i_{jk}(t,x')\right| \leq H\left(\left|x-x'\right|^{\alpha} + \left|t-t'\right|^{\frac{\alpha}{2}}\right)$$

for all (t,x), $(t',x') \in \overline{\Omega}$ and $j,k = 1,\ldots,m$.

Remark 1 (cf. [10]). *Let operators $\mathcal{F}^i$, $i \in S$ be uniformly parabolic and their coefficients satisfy the assuption* (**H**) *and* ($\mathbf{H_a}$)*, then $\Gamma^i(t,x;\tau,\xi)$ are the positive functions.*

Using the fundamental solutions and the Niemycki operator we can transform the differential problem (1), (2) into the following associated integral system

$$u^i(t,x) = \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int_0^t \int_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi d\tau \tag{3}$$

for $t > 0$, $x \in \mathbb{R}^m$.

Assumptions related to the initial data $\varphi = \{\varphi^i\}_{i\in S}$ and right hand side $f = \{f^i\}_{i\in S}$ are as follows:

(φ) $\varphi^i \in CB(\mathbb{R}^m)$ for all $i \in S$, i.e. $\varphi \in CB_S(\mathbb{R}^m)$;

(C_f) $f^i(t,x,s)$ are continuous in $\overline{\Omega} \times \mathbb{R}$ for all $i \in S$;

(B_f) $f^i(t,x,s)$ are commonly bounded in $\overline{\Omega} \times \mathbb{R}$ for all $i \in S$;

(I_f) $f^i(t,x,s)$ are increasing with respect to s for all $i \in S$.

Furthermore, let function function $f = \{f^i\}_{i\in S}$ satisfies the following additional conditions:

(H_f) locally Hölder continuous with respect to x;

(L_f) Lipchitz continuous in s.

Moreover, we assume that

(O) In the space $CB_S(\overline{\Omega})$ the following order (precisely *partial order*) is introduced for $z, \tilde{z} \in CB_S(\overline{\Omega})$, the inequality $z \le \tilde{z}$ means that $z^i(t,x) \le \tilde{z}^i(t,x)$ for all $(t,x) \in \overline{\Omega}$ and $i \in S$.

3. COMPARISON PRINCIPLES

The main goal of this section is to state some auxiliary results which will be our tools in the next section. We start this section with a simple comparison principle. The Gronwall lemma will be our main tool in proofs of this section.

Proposition 1. *Let functions $f^i = f^i(t,x,s)$ be Lipschitz continuous in s uniformly with respect to $i \in S$*

$$\left|f^i(t,x,s) - f^i(t,x,\tilde{s})\right| \le L \left\|s - \tilde{s}\right\|_{B(S)}.$$

If v, $w \in C_S(\overline{\Omega})$ satisfy the following systems

$$\mathcal{F}^i[v^i](t,x) = f^i(t,x,v(t,x)), \quad \mathcal{F}^i[w^i](t,x) = \bar{f}^i(t,x,w(t,x)), \quad i \in S, \tag{4}$$

and if there exists a nonnegative constant M independent of $i \in S$ such that

$$\left|f^i(t,x,s) - \bar{f}^i(t,x,s)\right| \le M,$$

then

$$\|v - w\|_{0,t} \le C \left\|v(0,\cdot) - w(0,\cdot)\right\|_0 e^{tCL} + CM \int_0^t e^{(t-\tau)CL}\, d\tau$$

provided that $v - w \in CB_S(\overline{\Omega})$.

Proof. We consider the auxiliary function $\tilde{z}(t) := \|z(t)\|_{B(S)}$, where $z(t) = \{z^i(t)\}_{i\in S}$ and $z^i(t) = \sup_{x\in\mathbb{R}^m,\, \bar{t}\le t} \left|v^i(\bar{t},x) - w^i(\bar{t},x)\right| =: \left\|v^i - w^i\right\|_{0,t}$.

At first we show a certain estimate for this function. Since the functions v, w satisfy the systems (4), then we have

$$\begin{aligned} v^i(t,x) - w^i(t,x) = &\int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\left[v^i(0,\xi) - w^i(0,\xi)\right] d\xi + \\ &+ \int_0^t \int_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)\left[f^i(\tau,\xi,v(\tau,\xi)) - \bar{f}^i(\tau,\xi,w(\tau,\xi))\right] d\xi\, d\tau \le \\ &\le C z^i(0) + \int_0^t C[M + L\|z(\tau)\|_{B(S)}] d\tau, \quad i \in S, \end{aligned}$$

therefore the function $\tilde{z}$ satisfies the following integral inequality

$$\tilde{z}(t) \leq C\tilde{z}(0) + \int_0^t C[M + L\tilde{z}(\tau)]d\tau.$$

Thus by the virtue of the Gronwall lemma, we obtain

$$\tilde{z}(t) \leq C\tilde{z}(0)e^{tCL} + \int_0^t CMe^{(t-\tau)CL}\ d\tau,$$

and the proof of Proposition 1 is complete. □

Remark 1. *In particular, if the right hand sides of the systems* (4) *are equal and* $v(0,x) = w(0,x)$*, then Proposition 1 yields the uniqueness.*

In the next section we construct the certain sequences of successive approximation as solutions of linear systems of differential equations. But first we formulate the result which ensures the uniform convergence of these sequences to the desired solution of the problem (1), (2).

Proposition 2 (Differential inequalities). *Let functions* $f^i = f^i(t,x,s)$ *satisfy the condition* (I_f) *and* $f^i = f^i(t,x,s)$ *also satisfy the one-sided Lipschitz condition with respect to* s (*uniformly with respect to* $i \in S$)

$$f^i(t,x,s) - f^i(t,x,\tilde{s}) \leq L\left\|s - \tilde{s}\right\|_{B(S)} \quad \textit{for} \quad s \geq \tilde{s}$$

If $v,\ w \in CB_S(\overline{\Omega})$ *satisfy the following systems of inequalities:*

$$\mathcal{F}^i[v^i](t,x) \leq f^i(t,x,v(t,x)), \quad \mathcal{F}^i[w^i](t,x) \geq f^i(t,x,w(t,x)), \quad i \in S, \tag{5}$$

and $v(0,x) \leq w(0,x)$ *in* $\mathbb{R}^m$ *then* $v(t,x) \leq w(t,x)$ *for* $(t,x) \in \overline{\Omega}$.

Proof. We define an auxiliary function $y(t) = \{y^i(t)\}_{i\in S}$ by setting $y^i(t) := \max\{0,\ z^i(t)\}$ where $z^i = \sup_{x\in\mathbb{R}^m,\bar{t}\leq t}\Big(v^i(\bar{t},x) - w^i(\bar{t},x)\Big)$. It is easy to see that $0 \leq y^i(t) < \infty$ and $z^i(t) \leq y^i(t)$ for all $i \in S$. Now, applying the condition (I_f) and the one-sided Lipschitz condition we estimate the differences

$$\begin{aligned} f^i(t,x,v(t,x)) - f^i(t,x,w(t,x)) = f^i(t,x,[w + (v-w)](t,x)) - f^i(t,x,w(t,x)) \leq \\ \leq f^i(t,x,w(t,x) + y(t)) - f^i(t,x,w(t,x)) \leq L\left\|y(t)\right\|_{B(S)} \end{aligned}$$

Since the functions $v,\ w$ satisfy the systems (5), then for all $i \in S$ we have the following inequalities

$$v^i(t,x) - w^i(t,x) \le \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\left[v^i(0,\xi) - w^i(0,\xi)\right] d\xi +$$

$$+\int\limits_0^t \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)\left[f^i(\tau,\xi,v(\tau,\xi)) - f^i(\tau,\xi,w(\tau,\xi))\right] d\xi\, d\tau \le \int\limits_0^t CL\left\|y(\tau)\right\|_{B(S)}\, d\tau.$$

Thus, for function $\tilde{y}(t) = \|y(t)\|_{B(S)}$ we have the inequality

$$\tilde{y}(t) \le \int\limits_0^t CL\tilde{y}(\tau)\; d\tau.$$

Having this fact, on the basis the Gronwall lemma we obtain $\tilde{y}(t) \equiv 0$, which means that $v(t,x) \le w(t,x)$ in $\overline{\Omega}$. Proposition 2 is proved. □

4. MONOTONE ITERATION. EXISTENCE AND UNIQUENESS SOLUTION

Before formulating the main theorem of this section we introduce the following notation. Namely, for every sufficiently smooth function β we denote by $\mathcal{P}$ the operator $\mathcal{P}\colon \beta \mapsto \gamma = \mathcal{P}[\beta]$, where γ is a unique solution of the following initial value problem:

$$\mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,x)), \quad i \in S,$$
$$\gamma(0,x) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m\,.$$

Remark 1. *Due to the fact that the functions $f^i = f^i(t,x,s)$, $i \in S$ are increasing with respect to s and $\Gamma^i(t,x;\tau,\xi)$ are positive functions the operator $\mathcal{P}$ is increasing.*

Functions $v = v(t,x)$ and $w = w(t,x) \in CB_S(\overline{\Omega})$ satysfying the system of the following inequalities:

$$\mathcal{F}^i[v^i](t,x) \le f^i(t,x,v(t,x)), \quad i \in S,$$
$$u(0,x) \le \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m,$$
$$\mathcal{F}^i[w^i](t,x) \ge f^i(t,x,w(t,x)), \quad i \in S,$$
$$u(0,x) \ge \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m\,.$$

are called, respectively, a *subsolution* and a *supersolution* for problem (1), (2) in $\overline{\Omega}$.

Assumption A. We assume that there exists at least one pair v_0, $w_0 \in CB_S(\overline{\Omega})$ of a subsolution and a supersolution of the problem (1), (2) in $\overline{\Omega}$ which are Hölder continuous in x uniformly with respect to t.

Remark 2. *If v and w are a subsolution and supersolution for the problem* (1), (2) *in $\overline{\Omega}$, respectively, and u is any classical solution of this problem, then Proposition 2 yields*

$$v(t,x) \le u(t,x) \le w(t,x) \quad for \quad (t,x) \in \overline{\Omega}.$$

In particular, we have

$$v_0(t,x) \le u(t,x) \le w_0(t,x) \quad for \quad (t,x) \in \overline{\Omega}.$$

Now we formulate the theorem on the existence and the uniqueness of a solution of the problem (1), (2) obtained by simple iterative method, i.e. starting the iteration procedure from the subsolution v_0 and the supersolution w_0 we define the iterative sequence by induction as follows $v_{n+1} := \mathcal{P}[v_n]$, $w_{n+1} := \mathcal{P}[w_n]$, $(n = 1, 2, \dots)$. Thus, on each step we have the infinite system of the linear equations.

Theorem. *Let the operators $\mathcal{F}^i$, $i \in S$ be uniformly parabolic in $\overline{\Omega}$. Let the assumptions* $(\mathbf{H})$, $(\mathbf{H_a})$, (φ), (C_f), (B_f), (I_f), (H_f), (L_f), (O) *and the assumption* A *hold. Consider the following infinite system of linear equations*

$$\mathcal{F}^i[v^i_{n+1}](t,x) = f^i(t,x,v_n(t,x)), \tag{6}$$

$$\mathcal{F}^i[w^i_{n+1}](t,x) = f^i(t,x,w_n(t,x)), \tag{7}$$

for $(t,x) \in \Omega$, $i \in S$, for $n = 1, 2, \dots$ with the initial condition (2) *and let $N_0 = \|w_0 - v_0\| < \infty$. Then:*

- (i) *there exist unique classical bounded solutions v_n and w_n $(n = 1, 2 \dots)$ of systems* (6) *and* (7) *with the initial condition* (2) *in $\overline{\Omega}$;*
- (ii) *functions v_n and w_n $(n = 1, 2)s$ are the subsolutions and supersolutions for problem* (1), (2) *in $\overline{\Omega}$, respectively;*
- (iii) *the following inequalities*

  $$v_0(t,x) \le \cdots \le v_n(t,x) \le v_{n+1}(t,x) \le \cdots \le w_{n+1}(t,x) \le w_n(t,x) \le \cdots \le w_0(t,x$$

  hold for $(t,x) \in \overline{\Omega}$, $(n = 1, 2 \dots)$;
- (iv) $\lim_{n\to\infty}[w^i_n(t,x) - v^i_n(t,x)] = 0$ *uniformly in $\overline{\Omega}$, $i \in S$;*
- (v) *the function $u(t,x) = \lim_{n\to\infty} v_n(t,x)$ is a unique bounded solution of problem* (1), (2) *in $\overline{\Omega}$.*

Proof of Theorem. (i) Starting the iteration procedure from the subsolution v_0 and the supersolution w_0 we define v_1, w_1 as the solutions of the systems of linear equations (6), (7) supplemented with initial condition (2), i.e. $v_1 = \mathcal{P}[v_0]$, $w_1 = \mathcal{P}[w_0]$. We observed that considered systems have the following property: the i-th equation depends on the i-th unknown function only, therefore since v_0, w_0 satisfy the assumption A, the classical theorems on existence and uniqueness of solution of linear parabolic Cauchy problem (cf. [10]) assert that there exist the unique solutions of the above problems v_1, $w_1 \in CB_S(\overline{\Omega})$ and functions v_1 and w_1 are Hölder continuous with respect to x (cf. [8]). Next we define by induction the succesive terms of the iteration sequences $\{v_n\}$, $\{w_n\}$ as solutions of the linear systems (6), (7) supplemented with initial condition (2), i.e. $v_n = \mathcal{P}[v_{n-1}]$, $w_n = \mathcal{P}[w_{n-1}]$. The preceding reasoning yields that v_n, w_n exist, are uniquely defined. Moreover, for each $i \in S$, $n = 1, 2, \ldots$ v_n^i, w_n^i belong to $C^{1,2}(\overline{\Omega})$ and they are Hölder continuous in x uniformly with respect to t.

(ii) We now show by induction argument that the functions v_n are subsolutions. v_0 is the subsolution by assumption (A). Let v_n be a subsolution of (1), (2), i.e.

$$\mathcal{F}^i[v_n^i](t,x) \le f^i(t,x,v_n(t,x)), \quad i \in S, \tag{8}$$

$$v_n(0,x) \le \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m . \tag{9}$$

From the definition of the operator $\mathcal{P}$ follows that

$$\mathcal{F}^i[v_{n+1}^i](t,x) = f^i(t,x,v_n(t,x)), \quad i \in S, \tag{10}$$

$$v_{n+1}(0,x) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^m . \tag{11}$$

Thus, Proposition 2 yields the inequality

$$[v_n - v_{n+1}](t,x) \le 0 \quad \text{for} \quad (t,x) \in \overline{\Omega},$$

i.e.

$$v_n(t,x) \le \mathcal{P}[v_n](t,x) \quad \text{for} \quad (t,x) \in \overline{\Omega}.$$

Now the monotonicity condition (I_f) enables us to obtain the following

$$\mathcal{F}^i[v_{n+1}^i](t,x) - f^i(t,x,v_{n+1}(t,x)) \;=\; f^i(t,x,v_n(t,x)) - f^i(t,x,\mathcal{P}[v_n](t,x)) \;\le\; 0$$

for all $i \in S$, $(t,x) \in \overline{\Omega}$. We conclude that function v_{n+1} is a subsolution as well. To proof that functions w_n are supersolutions we proceed in a similar way.

(iii) The monotonicity of the sequences $\{v_n\}$, $\{w_n\}$ is equivalent of the fact that functions v_n, w_n are subsolutions and supersolutions, respectively, whereas the inequality $v_n \le w_n$ is the consequence of the fact that $v_0 \le w_0$ and the fact that the operator $\mathcal{P}$ is increasing.

(iv) In this step of our proof we show by induction that

$$m_n^i(t,x) := w_n^i(t,x) - v_n^i(t,x) \ge 0$$

is estimated as follows:

$$m_n^i(t,x) \le N_0 \frac{(Lt)^n}{n!}, \qquad n = 0, 1 \ldots, \quad \text{for} \quad (t,x) \in \overline{\Omega},\ i \in S.$$

The inequality for m_0^i is obvious. Suppose it holds for m_n^i. Due to the condition (L_f) and the induction assumption we have the following estimation

$$\mathcal{F}^i[m_{n+1}^i](t,x) = f^i(t,x,w_n(t,x)) - f^i(t,x,v_n(t,x)) \leq L\,\|m_n\|_{B(S)} \leq N_0\frac{L^{n+1}t^n}{n!}$$

in Ω and $m_{n+1}^i(0,x) = 0$ for $x \in \mathbb{R}^m$, $i \in S$. In order to apply the theorem on differential inequality, let us consider the comparison system

$$\mathcal{F}^i[M_{n+1}^i](t,x) = N_0\frac{L^{n+1}t^n}{n!} \quad \text{for} \quad (t,x) \in \overline{\Omega},\ i \in S$$

supplemented with the initial condition $M_{n+1}^i(0,x) \geq 0$ for $x \in \mathbb{R}^m$, $i \in S$. The functions $M_{n+1}^i(t,x) = N_0\frac{(Lt)^{n+1}}{(n+1)!}$ are the solutions of comparison problem, therefore, owing to Proposition 2 we get

$$m_{n+1}^i(t,x) \leq M_{n+1}^i(t,x) = N_0\frac{(Lt)^{n+1}}{(n+1)!}, \quad \text{for} \quad (t,x) \in \overline{\Omega},\ i \in S,$$

so, the step induction is proved. Now, the uniform convergence of m_n^i in $\overline{\Omega}$ for all $i \in S$ is obvious. Since $\{v_n\}$ and $\{w_n\}$ are bounded and monotone sequences of continuous functions, there exist continuous functions $u^i = u^i(t,x)$ such that

$$\lim_{n\to\infty} v_n^i(t,x) = u^i(t,x),\ \lim_{n\to\infty} w_n^i(t,x) = u^i(t,x) \tag{12}$$

uniformly in $\overline{\Omega}$ for all $i \in S$ and the function $u = \{u^i\}_{i\in S}$ satisfies the initial condition (2).

(v) First, we prove that the function u defined above satisfies the system (1). It is enough to show that u fulfills (1) in every compact set contained in Ω. Consequently, due to the definition D_R we only need to prove it in D_R for any $R > 0$. Due to the condition (W) and the inequalities (iii) it follows that $f^i(t,x,v_{n-1})$ are uniformly bounded in D_R (with respect to n) therefore, the solution $v_n(t,x)$ of the linear system

$$\mathcal{F}^i[v_n^i](t,x) = f^i(t,x,v_{n-1}(t,x)), \quad i \in S \tag{13}$$

with the suitable initial condition is Hölder continuous with exponent α with respect to x with the constant independent of n (cf. [8]) and $i \in S$ (by B_f). Hence, we conclude by (12) that the boundary function $u(t,x)$ satisfies the Hölder condition with respect to x as well. □

Now, let us consider the following system of equations

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,u(t,x)), \quad i \in S \tag{14}$$

supplemented with the conditions:

$$z(t,x) = u(t,x) \quad \text{on} \quad \Gamma_R, \tag{15}$$

$$z(0,x) = \varphi(x) \quad \text{on} \quad S_R^0. \tag{16}$$

Owing to the facts that $u(t,x)$ is Hölder continuous with respect to x and the conditions (H_f) and (L_f) hold, the right hand sides of this system are continuous in D_R and locally Hölder continuous with respect to x.

Thus, on the basis the classical theorems on existence and uniqueness of solution of linear parabolic initial-boundary valued problem (cf. [10]) there exists the unique solutution $z(t,x)$ of the problem (14), (15), (16) in $\overline{D}_R$.

On the other hand, from (12) and (L_f) it follows that

$$\lim_{n\to\infty} f^i(t,x,v_{n-1}(t,x)) = f^i(t,x,u(t,x)) \quad \text{uniformly in} \quad \overline{D}_R.$$

Moreover, the boundary values $v_n(t,x)$ converge uniformly to $u(t,x)$ on Γ_R and initial values are equal, therefore using the theorem on the continuous dependence of the solution on the right hand sides of the system and on the initial-boundary values (cf. [18]) to systems (14) and (13) we conclude

$$\lim_{n\to\infty} v_n^i(t,x) = z^i(t,x) \quad \text{uniformly in} \quad \overline{D}_R.$$

Thus $z^i(t,x) = u^i(t,x)$ in $\overline{D}_R$ for all $i \in S$, for arbitrary $R > 0$, which means $z(t,x) = u(t,x)$ for all $(t,x) \in \overline{\Omega}$, i.e. $u(t,x)$ is the classical bounded solution of the problem (1), (2).

The uniqueness of the solution follows directly from Remark 2. This completes the proof of the Theorem. □

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Anna Pudełko
`fronczyk@wms.mat.agh.edu.pl`

AGH University of Science and Technology,
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

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