Opuscula Mathematica • Vol. 25 • No. 2 • 2005

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A NOTE ON SELF-COMPLEMENTARY 4-UNIFORM HYPERGRAPHS

Abstract. We prove that a permutation θ is complementing permutation for a 4-uniform hypergraph if and only if one of the following cases is satisfied:
(i) the length of every cycle of θ is a multiple of 8,
(ii) θ has 1, 2 or 3 fixed points, and all other cycles have length a multiple of 8,
(iii) θ has 1 cycle of length 2, and all other cycles have length a multiple of 8,
(iv) θ has 1 fixed point, 1 cycle of length 2, and all other cycles have length a multiple of 8,
(v) θ has 1 cycle of length 3, and all other cycles have length a multiple of 8.
Moreover, we present algorithms for generating every possible 3 and 4-uniform self-complementary hypergraphs.

Keywords: complementing permutation, self-complementary hypergraph, k-uniform hypergraph.

Mathematics Subject Classification: Primary 34A30, 34B05, 35B12, 35A15; Secondary 35J50, 35J65, 46T15, 34B15.

1. INTRODUCTION

Let V be a set of n elements. The set of all k-subsets of V is denoted $\binom{V}{k}$. A *k-uniform hypergraph* G consists of a *vertex-set* $V(G)$ and an *edge-set* $E(G) \subseteq \binom{V(G)}{k}$. Two k-uniform hypergraphs X and Y are *isomorphic*, if there is a bijection $\theta\colon V(X) \to V(Y)$ which maps $E(X)$ in to $E(Y)$. A k-uniform hypergraph G is called *self-complementary* if G is isomorphic with its complement $\overline{G}$. The complement $\overline{G}$, of any k-uniform hypergraph consists of all k-subsets of $V(G)$ not in $E(G)$. Isomorphism of a hypergraph to its complement is called *complementing permutation.*

2. SELF-COMPLEMENTARY 4-UNIFORM HYPERGRAPHS

In 1992 Kocay [1] studied 3-uniform hypergraphs providing the following result.

Theorem 1. *A permutation θ is a complementing permutation for a 3-uniform hypergraph if and only if either:*

(i) *every cycle of θ has even length, or*

(ii) *θ has 1 or 2 fixed points, and all other cycles have length a multiple of 4.*

We can now formulate analogue of Theorem 1 for 4-uniform hypergraph.

Theorem 2. *A permutation θ is complementing permutation for a 4-uniform hypergraph if and only if one of the following cases is satisfied:*

(i) *the length of every cycle of θ is a multiple of 8,*

(ii) *θ has 1, 2 or 3 fixed points, and all other cycles have length a multiple of 8,*

(iii) *θ has 1 cycle of length 2, and all other cycles have length a multiple of 8,*

(iv) *θ has 1 fixed point, 1 cycle of length 2, and all other cycles have length a multiple of 8,*

(v) *θ has 1 cycle of length 3, and all other cycles have length a multiple of 8.*

Proof. Let θ be a complementing permutation for 4-uniform hypergraph G. Any quadruple $\{i, j, k, l\}$ gives rise to a sequence of quadruples:

$$\{i, j, k, l\}, \theta\{i, j, k, l\}, \theta^2\{i, j, k, l\}, \ldots \tag{1}$$

Lemma 1. *θ is a complementing permutation for 4-uniform hypergraph if and only if every sequence* (1) *of quadruples has even length.*

We first analyze every possible quadruple:

I. Let i, j, k and l all are in the same cycle of θ, of length m. Suppose that $(v_1, v_2, \ldots, v_m)$ represents such a cycle.

- **a.** If m=4 then the sequence of quadruples will have length 1. This contradicts the fact that every sequence of quadruples has even length. Therefore θ has no cycles of length 4.
- **b.** If $m > 4$ then in general the sequence of quadruples will also have length m, so that m must by even. Thus every cycle of θ of length ≥ 4 must be of even length.
- **c.** Let $m = 2r$ and $i = v_1, j = v_{1+s}, k = v_{r+1}, l = v_{r+1+s}$, where $2r > 4$ and $s \leq (r-1)$. The length of sequence of quadruples will then be r, so that r must be even. This means that, every even cycle of θ of length > 4 must have length a multiple of 4. Moreover combining I.a and I.b we conclude that there are no other cycles of length ≥ 4.

d. If $m = 4r$,where $4r > 4$, and i, j, k, l are $v_1, v_{r+1}, v_{2r+1}, v_{3r+1}$, then the length of the sequence of quadruples will be r, which means that r must be even. Combining this with I.c we conclude that every cycle of θ of length ≥ 4 must have length a multiple of 8.

II. Let (v_1, v_2, v_3) be the cycle of θ, of length 3.

a. If $i = v_1, j = v_2, k = v_3$ and l is in another cycle of length $m \geq 4$, then quadruple $\{i, j, k, l\}$ will produce a sequence of length m, and therefore m will be even.

b. If $i = v_1, j = v_2$ and if k, l are both in different cycles of θ of lengths ≥ 4 (by I.c it is obvious that lengths of this cycles are multiple of 8), then the period of $\{i, j, k, l\}$ will be the last common multiple of even numbers, an even number.

c. If $i = v_1$ and j,k,l are all in different cycles of θ of lengths ≥ 4, then the length of the sequence of quadruples will be the last common multiple of 3 and even numbers, which is even.

d. Let $i = v_1, j = v_2, k = v_3$ and l, say, be a fixed point. Then the quadruple $\{i, j, k, l\}$ will produce a sequence of length 1, a contradiction. Therefore θ cannot contain both cycle of length 3 and fixed point.

e. If $i = v_1, j = v_2$, and k, l are in the cycle of θ of length 2, then the period of $\{i, j, k, l\}$ will be equal to 3, which is a contradiction. So θ cannot contain both cycle of length 2 and cycle of length 3.

f. If $i = v_1, j = v_2, k = v_3$, and l is in another cycle of length 3, then the length of the sequence of quadruples will be equal to 3, giving us a contradiction. Hence permutation θ can contain only one cycle of length 3.

III. Let (v_1, v_2) be the cycle of θ, of length 2.

a. If $i = v_1, j = v_2$ and k, l are both in another cycles of lengths ≥ 4, then quadruple $\{i, j, k, l\}$ will produce a sequence of even length.

b. Let $i = v_1$ and j, k, l are all in another cycles of θ of lengths ≥ 4. The length of the sequence of quadruples will then be the last common multiple of even numbers, an even number.

c. If $i = v_1, j = v_2$, k is fixed point, and l is in another cycle of length $m \geq 4$, then quadruple $\{i, j, k, l\}$ will produce a sequence of length m, which is even.

d. Let $i = v_1, j = v_2$ and k, l are both fixed points. So the quadruple $\{i, j, k, l\}$ will produce a sequence of length 1, a contradiction. Hence θ cannot contain 2 fixed points and cycle of length 2.

e. If $i = v_1, j = v_2$ and k, l are both in another cycle of length 2, then the period of $\{i, j, k, l\}$ will be equal to 1, which is a contradiction. So θ can contain only one cycle of length 2.

IV. Let i be a fixed point of θ.

a. If j, k, l are all in cycles of θ of lengths ≥ 4, then the length of the sequence of quadruples will be the last common multiple of even numbers, an even number.

b. If j a fixed point of θ, and k, l are both in another cycles of lengths ≥ 4, then the length of the sequence of quadruples will be the last common multiple of even numbers, which is an even number.

c. If j and k are both fixed points, and l is in another cycle of θ of length $m \geq 4$, then quadruple $\{i, j, k, l\}$ will produce a sequence of even length m.

d. Let j, k, l are all fixed points of θ. The quadruple $\{i, j, k, l\}$ would then be fixed, giving us a contradiction. Therefore θ cannot contain more then 3 fixed points.

It is evident that there are no other cases and we are now in position to finish the proof. I.d implies (i). I.d, II.d and IV.d gives (ii). From I.d, II.e and III.e we obtain (iii). I.d, II.e, III.d, III.e implies (iv), and (v) follows from I.d, II.d, II.e,II.f. □

Corollary 1. *If G is a self-complementary 4-uniform hypergraph on n vertices, then $n = 0, 1, 2, 3 \pmod 8$.*

3. GENERATING SELF-COMPLEMENTARY 3- AND 4-UNIFORM HYPERGRAPHS

Sachs [3] and Ringel [2] described algorithm for generating every possible self-complementary graphs. We can now use theorems 1 and 2, to state the algorithms for generating 3 and 4-uniform self-complementary hypergraphs.

Algorithm 1 (for 3-uniform hypergraphs). *Let H_n be the complete 3-uniform hypergraph on n vertices. We choose permutation $\theta \in S_n$, with cycle structure as in theorem 1.*

1. *Take an arbitrary uncoloured edge $\{i, j, k\}$, and create sequence of triples: $\{i, j, k\}, \theta\{i, j, k\}, \theta^2\{i, j, k\}, \ldots$ Colour all the edges $\theta^{2i}\{i, j, k\} = \{\theta^{2i}(i), \theta^{2i}(j), \theta^{2i}(k)\}$ red and all the edges $\theta^{2i+1}\{i, j, k\}$ blue, for each integer i. Theorem 1 ensures that there are no edges coloured both red and blue.*
2. *Repeat step **1.** for any uncoloured edges, until all edges have been coloured.*
3. *From each sequence of triples, choose either the red edges or the blue edges (we may choose red edges from one orbit and blue from another). The chosen edges then form a self-complementary 3-uniform hypergraph.*

Algorithm 2 (for 4-uniform hypergraphs). *Let H_n be the complete 4-uniform hypergraph on n vertices. We choose permutation $\theta \in S_n$, with cycle structure as in theorem 2.*

1. *Take an arbitrary uncoloured edge $\{i, j, k, l\}$, and create sequence of quadruples: $\{i, j, k, l\}, \theta\{i, j, k, l\}, \theta^2\{i, j, k, l\}, \ldots$ Colour all the edges $\theta^{2i}\{i, j, k, l\}$ red and all the edges $\theta^{2i+1}\{i, j, k, l\}$ blue, for each integer i. Theorem 2 ensures that there are no edges coloured both red and blue.*
2. *Repeat step **1.** for any uncoloured edges, until all edges have been coloured.*

3. *From each sequence of quadruples, choose either the red edges or the blue edges (we may choose red edges from one orbit and blue from another). The chosen edges then form a self-complementary 4-uniform hypergraph.*

If there are s sequences of triples, then there are 2^s different ways of making the choices in step 3. of algorithms, though some of them may give us isomorphic 3(4)-uniform hypergraphs, and some self-complementary 3(4)-uniform hypergraph may also be associated with other permutation with different cycle lengths. Evidently there are no other self-complementary 3(4)-uniform hypergraphs with permutation θ, apart from the ones produced here.

4. HYPERGRAPHS

Since in the general complete hypergraph $(X, 2^X)$ there is exactly one edge on $n = |X|$ vertices, we call hypergraph G on n vertices self-complementary if it is isomorphic with $\overline{G}$, where $\overline{G}$ is complement of G without this n-vertices edge. We state the following conjecture:

Conjecture 1. *Self-complementary hypergraph on n vertices exist if and only if $n = 2^k$, $k \in \mathcal{N}$.*

This conjecture has been checked by computer for all n up to 1000.

REFERENCES

[1] Kocay W.: *Reconstructing Graphs as Subsumed Graphs of Hypergraphs, and Some Self-Complementary Triple Systems*. Graphs and Combinatorics 8 (1992) 259–276.

[2] Ringel G.: *Selbstkomplementäre Graphen*. Arch. Math. 14 (1963) 354–358.

[3] Sachs H.: *Über selbstkomplementäre graphen*. Publ. Math. Debrecen 9 (1962) 270–288.

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Received: December 23, 2004.