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## BIPARTITE EMBEDDING OF $(p, q)$ -TREES

**Abstract.** A bipartite graph  $G = (L, R; E)$  where  $V(G) = L \cup R$ ,  $|L| = p$ ,  $|R| = q$  is called a  $(p, q)$ -tree if  $|E(G)| = p + q - 1$  and  $G$  has no cycles. A bipartite graph  $G = (L, R; E)$  is a subgraph of a bipartite graph  $H = (L', R'; E')$  if  $L \subseteq L'$ ,  $R \subseteq R'$  and  $E \subseteq E'$ .

In this paper we present sufficient degree conditions for a bipartite graph to contain a  $(p, q)$ -tree.

**Keywords:** bipartite graph, tree, embedding graph.

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### 1. TERMINOLOGY

We shall use standard graph theory notation. We consider only finite, undirected graphs. All graphs will be assumed to have neither loops nor multiple edges.

Let  $G = (L, R; E)$  be a bipartite graph with a partition  $L, R$  and an edge set  $E$ . That means,  $L$  and  $R$  are two disjoint sets of independent vertices of the graph  $G$  such that  $L \cup R = V(G)$ . We call  $L = L(G)$  and  $R = R(G)$  the left and right set of bipartition. Note that the graphs  $G = (L, R; E)$  and  $G' = (R, L; E)$  are different.

For a vertex  $x \in V(G)$ ,  $N(x, G)$  denotes the set of its neighbors in  $G$ . The degree  $d(x, G)$  of the vertex  $x$  in  $G$  is the cardinality of the set  $N(x, G)$ .

$\Delta_L(G)$  ( $\delta_L(G)$ ) and  $\Delta_R(G)$  ( $\delta_R(G)$ ) are maximum (minimum) vertex degree in the set  $L(G)$  and  $R(G)$ , respectively. A vertex  $x$  of  $G$  is said to be pendant if  $d(x, G) = 1$ . For subsets  $A$  and  $B$  of  $V(G)$ , let  $N(A, B; G)$  denote the set of edges  $xy \in E(G)$  such that  $x \in A$  and  $y \in B$ .  $K_{p,q}$  is the complete  $(p, q)$ -bipartite graph.  $\bar{G}$  is the complement of  $G$  into  $K_{p,q}$ .

A bipartite graph  $G = (L, R; E)$  is a *subgraph* of a bipartite graph  $H = (L', R'; E')$  if  $L \subseteq L'$ ,  $R \subseteq R'$  and  $E \subseteq E'$ . If  $G$  is a subgraph of  $H$ , then we write  $G \leq H$ . Observe that the meaning of the word *subgraph* is different from the usual

one (see [3] and [1] page 1282). For instance, the graph  $K_{1,2} = (\{a\}, \{b, c\}; \{ab, ac\})$  is not a subgraph of  $K_{2,1} = (\{d, e\}, \{f\}; \{df, ef\})$ . We say that a bipartite graph  $G = (L, R; E)$  is *bipartite embeddable* or simply *embeddable* into bipartite graph  $H = (L', R'; E')$  if there is an injection  $f$  such that  $f : L \cup R \rightarrow L' \cup R'$ ,  $f(L) = L'$  and for every edge  $xy \in E$ ,  $f(x)f(y)$  is an edge of  $H$ . The function  $f$  is called the *bipartite embedding* (or *embedding*) of  $G$  into  $H$ . In other words, a bipartite graph  $G = (L, R; E)$  is said to be *embedded* into bipartite graph  $H = (L', R' : E')$  when there exists a pair  $(f_1, f_2)$  of injective mappings  $f_1 : L \rightarrow L'$  and  $f_2 : R \rightarrow R'$  such that if  $x \in L$  and  $y \in R$  are adjacent in  $G$ , then  $f_1(x)$  and  $f_2(y)$  are adjacent in  $H$  (see [3]). It follows easily that  $G$  is embeddable into  $H$  if and only if  $G$  is a subgraph of  $H$ . Note that  $K_{1,2}$  is not embeddable into  $K_{2,1}$ .

A  $(p, q)$ -bipartite graph  $G$  is called a  $(p, q)$ -*tree* if  $G$  is connected and  $|E(G)| = p + q - 1$ . Thus each  $(p, q)$ -tree is a tree and for each tree  $T$  there exist integers  $p$  and  $q$  such that  $T$  is a  $(p, q)$ -tree. If  $G$  is a  $(p, q)$ -bipartite and  $|E(G)| = p + q - k$  and  $G$  has no cycles then  $G$  is called a  $(p, q, k)$ -*forest*. So, a  $(p, q, 1)$ -forest is a  $(p, q)$ -tree. Let  $T$  be a  $(p, q)$ -tree and  $y \in V(T)$ . Let us denote by  $U_y$  the set of all  $z \in N(y, T)$  such that  $d(z, T) = 1$ . We shall call  $U_y$  the *bough with the center  $y$* . The vertex  $x \in V(T)$  is called *penultimate vertex* if  $U_x \neq \emptyset$  and  $d(x, T) = |U_x| + 1$  and there is the longest path  $P$  in  $T$  such that  $x \in V(P)$ .

## 2. RESULTS

First we shall give some results concerning the subgraphs of general graphs.

In 1963, Erdős and Sós (see [5]) stated the following conjecture, which was proved by Brandt in [2].

**Theorem 1.** *Let  $G$  be a graph with  $n$  vertices and more than*

$$f(k, n) = \max \left\{ \binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1) \right\}$$

*edges. Then  $G$  contains every forest with  $k$  edges and without isolated vertices as a subgraph.*

The following well-known result was attributed by Chvátal to graph-theoretical folklore [4]:

**Theorem 2.** *Suppose  $G$  is a graph with the minimum degree not less than  $k$ . Then  $G$  contains every tree with  $k$  edges.*

S. Brandt in [2] proved:

**Theorem 3.** *Suppose  $F$  is a forest with  $k$  edges and order  $n$  and  $G$  is a graph with at least  $n$  vertices. If  $\delta(G) \geq k$ , then  $F$  is a subgraph of  $G$ .*

We shall consider bipartite embedding problem, analogous to the classical embedding problem, the first general condition for a bipartite graph to be a subgraph of another bipartite graph was given by Rado in [6] (See also [3]).

In this paper we present sufficient degree conditions for a bipartite graph to contain every  $(p, q)$ -tree.

The following lemma, proved in Section 3, is an easy bipartite equivalent of Theorem 2.

**Lemma A.** *Let  $G = (L', R'; E')$  be a  $(p', q')$ -bipartite graph such that  $\delta_L(G) \geq q$  and  $\delta_R(G) \geq p$ . Then every  $(p, q)$ -tree  $T = (L, R; E)$  is a subgraph of  $G$ .*

Observe that if  $\Delta_L(T) = q$  (or  $\Delta_R(T) = p$ ), then Lemma A is best possible in the sense that it cannot be improved by decreasing the minimum degree of the graph  $G$ .

Hence, now we shall consider a  $(p, q)$  - tree  $T$  such that  $K_{1,q}$  is not a subgraph of  $T$ .

The main results are the following theorem and its obvious corollaries:

**Theorem B.** *Let  $T = (L, R; E)$  be a  $(p, q)$ -tree,  $\Delta_L(T) \leq q-1$  and let  $G = (L', R'; E')$  be a connected  $(p', q')$ -bipartite graph such that  $q' \geq q$ ,  $\delta_L(G) \geq q-1$  and  $\delta_R(G) \geq p$ . Then  $T$  is a subgraph of  $G$ .*

Note that if  $\Delta_L(T) = q-1$  (or  $\Delta_R(T) = p$ ), then Theorem B is best possible.

Let  $P_k$  be a path with  $k$  edges and let  $k$  be even,  $k \geq 4$ . By Theorem 3,  $P_k$  is a subgraph of a graph  $G$  if  $\delta(G) \geq k$ , but by Theorem B,  $P_k$  is a subgraph of a bipartite graph  $G'$ , if  $\delta(G') \geq k/2$ .

**Corollary C.** *Let  $T = (L, R; E)$  be a  $(p, q)$ -tree,  $\Delta_L(T) \leq q-1$  and let  $G = (L', R'; E')$  be a  $(p', q')$ -bipartite graph such that  $\delta_L(G) \geq q-1$ ,  $\delta_R(G) \geq p$  and every connected component  $G_1$  has at least  $p$  and  $q$  vertices in  $L(G_1)$  and  $R(G_1)$ , respectively. Then  $T$  is a subgraph of each component of  $G$ .*

**Corollary D.** *Let  $F = (L, R; E)$  be a  $(p, q, k)$ -forest,  $k \geq 2$  and let  $G = (L', R'; E')$  be a  $(p', q')$ -bipartite graph such that  $q' \geq q$ ,  $\delta_L(G) \geq q-1$ ,  $\delta_R(G) \geq p$ . Then  $F$  is a subgraph of  $G$ .*

### 3. PROOFS

To prove Lemma A and Theorem B, we shall need two lemmas.

**Lemma 3.1** *Let  $T = (L, R; E)$  be a  $(p, q)$ -tree, let  $U_y \neq \emptyset$  be a bough in  $T$  and let  $G$  be a  $(p', q')$ -bipartite graph,  $\delta_L(G) \geq q$  and  $\delta_R(G) \geq p$ . If  $T \setminus U_y \leq G$  then  $T \leq G$ .*

*Proof.* Let  $T = (L, R; E)$  be a  $(p, q)$ -tree,  $y \in V(T)$ ,  $U_y \neq \emptyset$  and let  $G = (L', R'; E')$  be a  $(p', q')$ -bipartite graph verifying the assumptions of the lemma. Without loss of generality we may assume that  $y \in L$ . Let us denote by  $T_1$  the tree  $T \setminus U_y$ . Let  $|U_y| = k$ . If  $k = q$  then  $T = K_{1,q}$  and  $T \leq G$ . We now assume that  $k \leq q - 1$ . Note that  $T_1 = (L_1, R_1; E_1)$  is a  $(p, q - k)$ -tree and  $1 \leq d(y, T_1) \leq q - k$ . By assumptions of the lemma, there exists an embedding  $f$  of  $T_1$  into  $G$ . Let  $f(y) = z$ . We will denote by  $N^*(z)$  the set  $\{w \in N(z, G) \text{ such that } w \in f[R_1]\}$ . Hence  $|N^*(z)| \leq q - k$ . Since  $\delta_L(G) \geq q$ , there are  $k$  vertices  $w'_i$  such that  $w'_i \in (N(z, G) \setminus N^*(z))$ . If  $W^* = \{w'_i, i = 1, \dots, k\}$  then the function  $f^*$  such that  $f^*(v) = f(v)$  for  $v \in V(T_1)$  and  $f^*[U_y] = W^*$  is an embedding of  $T$  into  $G$ .  $\square$

*The Proof of Lemma A.* The proof is by induction on  $p + q$ . If  $T$  is a  $(p, q)$ -tree such that  $p + q \leq 4$  and  $G$  is a  $(p', q')$ -bipartite graph verifying the assumptions of the lemma, then the lemma is easy to check.

So, let us suppose  $p + q \geq 5$  and the lemma is true for all integers  $p_1, q_1$  with  $p_1 + q_1 < p + q$ . Let  $T$  be a  $(p, q)$ -tree and let  $G$  be a  $(p', q')$ -bipartite graph such that  $\delta_L(G) \geq q$  and  $\delta_R(G) \geq p$ . There exists a vertex  $y$  in  $V(T)$  such that  $|U_y| = k > 0$ . Without loss of generality we may assume that  $y \in L$ . If  $k = q$ , then the lemma is obvious. If  $k \leq q - 1$ , then let us denote by  $T_1$  the tree  $T \setminus U_y$ . Since  $T_1$  is  $(p, q - k)$ -tree it follows, by the induction hypothesis, that  $T_1 \leq G$ . We obtain an embedding of  $T$  into  $G$  by Lemma 3.1.  $\square$

**Lemma 3.2** *Let  $T$  be a  $(p, q)$ -tree such that  $T \neq K_{1,q}$  and  $T \neq K_{p,1}$ . Then there exist at least two penultimate vertices in  $V(T)$ .*

The proof of Lemma 3.2 is trivial.

*The Proof of Lemma B.* Let  $T = (L, R; E)$  be a  $(p, q)$ -tree such that  $\Delta_L(T) \leq q - 1$  and let  $G = (L', R'; E')$  be a  $(p', q')$ -bipartite graph verifying assumptions of Theorem B. The proof will be divided into two steps.

**Case 1.** *Let us first assume that there exists a penultimate vertex, say  $y$ , in  $L$ .*

Let  $|U_y| = k > 0$ ,  $\{x\} = N(y, T) \setminus U_y$  and let us denote by  $T_1$  the tree  $T \setminus U_y = (L_1, R_1; E_1)$ .  $T_1$  is a  $(p, q - k)$ -tree. By Lemma A, there exists an embedding  $f$  of  $T_1$  into  $G$ . Let  $f(y) = w$ ,  $f(x) = z$ ,  $f[L_1] = L'_1$  and  $f[R_1] = R'_1$ . If  $d(w, G) \geq q$  or  $R'_1 \not\subset N(w, G)$ , then there are  $k$  vertices,  $v_1, \dots, v_k \in (N(w, G) \setminus R'_1)$ . The function  $f^*$  such that

$$\begin{aligned} f^*(v) &= f(v), \text{ for } v \in V(T_1) \\ f^*(x_i) &= v_i, \text{ for } x_i \in U_y, \quad i = 1, \dots, k \end{aligned}$$

is an embedding of  $T$  into  $G$ . So, we may assume that  $d(w, G) = q - 1$  and  $R'_1 \subset N(w, G)$ . Write  $R'_2 = N(w, G) \setminus R'_1$ .

**Subcase 1.1** *There exists a vertex  $w_1 \in N(z, G)$  such that  $d(w_1, G) \geq q$  or  $|N(w_1, G) \cap R'_1| < q - k$ .*

Then, the vertex  $w_1$  has  $k$  neighbors, say  $z'_1, \dots, z'_k$ , which are not elements of  $R'_1$ . Thus we conclude that the function  $f_1^*$  given by

$$\begin{aligned} f_1^*(v) &= f(v), \text{ for } v \in V(T_1) \setminus \{y\}, \\ f_1^*(y) &= w_1, \\ f_1^*(x_i) &= z'_i, \text{ for } x_i \in U_y, \quad i = 1, \dots, k, \end{aligned}$$

and, if  $w_1 \in f[L_1]$  and  $w_1 = f(v^*)$  then  $f_1^*(v^*) = w_1$ , is the embedding of  $T$  into  $G$ .

**Subcase 1.2** *Now we assume that for each vertex  $w' \in N(z, G)$  there is  $|N(w', G) \cap R'_1| = q - k$  and  $d(w', G) = q - 1$ .*

Observe that in this case  $G$  has a subgraph  $H$  such that  $H$  is a  $(p'_1, q - k)$ -complete bipartite graph,  $L(H) = N(z, G)$ ,  $R(H) = R'_1$  and  $p'_1 = d(z, G) \geq p$ .

**Subcase 1.2.1** *There is a vertex  $w'_1 \in N(z, G)$  such that  $N(w'_1, G) \neq N(w, G)$ .*

Thus there exist vertices  $z_1 \in R' \setminus N(w, G)$ ,  $z_2 \in R'_2$  such that  $z_1 w'_1 \in E'$  and  $z_2 w'_1 \notin E'$ . By Lemma 3.2, there is a penultimate vertex  $y' \neq y$  in  $V(T)$ . First we assume that  $y' \in L$ . We will denote by  $F_2$  the forest  $T \setminus U_y \setminus \{y, y', x'_1\}$ , where  $x'_1 \in U_{y'}$ . By Lemma A,  $F_2 \leq H_1 = H \setminus \{z_3, w, w'_1\}$ , where  $z_3 \in R(H)$ . If  $f_2$  is an embedding of  $F_2$  into  $H_1$  then the embedding  $f_2^*$  of  $T$  into  $G$  is defined as follows:

$$\begin{aligned} f_2^*(v) &= f_2(v), \text{ for } v \in V(F_2), \\ f_2^*[U_y] &= R'_2 \cup \{z_3\}, \\ f_2^*(y') &= w'_1, \\ f_2^*(x'_1) &= z_1, \\ f_2^*(y) &= w. \end{aligned}$$

Let now  $y' \in R$  and  $|U_{y'}| = k'$  and let  $x' \in (N(y', T) \setminus U_{y'})$ . Let us denote by  $T_3$  the tree  $T \setminus U_{y'} \setminus U_y \setminus \{y, x', y'\}$ , and by  $H_2$  a bipartite graph such that  $L(H_2) = L(H) \setminus \{w, w'_1\} \setminus L'_3$ , where  $L'_3 \subset N(z_1, G) \setminus \{w, w'_1\}$ ,  $|L'_3| = k'$ ,  $R(H_2) = R(H) \setminus \{z_3\}$ , and  $z_3 \in R(H)$ . By Lemma A, there is an embedding  $f_3$  of  $T_3$  into  $H_2$ . Let  $f_3^*$  be given as follows:

$$\begin{aligned} f_3^*(v) &= f_3(v), \quad v \in V(T_3), \\ f_3^*(y) &= w, \\ f_3^*(x') &= w'_1, \\ f_3^*[U_y] &= R'_2 \cup \{z_3\}, \\ f_3^*(y') &= z_1, \end{aligned}$$

$$f_3^*[U_{y'}] = L'_3.$$

Therefore,  $T \leq G$ .

**Subcase 1.2.2** Each vertex  $w' \in N(z, G)$  has the degree  $q - 1$  and  $N(w, G) = N(w', G)$ . It follows that  $G$  has a subgraph  $H_3 = K_{p'_1, q-1}$ , where  $L(H_3) = N(z, G)$ ,  $R(H_3) = N(w, G)$ . Observe that  $R \setminus R(H_3) \neq \emptyset$  and  $N(L(H_3), R \setminus R(H_3); G) = \emptyset$ . Let  $z_4$  be a vertex in  $R \setminus R(H_3)$ . By assumption of the theorem, there are vertices  $z_5 \in R(H_3)$  and  $w_2 \in L(G) \setminus L(H_3)$  such that  $z_4 w_2 \in E(G)$  and  $w_2 z_5 \in E(G)$ . It is easily seen that  $T_4 = (T \setminus U_y \setminus \{x, y\}) \leq (H_3 \setminus \{z_5\} \setminus R'_4)$ , where  $R'_4 \subset (N(w_2, G) \setminus \{z_5\})$  and  $|R'_4| = k$  and  $z_4 \in R'_4$ . Obviously,  $T \leq G$ , again.

**Case 2** Let us assume there is no penultimate vertex in  $L$ .

Thus, by Lemma 3.2, there exist at least two penultimate vertices in  $R$ . Let  $y_1$  be a penultimate vertex in  $R$  and let  $\{x''\} = N(y_1, T) \setminus U_{y_1}$ .

Consider a tree  $T_5$  obtained from the tree  $T$  by deleting pendant vertices  $x_1, x_2, \dots, x_k$ , so that the vertex  $x''$  may be penultimate vertex in  $L$ .

By Case 1,  $T_5 \leq G$  and by assumption  $\delta_R(G) \geq p$  we deduce that  $T \leq G$  and the theorem is proved.  $\square$

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