Semyon B. Yakubovich

A DISTRIBUTION ASSOCIATED WITH THE KONTOROVICH–LEBEDEV TRANSFORM

Abstract. We show that in a sense of distributions

$$\lim_{\varepsilon\to 0+} \frac{1}{\pi^2}\tau \sinh \pi\tau \int_{\varepsilon}^{\infty} K_{i\tau}(y)K_{ix}(y)\frac{dy}{y} = \delta(\tau - x),$$

where δ is the Dirac distribution, τ, $x \in \mathbb{R}$ and $K_\nu(x)$ is the modified Bessel function. The convergence is in $\mathcal{E}'(\mathbb{R})$ for any even $\varphi(x) \in \mathcal{E}(\mathbb{R})$ being a restriction to $\mathbb{R}$ of a function $\varphi(z)$ analytic in a horizontal open strip $G_a = \{z \in \mathbb{C}\colon\ |\mathrm{Im}\ z| < a,\ a > 0\}$ and continuous in the strip closure. Moreover, it satisfies the condition $\varphi(z) = O\big(|z|^{-\mathrm{Im}\ z - \alpha}\, e^{-\pi|\mathrm{Re}\ z|/2}\big)$, $|\mathrm{Re}\ z| \to \infty$, $\alpha > 1$ uniformly in $\overline{G_a}$. The result is applied to prove the representation theorem for the inverse Kontorovich-Lebedev transformation on distributions.

Keywords: Kontorovich–Lebedev transform, distributions, modified Bessel functions.

Mathematics Subject Classification: 46F12, 44A15, 33C10.

1. INTRODUCTION

In this paper we study a natural extension, to spaces of distributions, of the inverse Kontorovich–Lebedev transform [4, 7]

$$F(y) = \int_{-\infty}^{\infty} K_{i\tau}(y) f(\tau) d\tau,\ \ y > 0, \tag{1.1}$$

modifying our previous version of this transformation given in [5], which is based on the following expansion

$$f(x) = \frac{1}{\pi^2} \lim_{\epsilon\to 0+} \int_0^{\infty} K_{ix}(y) \frac{1}{y^{1-\epsilon}} \int_{-\infty}^{+\infty} \tau \sinh(\pi\tau) K_{i\tau}(y) f(\tau) d\tau\, dy, \tag{1.2}$$

where the limit is understood in the weak topology of distributions with compact supports, $\mathcal{E}'(\mathbb{R})$. This modification has been initiated by the pioneer paper [8], where a similar expansion was established for the direct Kontorovich–Lebedev transformation on distributions. Furthermore, these results can be used to evaluate concrete distributions useful in various applications of the Kontorovich–Lebedev transform (cf. in [2]).

As it is known $[1,4,6,7]$, the kernel $K_{i\tau}(y)$ belongs to a class of the modified Bessel functions $K_\nu(z)$, $I_\nu(z)$ which are linear independent solutions of the Bessel differential equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2+\nu^2)u=0. \tag{1.3}$$

They can be given by formulas

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{\Gamma(\nu+k+1)k!}, \tag{1.4}$$

where $\Gamma(w)$ is Euler's Gamma-function [1],

$$K_\nu(z)=\frac{\pi}{2\sin\pi\nu}\left[I_{-\nu}(z)-I_\nu(z)\right], \tag{1.5}$$

when $\nu\neq 0,\pm1,\pm2,\ldots$, and $K_n(z)=\lim_{\nu\to n}K_\nu(z),\ n=0,\pm1,\pm2,\ldots$. The function $K_\nu(z)$ is also called the Macdonald function. It is even with respect to ν and has the following integral representations (cf., in $[1,3]$)

$$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt=\frac{1}{2}\int_0^\infty e^{-z(t+t^{-1})/2}t^{\nu-1}dt. \tag{1.6}$$

Useful relations are [1]

$$z\frac{\partial}{\partial z}K_\nu(z)=\nu K_\nu(z)-zK_{\nu+1}(z), \tag{1.7}$$

$$\int_0^\infty I_\xi(x)K_\nu(x)\frac{dx}{x}=\frac{1}{\xi^2-\nu^2},\ \ \mathrm{Re}\,\xi>|\,\mathrm{Re}\,\nu|, \tag{1.8}$$

$$I_\nu(z)=\frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{z}{2}\right)^\nu\int_{-1}^{1}e^{-zx}(1-x^2)^{\nu-\frac{1}{2}}dx,\ \ \mathrm{Re}\ \ \nu>-\frac{1}{2}. \tag{1.9}$$

These functions reveal the following asymptotic behaviour $[1,7]$

$$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(1/z)],\ \ |\arg z|<\frac{3\pi}{2},\qquad z\to\infty, \tag{1.10}$$

$$I_\nu(z)=\frac{e^z}{\sqrt{2\pi z}}[1+O(1/z)],\ \ -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\qquad z\to\infty, \tag{1.11}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\operatorname{Re}\nu|}\right),\ z \to 0, \tag{1.12}$$

$$K_0(z) = O(\log|z|),\ z \to 0, \tag{1.13}$$

$$I_\nu(z) = O(z^{\operatorname{Re}\nu}),\ \nu \neq 0,\ z \to 0. \tag{1.14}$$

We also mention the value of their Wronskian [1]

$$W(K_\nu(z), I_\nu(z)) = K_\nu(z)I'_\nu(z) - I_\nu(z)K'_\nu(z) = -\frac{1}{z}, \quad z \neq 0, \quad \nu \in \mathbb{C} \tag{1.15}$$

where the symbol "′" denotes the derivative with respect to z. When the index of the Macdonald function is pure imaginary, i.e., $\nu = i\tau$, $\tau \in \mathbb{R}$ then $K_{i\tau}(y)$, $y > 0$ is real-valued.

The main object of this work is to study a distributional version of the Kontorovich–Lebedev transformation (1.1) and to prove a representation theorem involving the following kernel function

$$\mathcal{K}_\varepsilon(\tau, x) = \frac{1}{\pi^2}\tau \sinh \pi\tau \int_\varepsilon^\infty K_{i\tau}(y)K_{ix}(y)\frac{dy}{y},\ \varepsilon > 0. \tag{1.16}$$

We will prove that $\mathcal{K}_\varepsilon(\tau, x)$ converges to a shifted Dirac distribution $\delta(\tau - x)$ when $\varepsilon \to 0+$ in the sense of the convergence in $\mathcal{E}'(\mathbb{R})$. This property can be interpreted as a certain orthogonality of the modified Bessel functions with pure imaginary subscripts.

We note that $\mathcal{E}'(\mathbb{R})$ is a dual space of $\mathcal{E}(\mathbb{R})$, which, in turn, is a metrizable locally convex countably multinorm space of infinitely differentiable functions $\varphi(x)$ with the topology generated by the collection of seminorms

$$\gamma_{p,K}(\varphi) \equiv \sup_{x\in K} |D_x^p \varphi(x)| < \infty, \tag{1.17}$$

where p is a non-negative integer number, K is a compact set in $\mathbb{R}$, and $D_x = \frac{d}{dx}$.

Throughout this paper we will denote by C a positive constant, not necessarily the same in each occurrence.

2. ORTHOGONALITY OF THE MACDONALD FUNCTIONS

The main result of this section is the following

Theorem 1. *The equality*

$$\lim_{\varepsilon\to 0+} \mathcal{K}_\varepsilon(\tau, x) = \delta(\tau - x),\ \tau, x \in \mathbb{R} \tag{2.1}$$

holds in $\mathcal{E}'(\mathbb{R})$. More precisely, for any even $\varphi(x) \in \mathcal{E}(\mathbb{R})$ being a restriction to $\mathbb{R}$ of a function $\varphi(z)$ analytic in a horizontal open strip $G_a = \{z \in \mathbb{C}\colon\ |\mathrm{Im}\ z| < a,\ a > 0\}$, continuous in the strip's closure and satisfying the condition

$$\varphi(z) = O\left(|z|^{-\mathrm{Im}\ z-\alpha} e^{-\pi|\mathrm{Re}\ z|/2}\right),\ |\mathrm{Re}\ z| \to \infty,\ \alpha > 1 \tag{2.2}$$

uniformly in $\overline{G_a}$, there is

$$\lim_{\varepsilon \to 0+} \langle \mathcal{K}_\varepsilon(\cdot, x), \varphi\rangle = \varphi(x),\ x \in \mathbb{R}, \tag{2.3}$$

where the convergence is in $\mathcal{E}(\mathbb{R})$.

Proof. By using relation (1.12.3.3) in [3] we calculate the integral with respect to y in (1.16). Then invoking (1.5) with the definition of Wronskian, the kernel $\mathcal{K}_\varepsilon(\tau, x)$ can be represented as follows ($|\tau| \neq |x|$):

$$\mathcal{K}_\varepsilon(\tau, x) = \frac{\varepsilon\tau \sinh \pi\tau}{\pi^2(\tau^2 - x^2)} \left[K_{ix}(\varepsilon)K'_{i\tau}(\varepsilon) - K_{i\tau}(\varepsilon)K'_{ix}(\varepsilon)\right] =$$

$$= \frac{\varepsilon\tau \sinh \pi\tau}{\pi^2(\tau^2 - x^2)} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right) =$$

$$= \frac{\varepsilon i\tau}{2\pi(\tau^2 - x^2)} \left[W\left(K_{ix}(\varepsilon), I_{-i\tau}(\varepsilon)\right) - W\left(K_{ix}(\varepsilon), I_{i\tau}(\varepsilon)\right)\right]. \tag{2.4}$$

Diagonal values $|\tau| = |x|$ of kernel (1.16) can easily be found by its continuity on $\mathbb{R}^2$ as a function of two variables. In fact, for each $\varepsilon > 0$ the integral with respect to y is absolutely and uniformly convergent with respect to (τ, x) on any compact subset of $\mathbb{R}^2$ by virtue of the inequality $|K_\nu(y)| \leq K_{\mathrm{Re}\,\nu}(y)$ (see (1.6)) and asymptotic relation (1.10). Our goal is to prove that, under conditions of the theorem, for any nonnegative integer r, there holds

$$\max_{0 \leq p \leq r} \sup_{x \in [-x_0, x_0]} |D_x^p(\varphi - \varphi_\varepsilon)| \to 0,\ \varepsilon \to 0+, \tag{2.5}$$

where $x_0 > 0$ and

$$\varphi_\varepsilon(x) = \langle \mathcal{K}_\varepsilon(\cdot, x), \varphi\rangle,\ \varepsilon > 0. \tag{2.6}$$

We show that (2.6) is a regular distribution, i.e., $\mathcal{K}_\varepsilon(\cdot, x)$ is locally integrable with respect to τ and $\varphi_\varepsilon(x)$ can be represented as an integral. Indeed, taking into account the evenness of φ we can write it in the form

$$\varphi_\varepsilon(x) = \frac{\varepsilon}{\pi^2} \int_{-\infty}^{\infty} \frac{\tau \sinh \pi\tau}{\tau^2 - x^2} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right) \varphi(\tau)\, d\tau =$$

$$= \frac{\varepsilon}{2\pi^2} \int_{-\infty}^{\infty} \left[\frac{1}{\tau - x} + \frac{1}{\tau + x}\right] \sinh \pi\tau W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right) \varphi(\tau) d\tau =$$

$$= \frac{\varepsilon}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh \pi\tau}{\tau - x} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right) \varphi(\tau) d\tau =$$

$$= \frac{\varepsilon}{2\pi i} P.V. \int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{i\tau}(\varepsilon)\right) \frac{\varphi(\tau)}{\tau - x} d\tau =$$

$$= -\frac{\varepsilon}{2\pi i} P.V. \int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{-i\tau}(\varepsilon)\right) \frac{\varphi(\tau)}{\tau - x} d\tau = \varphi_{1\varepsilon}(x) - \varphi_{2\varepsilon}(x), \tag{2.7}$$

where both integrals $\varphi_{j\varepsilon}(x)$, $j = 1, 2$, are understood as the principal Cauchy values. We also verify their absolute convergence. We take, for instance, integral $\varphi_{1\varepsilon}(x)$. There is

$$\varphi_{1\varepsilon}(x) = \frac{\varepsilon}{2\pi i} \lim_{\delta \to 0+} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} d\tau.$$

Hence it is sufficient that

$$\int_{|\tau| \geq M} \left| W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} \right| d\tau < \infty, \ \varepsilon > 0, \ x \in [-x_0, x_0],$$

where $M > x_0 \geq |x|$ is large enough. In fact, from (1.9) we immediately obtain the following estimates for the modified Bessel function and its derivative with respect to the argument

$$|I_\nu(y)| \leq \frac{\Gamma(\operatorname{Re}\nu + 1/2)}{\Gamma(\operatorname{Re}\nu + 1)|\Gamma(\nu + 1/2)|} e^y \left(\frac{y}{2}\right)^{\operatorname{Re}\nu}, \quad y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}, \tag{2.8}$$

$$|I'_\nu(y)| \leq \frac{\Gamma(\operatorname{Re}\nu + 1/2)}{\Gamma(\operatorname{Re}\nu + 1)|\Gamma(\nu + 1/2)|} \left(\frac{|\nu|}{y} + 1\right) e^y \left(\frac{y}{2}\right)^{\operatorname{Re}\nu}. \tag{2.9}$$

Consequently, taking into account asymptotic condition (2.2), we derive

$$\int_{|\tau| \geq M} \left| W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} \right| d\tau \leq C e^{\varepsilon} \left[\left(\frac{2}{\varepsilon} + \frac{1}{M} \right) |K_{ix}(\varepsilon)| + \frac{|K'_{ix}(\varepsilon)|}{M} \right] \times$$

$$\times \int_{|\tau| \geq M} \frac{d\tau}{(|\tau| - x_0)^\alpha} < \infty, \ \alpha > 1.$$

We treat analogously $\varphi_{2\varepsilon}(x)$. Thus (2.6) is a regular distribution and we have the representation (2.7). It is easily seen by an elementary substitution in the integral that $\varphi_{1\varepsilon}(x) = -\varphi_{2\varepsilon}(-x)$. Hence, $\varphi_\varepsilon(x) = -\varphi_{2\varepsilon}(x) - \varphi_{2\varepsilon}(-x)$ and we prove that

$$\max_{0 \leq p \leq r} \sup_{x \in [-x_0, x_0]} \left| D_x^p \left(\frac{\varphi}{2} + \varphi_{2\varepsilon} \right) \right| \to 0, \ \varepsilon \to 0+. \tag{2.10}$$

Taking then into account the evenness of φ, we will conclude (2.5) and thus achieve our goal.

In order to establish (2.10), we will use analytic properties of $\varphi(z)$ in the strip G_a. Precisely, via Cauchy's theorem we take a big positive R and a small $\delta > 0$ to write the equality

$$\frac{\varepsilon}{2\pi i}\left(\int_{-R}^{-\delta} + \int_{\delta}^{R} + \int_{R}^{R+ia} + \int_{R+ia}^{-R+ia} + \int_{-R+ia}^{-R}\right) W\left(K_{ix}(\varepsilon), I_{-i(z+x)}(\varepsilon)\right)\frac{\varphi(z+x)}{z}dz +$$

$$+\frac{\varepsilon}{2\pi}\int_{\pi}^{0} W\left(K_{ix}(\varepsilon), I_{-i(\delta e^{i\theta}+x)}(\varepsilon)\right)\varphi\left(\delta e^{i\theta}+x\right)d\theta = 0. \tag{2.11}$$

Hence, letting $R \to \infty$ we observe that integrals over $(R, R+ia)$ and $(-R+ia, -R)$ tend to zero due to asymptotic behavior of the function φ in the strip G_a. Then we let $\delta \to 0$ to obtain (see (2.7))

$$\varphi_{2\varepsilon}(x) = -\frac{\varepsilon}{2\pi}\lim_{\delta\to 0+}\int_{\pi}^{0} W\left(K_{ix}(\varepsilon), I_{-i(\delta e^{i\theta}+x)}(\varepsilon)\right)\varphi\left(\delta e^{i\theta}+x\right)d\theta +$$

$$+\frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)\frac{\varphi(\tau+ia)}{\tau-x+ia}d\tau, \quad a>0. \tag{2.12}$$

We can also pass to the limit when $\delta \to 0+$ under the integral sign in (2.12) via the dominated convergence theorem. Hence, using the evenness of the function $K_{ix}(\varepsilon)$ with respect to x and the value of the Wronskian (1.15), we derive the equality

$$\varphi_{2\varepsilon}(x) + \frac{\varphi(x)}{2} = \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)\frac{\varphi(\tau+ia)}{\tau-x+ia}d\tau. \tag{2.13}$$

Hence differentiating (2.13) with respect to x we put derivatives inside the integral via the uniform convergence on the compact $[-x_0, x_0]$ to find

$$D_x^p\left(\frac{\varphi}{2}+\varphi_{2\varepsilon}\right) = \frac{\varepsilon}{2\pi i}\sum_{l=0}^{p}\frac{p!}{(p-l)!}\int_{-\infty}^{\infty} W\left(D_x^{p-l}K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)\times$$

$$\times\frac{\varphi(\tau+ia)}{(\tau-x+ia)^{l+1}}d\tau. \tag{2.14}$$

Now using representations (1.6) and assuming $0 < \varepsilon < 1$, we obtain the uniform estimates

$$\left|D_x^{p-l}K_{ix}(\varepsilon)\right| \le \int_0^{\infty} e^{-\frac{\varepsilon}{2}e^t}t^{p-l}dt = \int_{\varepsilon/2}^{\infty} e^{-u}\log^{p-l}\left(\frac{2u}{\varepsilon}\right)\frac{du}{u} \le$$

$$\leq \int_{\varepsilon/2}^{1/2} \log^{p-l}\left(\frac{2u}{\varepsilon}\right)\frac{du}{u} + \int_{1/2}^{\infty} e^{-u}\left(\log 2u + \log\varepsilon^{-1}\right)^{p-l}\frac{du}{u} = O\left(\log^{p-l+1}\varepsilon^{-1}\right).$$

Analogously, for $p > l$ we obtain

$$\varepsilon\left|D_x^{p-l}K'_{ix}(\varepsilon)\right| \leq \varepsilon\int_0^\infty e^{-\frac{\varepsilon}{2}e^t}e^t t^{p-l}dt = 2(p-l)\int_0^\infty e^{-\frac{\varepsilon}{2}e^t}t^{p-l-1}dt =$$

$$= O\left(\log^{p-l}\varepsilon^{-1}\right).$$

When $p = l$, then obviously (see (1.6), (1.12))

$$\varepsilon\left|D_x^{p-l}K'_{ix}(\varepsilon)\right| = \varepsilon\left|K'_{ix}(\varepsilon)\right| < \varepsilon\int_0^\infty e^{-\varepsilon\sinh t}\cosh t\,dt = 1.$$

Combining this with estimates (2.2), (2.8), (2.9), we return to (2.14) to derive

$$\max_{0\leq p\leq r}\sup_{x\in[-x_0,x_0]}\left|D_x^p\left(\frac{\varphi}{2}+\varphi_{2\varepsilon}\right)\right| \leq \max_{0\leq p\leq r}\sup_{x\in[-x_0,x_0]}\frac{\varepsilon}{2\pi}\sum_{l=0}^{p}\frac{p!}{(p-l)!}\times$$

$$\times\int_{-\infty}^{\infty}\left|D_x^{p-l}K_{ix}(\varepsilon)I'_{a-i\tau}(\varepsilon)\right|\frac{|\varphi(\tau+ia)|}{|\tau-x+ia|^{l+1}}d\tau + \max_{0\leq p\leq r}\sup_{x\in[-x_0,x_0]}\frac{\varepsilon}{2\pi}\sum_{l=0}^{p}\frac{p!}{(p-l)!}\times$$

$$\times\int_{-\infty}^{\infty}\left|D_x^{p-l}K'_{ix}(\varepsilon)I_{a-i\tau}(\varepsilon)\right|\frac{|\varphi(\tau+ia)|}{|\tau-x+ia|^{l+1}}d\tau \leq C\left(\frac{\varepsilon}{2}\right)^a e^{\varepsilon}\log\varepsilon^{-1}$$

$$\max_{0\leq p\leq r}\sum_{l=0}^{p}\frac{p!}{(p-l)!}a^{-l}\log^{p-l}\varepsilon^{-1}\int_{-\infty}^{\infty}\frac{(|a-i\tau|+\varepsilon)\,e^{-\pi|\tau|/2}}{|\Gamma(a-i\tau+1/2)|\,|\tau+ia|^{\alpha+a}\,|\tau-x+ia|}d\tau +$$

$$+C\left(\frac{\varepsilon}{2}\right)^a e^{\varepsilon}\max_{0\leq p\leq r}\sum_{l=0}^{p}\frac{p!}{(p-l)!}a^{-l}\log^{p-l}\varepsilon^{-1}\int_{-\infty}^{\infty}\frac{e^{-\pi|\tau|/2}}{|\Gamma(a-i\tau+1/2)|\,|\tau+ia|^{\alpha+a}}d\tau <$$

$$< C\varepsilon^a\left(\log\varepsilon^{-1}+\frac{1}{a}\right)^r\left(\log\varepsilon^{-1}+1\right)\to 0,\ \varepsilon\to 0+.$$

The latter integrals are indeed bounded, since, due to Stirling's formula for Gamma-functions [1]

$$\frac{e^{-\pi|\tau|/2}}{|\Gamma(a-i\tau+1/2)|} = O(|\tau|^{-a}),\quad |\tau|\to\infty,$$

and therefore,

$$\int_{-\infty}^{\infty} \frac{e^{-\pi|\tau|/2}}{|\Gamma(a-i\tau+1/2)|\,|\tau+ia|^{\alpha+a}} d\tau = O(1) + O\left(\int_{|\tau|\geq M} \frac{d\tau}{|\tau|^{2a+\alpha}}\right) = O(1).$$

Also,

$$\int_{-\infty}^{\infty} \frac{(|a-i\tau|+\varepsilon)\, e^{-\pi|\tau|/2}}{|\Gamma(a-i\tau+1/2)|\,|\tau+ia|^{\alpha+a}\,|\tau-x+ia|} d\tau = O(1)+$$

$$+O\left(\int_{|\tau|\geq M>x_0} \frac{d\tau}{|\tau|^{2a+\alpha-1}(|\tau|-x_0)}\right) = O(1).$$

Thus we establish (2.10), which implies (2.5). Theorem 1 is proved.

3. REPRESENTATION THEOREM

We define a complex analogue of the Kontorovich–Lebedev transform (1.1) on distributions $f \in \mathcal{E}'(\mathbb{R})$ by

$$F(z) = \langle f, K_{i\cdot}(z)\rangle\,,\ \ z \in \mathbb{C}. \tag{3.1}$$

From representations (1.6) it follows that $K_{i\tau}(z)$ is infinitely differentiable with respect to $\tau \in \mathbb{R}$ and analytic with respect to z in the right half-plane $\mathrm{Re}\ z > 0$. Thus $\mathcal{E}(\mathbb{R})$ contains $K_{i\tau}(z)$ for various values of the complex parameter z. We will prove that $F(z)$ is an analytic function in the right-half plane and satisfies an appropriate estimate there. Precisely, the following theorem is true.

Theorem 2. *For each $f \in \mathcal{E}'(\mathbb{R})$ $F(z)$ is analytic in the right half-plane* $\mathrm{Re}\, z > 0$ *and its derivatives are of the form*

$$D_z^p F := \frac{(-1)^p}{2^p} \sum_{l=0}^{p} \binom{p}{l} \langle f, K_{i\cdot -p+2l}(z)\rangle\,,\ \ p \in \mathbb{N}_0. \tag{3.2}$$

Furthermore, the following estimates are true

$$|F(z)| = O\left(\log^{r+1}\left(\frac{1}{\mathrm{Re}\, z}\right)\right),\ \ \mathrm{Re}\ z \to 0+,\ \ r \in \mathbb{N}_0, \tag{3.3}$$

$$|F(z)| = O\left(\frac{e^{-\,\mathrm{Re}\ z}}{\sqrt{\mathrm{Re}\ z}}\right),\ \ \mathrm{Re}\ z \to +\infty. \tag{3.4}$$

Proof. Let z be an arbitrary fixed point in the right half-plane with $\mathrm{Re}\ z \geq y_0 > 0$. Taking a complex increment $\Delta z \neq 0$ such that z, $z + \Delta z$ belong to the right half-plane, we show that $F(z)$ admits a derivative in each inner half-plane. In view of our freedom to choose y_0 arbitrarily close to zero, we will establish the analyticity of $F(z)$ in the right half-plane.

Indeed, using definition (3.1) of $F(z)$ we write

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - \langle f, D_z K_{i\cdot}(z)\rangle = \langle f, \Psi_{\Delta z}(\cdot)\rangle , \tag{3.5}$$

where

$$\Psi_{\Delta z}(\tau) = \frac{1}{\Delta z}\left[K_{i\tau}(z+\Delta z) - K_{i\tau}(z)\right] - D_z K_{i\tau}(z).$$

Thus our aim is to verify that for any integer $r \in \mathbb{N}_0$ and for any compact $T \subset \mathbb{R}$

$$\max_{0\le p\le r}\sup_{\tau\in T} |D_\tau^p \Psi_{\Delta z}(\tau)| \to 0,\ \ |\Delta z| \to 0. \tag{3.6}$$

To do this, we again employ representations (1.6). Hence we put derivatives inside of the integral via its uniform convergence and after simple manipulations we arrive at the estimate

$$|D_\tau^p \Psi_{\Delta z}(\tau)| \le \int_0^\infty t^p e^{-y_0\cosh t}\frac{\left|e^{-\Delta z\cosh t} - 1 + \Delta z\cosh t\right|}{|\Delta z|}dt =$$

$$= \int_0^\infty t^p e^{-y_0\cosh t}\left|\sum_{n=2}^\infty \frac{(\Delta z)^{n-1}\cosh^n t}{n!}\right| dt \le \int_0^\infty t^p e^{-y_0\cosh t}\sum_{n=2}^\infty \frac{|\Delta z|^{n-1}\cosh^n t}{n!}dt.$$

The latter series can be taken out of the integral by virtue of the Levi theorem and we find

$$|D_\tau^p \Psi_{\Delta z}(\tau)| \le \sum_{n=2}^\infty \frac{|\Delta z|^{n-1}}{n!}\left(\int_0^1 + \int_1^\infty\right) t^p e^{-y_0 e^t/2} e^{nt} dt \le$$

$$\le \sum_{n=2}^\infty \frac{|\Delta z|^{n-1} e^n}{n!} + \sum_{n=2}^\infty \frac{|\Delta z|^{n-1}(n+1)}{(y_0/2)^{n+1}}\int_1^\infty e^{-t} t^p dt < C|\Delta z| \to 0,\ \ |\Delta z| \to 0.$$

Thus we establish (3.6). Hence by using an inductive argument, we get the existence of the p-th derivative with respect to z. Finally we use the relation (cf. [1, 3])

$$D_z^p K_\mu(z) = \frac{(-1)^p}{2^p}\sum_{l=0}^p \binom{p}{l} K_{\mu-p+2l}(z),$$

and we come out with (3.2).

In order to prove (3.3) we refer to the fact that $F(z)$ is a continuous linear functional on countably multinormed space $\mathcal{E}(\mathbb{R})$. Hence, there exists a positive constant C and a nonnegative integer r, which depend on f, such that for $0 < \operatorname{Re} z < 1$ we derive

$$|F(z)| \le C \max_{0\le p\le r} \sup_{\tau\in T} |D^p_\tau K_{i\tau}(z)| \le C \max_{0\le p\le r} \int_0^\infty e^{-\operatorname{Re} z \cosh t} t^p dt \le$$

$$\le C \max_{0\le p\le r} \left[\int_{\operatorname{Re} z/2}^{1/2} \log^p\left(\frac{2u}{\operatorname{Re} z}\right)\frac{du}{u} + \int_{1/2}^\infty e^{-u} \log^p\left(\frac{2u}{\operatorname{Re} z}\right)\frac{du}{u}\right] =$$

$$= O\left(\log^{r+1}\left(\frac{1}{\operatorname{Re} z}\right)\right), \ \operatorname{Re} z \to 0+ .$$

Analogously, since

$$\int_0^\infty e^{-\operatorname{Re} z\cosh t} t^p dt = e^{-\operatorname{Re} z} \int_0^\infty e^{-2\operatorname{Re} z \sinh^2(t/2)} t^p dt \le$$

$$\le e^{-\operatorname{Re} z} \int_0^\infty e^{-\operatorname{Re} z t^2/2} t^p dt = 2^{(p-1)/2}\Gamma\left(\frac{p+1}{2}\right) e^{-\operatorname{Re} z} (\operatorname{Re} z)^{-(p+1)/2},$$

we easily get (3.4). Theorem 2 is proved. □

Now we are ready to prove the representation theorem for the Kontorovich–Lebedev transform (3.1) of real positive variable.

Theorem 3. *Let $f \in \mathcal{E}'(\mathbb{R})$ and $\varphi \in \mathcal{E}(\mathbb{R})$ satisfy conditions of Theorem 1 with $\alpha > 2$. Then*

$$\lim_{\varepsilon\to 0+} \left\langle \frac{1}{\pi^2}\tau \sinh \pi\tau \int_\varepsilon^\infty F(y) K_{i\tau}(y)\frac{dy}{y}, \varphi(\tau)\right\rangle = \langle f, \varphi\rangle . \tag{3.7}$$

Proof. Taking into account asymptotic properties (1.10), (1.13) of the Macdonald function, estimates (3.3), (3.4) and elementary inequality $K_{i\tau}(y) \le K_0(y)$ (see (1.6)), we conclude that the latter integral in (3.7) is absolutely and uniformly convergent with respect to $\tau \in \mathbb{R}$ for each $\varepsilon > 0$. Moreover, it can be treated as an improper Riemann integral. Furthermore, we show that (3.7) is a regular distribution if φ satisfies (2.2) with $\alpha > 2$. In fact, by using (1.5) and the evenness of φ, we write

$$\frac{1}{\pi^2}\left\langle \tau \sinh \pi\tau \int_\varepsilon^\infty F(y) K_{i\tau}(y)\frac{dy}{y}, \varphi(\tau)\right\rangle = \frac{1}{2\pi} \int_{-\infty}^\infty \tau\varphi(\tau) \int_\varepsilon^\infty I_{i\tau}(y) F(y)\frac{dy}{y} d\tau. \tag{3.8}$$

Hence, using estimates (2.8), (3.3), (3.4) we easily verify the absolute convergence of the iterated integral in the right-hand side of (3.8). Precisely, we obtain

$$\int_{-\infty}^\infty |\tau\varphi(\tau)| \int_\varepsilon^\infty |I_{i\tau}(y) F(y)| \frac{dy}{y} d\tau < C \int_{-\infty}^\infty \frac{|\tau\varphi(\tau)|}{|\Gamma(i\tau+1/2)|} d\tau \int_\varepsilon^\infty e^y |F(y)| \frac{dy}{y} <$$

$$< C\int_{\varepsilon}^{\infty}\frac{dy}{y^{3/2}}\left[\int_{|\tau|<M}\frac{|\tau\varphi(\tau)|}{|\Gamma(i\tau+1/2)|}d\tau+\int_{|\tau|>M}\frac{d\tau}{|\tau|^{\alpha-1}}\right]<\infty,\ \alpha>2.$$

Thus the left-hand side of (3.8) is a regular distribution and the corresponding integral in its right-hand side can be approximated by Riemann's sums. Therefore, from (3.1) there follows:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau\varphi(\tau)\int_{\varepsilon}^{\infty}I_{i\tau}(y)F(y)\frac{dy}{y}d\tau=\lim_{N\to\infty}\frac{1}{2\pi}\sum_{m=0}^{N}x_m\varphi(x_m)\int_{\varepsilon}^{\infty}I_{ix_m}(y)F(y)\frac{dy}{y}=$$

$$=\lim_{N\to\infty}\left\langle f_x,\frac{1}{2\pi}\sum_{m=0}^{N}\tau_m\varphi(\tau_m)\int_{\varepsilon}^{\infty}I_{i\tau_m}(y)K_{ix}(y)\frac{dy}{y}\right\rangle. \tag{3.9}$$

But when $N\to\infty$,

$$\frac{1}{2\pi}\sum_{m=0}^{N}\tau_m\varphi(\tau_m)\int_{\varepsilon}^{\infty}I_{i\tau_m}(y)K_{ix}(y)\frac{dy}{y}=\varphi_{N,\varepsilon}(x)$$

converges in $\mathcal{E}(\mathbb{R})$ to $\varphi_\varepsilon(x)$, which, in turn, is defined by (2.6). Indeed, refering to the proofs of Theorems 1, 2, we find (cf. (1.16))

$$\max_{0\le p\le r}\sup_{x\in[-x_0,x_0]}|D_x^p(\varphi_{N,\varepsilon}(x)-\varphi_\varepsilon(x))|=\max_{0\le p\le r}\sup_{x\in[-x_0,x_0]}\left|\int_{|\tau|>N}D_x^p\mathcal{K}_\varepsilon(\tau,x)\varphi(\tau)d\tau\right|<$$

$$< C\int_{\varepsilon}^{\infty}\frac{dy}{y^{3/2}}\int_{|\tau|>N}\frac{d\tau}{|\tau|^{\alpha-1}}\to 0,\ N\to\infty,\ \alpha>2.$$

Combining this with (3.9), we get

$$\lim_{N\to\infty}\left\langle f_x,\frac{1}{2\pi}\sum_{m=0}^{N}\tau_m\varphi(\tau_m)\int_{\varepsilon}^{\infty}I_{i\tau_m}(y)K_{ix}(y)\frac{dy}{y}\right\rangle=\langle f,\varphi_\varepsilon\rangle.$$

Hence, using Theorem 1 we arrive at representation (3.7). Theorem 3 is proved. □

As a corollary, this immediately yields the uniqueness property for the Kontorovich–Lebedev transform (3.1).

Corollary 1. *If $F(y)=G(y)$, $y>0$, where F,G are Kontorovich–Lebedev transforms of f and g, respectively, then $f=g$ in the sense of equality in $\mathcal{E}'(\mathbb{R})$ for all φ from Theorem 1.*

Acknowledgments

The presented research was in part supported by the "Centro de Matemática" of the University of Porto.

REFERENCES

[1] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G., *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, London and Toronto, 1953.

[2] Forristall G. Z., Ingram J. D., *Evaluation of distributions useful in Kontorovich–Lebedev transform theory*, SIAM J: Math. Anal. **3** (1972), 561–566.

[3] Prudnikov A. P., Brychkov Yu. A., Marichev O. I., *Integrals and Series, Special Functions*, Gordon and Breach, New York, 1986.

[4] Sneddon I. N., *The Use of Integral Transforms*, McGraw–Hill, New York, 1972.

[5] Yakubovich S. B., Fisher B., *On the theory of the Kontorovich–Lebedev transformation on distributions*, Proc. of the Amer. Math. Soc., **122** (1994) 3, 773–777.

[6] Yakubovich S. B., Luchko Yu. F., *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Kluwers Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London, 1994.

[7] Yakubovich S. B., *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.

[8] Zemanian A. H., *The Kontorovich–Lebedev transformation on distributions of compact support and its inversion*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 139–143.

S. B. Yakubovich
syakubov@fc.up.pt

University of Porto,
Faculty of Sciences,
Department of Pure Mathematics,
Campo Alegre st., 687, 4169-007 Porto, Portugal

Received: January 26, 2006.