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## ON OSCILLATORY SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS

**Abstract.** Some difference equations with deviating arguments are discussed in the context of the oscillation problem. The aim of this paper is to present the sufficient conditions for oscillation of solutions of the equations discussed.

**Keywords:** difference equations, oscillation.

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In this paper we deal with the oscillatory behavior of solutions of the nonlinear difference equation

$$(-1)^v \Delta^m x(n) = p(n) \prod_{i=1}^w |x(g_i(n))|^{\alpha_i} \operatorname{sgn} x(n), \quad (N_v)$$

where  $n \in N = \{1, 2, \dots\}$ ,  $m \geq 2$ ,  $v \in \{1, 2, m\}$ ,  $w \geq 1$ ,  $p: N \rightarrow R_+ \cup \{0\}$ ,  $g_i: N \rightarrow N$ ,  $\lim_{n \rightarrow \infty} g_i = \infty$ ,  $g_i(n) \leq n$  on  $N$  ( $i = 1, \dots, w$ ),  $\alpha_i \in [0, 1]$  ( $i = 1, \dots, w$ )

and  $\sum_{i=1}^w \alpha_i = 1$ . Moreover, let

$$\sum_{s=n_0}^{\infty} p(s) \prod_{i=1}^w [(g_i(s))^{(m-1)}]^{\alpha_i} = \infty. \quad (C)$$

As usual, we define the forward difference operator as follows:

$$\Delta^0 x(n) = x(n), \quad \Delta^m x(n) = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} x(n+i),$$

for  $m \in N$ . For each  $r \in R$  and a nonnegative integer  $s$  the factorial expression is defined as  $r^{(s)} = r(r-1) \cdot \dots \cdot (r-s+1)$  with  $r^{(0)} = 1$ . By a solution of equation

$(N_v)$  we mean a function  $x : N \rightarrow R$  such that  $\sup_{n \geq n_0} |x(n)| > 0$  for any  $n_0 \in N$  and  $x$  satisfies  $(N_v)$  for  $n \in N$ . A nontrivial solution  $x$  is called oscillatory, if for every  $n_0$  there exists  $n \geq n_0$  such that  $x(n)x(n+1) \leq 0$ . Otherwise, it is nonoscillatory.

Hereinafter we use the following lemmas, which are the discrete analogues of the theorems for differential equations due to Kiguradze [4] and Koplatadze and Chanturia [5], (see also [1, 3]).

**Lemma 1.** *Let  $x$  be a nonoscillatory solution of  $(N_v)$ . Then there exist  $n_0 \in N$  and an integer  $l \in \{0, 1, \dots, m\}$ , with  $m + l + v$  even, such that for  $n \geq n_0$*

$$\begin{cases} x(n)\Delta^i x(n) > 0 & \text{for } i = 0, 1, \dots, l - 1 \\ (-1)^{i+l}x(n)\Delta^i x(n) > 0 & \text{for } i = l, \dots, m - 1. \end{cases} \tag{1}$$

**Lemma 2.** *Let  $x$  be a nonoscillatory solution of  $(N_v)$  satisfying inequalities (1) with  $l \in \{1, 2, \dots, m - 1\}$  and  $m + l + v$  even. In addition, let*

$$\sum_{n=1}^{\infty} n^{(m-l)} |\Delta^m x(n)| = \infty. \tag{2}$$

Then the following inequalities hold for sufficiently large  $n \geq n_1$ :

$$|\Delta^{l-1}x(n)| \geq (n - m + l) |\Delta^l x(n)| + \frac{1}{(m - l)!} \sum_{s=n_1}^{n-1} s^{(m-l)} |\Delta^m x(s)| \tag{3}$$

and

$$|x(n)| \geq \frac{(n - m + l - 1)^{(l-1)}}{l!} |\Delta^{l-1}x(n)|. \tag{4}$$

**Lemma 3.** *Let  $x$  be a nonoscillatory solution of  $(N_v)$  satisfying inequalities (1) with  $l \in \{1, 2, \dots, m - 1\}$  and  $m + l + v$  even. Then*

$$|\Delta^l x(n)| \geq \frac{1}{(m - l - 1)!} \sum_{s=n}^{\infty} (s - n + m - l - 1)^{(m-l-1)} |\Delta^m x(s)| \tag{5}$$

for sufficiently large  $n \geq n_0$ .

The following theorem characterizes the oscillatory behavior of bounded solutions of  $(N_m)$ .

**Theorem 1.** *If for some  $h \in \{0, 1, \dots, m - 1\}$*

$$\limsup_{n \rightarrow \infty} \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^{n-1} (s - g_j(n) + m - h - 1)^{(m-h-1)} p(s) \times \prod_{i=1}^w \left[ (g_i(n) - g_i(s) + h - 1)^{(h)} \right]^{\alpha_i} \right\}^{\alpha_j} > h!(m - h - 1)!, \tag{6}$$

then  $(N_m)$  has no solution  $x$  for which

$$(-1)^i x(n) \Delta^i x(n) > 0$$

$$\text{for } i = 0, 1, \dots, m - 1 \text{ and } (-1)^m x(n) \Delta^m x(n) \geq 0 \quad (n \geq n_0). \quad (7)$$

*Proof.* Suppose that equation  $(N_m)$  has a nonoscillatory solution  $x$  satisfying inequalities (7).

From the discrete Taylor formula:

$$\Delta^i x(n) = \sum_{j=i}^{m-1} \frac{(n-s)^{(j-i)}}{(j-i)!} \Delta^j x(s) + \frac{1}{(m-i-1)!} \sum_{r=s}^{n-m+i} (n-r-1)^{(m-i-1)} \Delta^m x(r), \quad (8)$$

for  $i = h \in \{0, 1, \dots, m - 1\}$ , we derive

$$|\Delta^h x(n)| = \sum_{j=h}^{m-1} \frac{(s-n+j-h-1)^{(j-h)}}{(j-h)!} |\Delta^j x(s)| + \frac{1}{(m-h-1)!} \sum_{r=n}^{s-1} (r-n+m-h-1)^{(m-h-1)} |\Delta^m x(r)|,$$

that is

$$|\Delta^h x(n)| \geq \frac{1}{(m-h-1)!} \sum_{r=n}^{s-1} (r-n+m-h-1)^{(m-h-1)} |\Delta^m x(r)| \quad (9)$$

for  $h \in \{0, 1, \dots, m - 1\}$  and  $s \geq n \geq n_0$ .

From (8) with  $i = 0$  there follows:

$$x(n) = \sum_{j=0}^{m-1} \frac{(n-s)^{(j)}}{j!} \Delta^j x(s) + \frac{1}{(m-1)!} \sum_{r=s}^{n-m} (n-r-1)^{(m-1)} \Delta^m x(r).$$

Next we see that

$$|x(n)| = \sum_{j=0}^{m-1} \frac{(s-n+j-1)^{(j)}}{j!} |\Delta^j x(s)| + \frac{1}{(m-1)!} \sum_{r=n}^{s-1} (r-n+m-1)^{(m-1)} |\Delta^m x(r)|,$$

which gives

$$|x(n)| \geq \frac{(s-n+h-1)^{(h)}}{h!} |\Delta^h x(s)|$$

for  $j = h$ ,  $h \in \{0, 1, \dots, m - 1\}$  and  $s \geq n \geq n_0$ .

From (9),  $(N_v)$  and the last inequality we get

$$\begin{aligned}
 |\Delta^h x(g_j(n))| &\geq \\
 &\geq \frac{1}{(m-h-1)!} \sum_{s=g_j(n)}^{n-1} (s-g_j(n)+m-h-1)^{(m-h-1)} |\Delta^m x(s)| = \\
 &= \frac{1}{(m-h-1)!} \sum_{s=g_j(n)}^{n-1} (s-g_j(n)+m-h-1)^{(m-h-1)} p(s) \times \prod_{i=1}^w |x(g_i(s))|^{\alpha_i} \geq \\
 &\geq \frac{1}{(m-h-1)!} \sum_{s=g_j(n)}^{n-1} (s-g_j(n)+m-h-1)^{(m-h-1)} \times \\
 &\quad \times p(s) \prod_{i=1}^w \left[ \frac{(g_i(n)-g_i(s)+h-1)^{(h)}}{h!} \right]^{\alpha_i} |\Delta^h x(g_i(n))|^{\alpha_i},
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 (m-h-1)!h! |\Delta^h x(g_j(n))| &\geq \\
 &\geq \prod_{i=1}^w |\Delta^h x(g_j(n))|^{\alpha_i} \sum_{s=g_j(n)}^{n-1} (s-g_j(n)+m-h-1)^{(m-h-1)} p(s) \times \\
 &\quad \times \prod_{i=1}^w \left[ (g_i(n)-g_i(s)+h-1)^{(h)} \right]^{\alpha_i}.
 \end{aligned}$$

Raising both sides of the above inequality to the power  $\alpha_j$  and then multiplying the resulting inequalities, we obtain

$$\begin{aligned}
 (m-h-1)!h! \prod_{j=1}^w |\Delta^h x(g_j(n))|^{\alpha_j} &\geq \prod_{i=1}^w |\Delta^h x(g_j(n))|^{\alpha_i} \times \\
 &\times \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^{n-1} (s-g_j(n)+m-h-1)^{(m-h-1)} p(s) \times \right. \\
 &\quad \left. \times \prod_{i=1}^w \left[ (g_i(n)-g_i(s)+h-1)^{(h)} \right]^{\alpha_i} \right\}^{\alpha_j},
 \end{aligned}$$

which contradicts (6). Therefore,  $(N_m)$  has no solution  $x$  with property (7). □

**Corollary 1.** *Consider difference equation  $(N_m)$  subject to condition (6). Then all bounded solutions of  $(N_m)$  (if exist) are oscillatory.*

We show that condition (6) is essential for a bounded solution of  $(N_m)$  to be oscillatory. Let us consider the following equation:

$$\Delta^2 x(n) = 2^{-10} [x(n-5)]^{2/5} [x(n-10)]^{3/5},$$

which has a bounded nonoscillatory solution  $x(n) = 2^{-n}$ . In this case condition (6) is not fulfilled, because, for  $h = 0$ :

$$\begin{aligned} \limsup_{n \rightarrow \infty} \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^{n-1} (s - g_j(n) + 1)p(s) \right\}^{\alpha_j} &= \\ &= \limsup_{n \rightarrow \infty} \left\{ \sum_{s=n-5}^{n-1} (s - n + 6)2^{-10} \right\}^{2/5} \left\{ \sum_{s=n-10}^{n-1} (s - n + 11)2^{-10} \right\}^{3/5} = \\ &= \frac{(15)^{2/5}(55)^{3/5}}{2^{10}} < 1, \end{aligned}$$

and for  $h = 1$ :

$$\begin{aligned} \limsup_{n \rightarrow \infty} \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^{n-1} p(s) \prod_{i=1}^w (g_i(n) - g_i(s))^{\alpha_i} \right\}^{\alpha_j} &= \\ &= \limsup_{n \rightarrow \infty} \left\{ \sum_{s=n-5}^{n-1} 2^{-10}(n - s)^{2/5}(n - s)^{3/5} \right\}^{2/5} \times \\ &\times \left\{ \sum_{s=n-10}^{n-1} 2^{-10}(n - s)^{2/5}(n - s)^{3/5} \right\}^{3/5} = \frac{(15)^{2/5}(55)^{3/5}}{2^{10}} < 1, \end{aligned}$$

Now, we present the theorem on behavior of solutions of  $(N_1)$ .

**Theorem 2.** Assume that (C) and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^{n-1} (s - g_j(n) + m - l - 1)^{(m-l-1)} \times \right. \\ \left. \times p(s) \prod_{i=1}^w [(g_i(s) - m + l)^{(l)}]^{\alpha_i} + \right. \\ \left. + \prod_{i=1}^w \left[ (g_i(n) - m + l) + \frac{1}{(m-1)!} \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} p(s) \prod_{k=1}^w [(g_k(s) - m + l)^{(l)}]^{\alpha_k} \right]^{\alpha_i} \right\} \times \\ \times \sum_{s=n+1}^{\infty} (s - g_j(n) - m - l - 1)^{(m-l-1)} p(s) \times \\ \times \prod_{i=1}^w [(g_i(s) - m + l - 1)^{(l-1)}]^{\alpha_i} \Big\}^{\alpha_j} > (m-1)!, \quad (10) \end{aligned}$$

are satisfied for all  $l \in \{1, 2, \dots, m - 1\}$  with  $m + l$  odd. Then for an even  $m$  every solution of  $(N_1)$  is oscillatory and for an odd  $m$  every solution of  $(N_1)$  is either oscillatory or  $\lim_{n \rightarrow \infty} \Delta^i x(n) = 0$  monotonically for  $i = 0, 1, \dots, m - 1$ .

*Proof.* Assume that  $(N_1)$  has a nonoscillatory solution  $x(n) \neq 0$  for  $n \geq n_0$ . Then, by Lemma 1,  $x$  satisfies (1) for some  $l \in \{0, 1, \dots, m-1\}$  with  $m+l$  odd.

First consider the case when  $m$  is even; then  $l \in \{1, 3, \dots, m-1\}$ . From Lemma 3 we get

$$|\Delta^l x(g_j(n))| \geq \frac{1}{(m-l-1)!} \sum_{s=g_j(n)}^{\infty} (s-g_j(n)+m-l-1)^{(m-l-1)} |\Delta^m x(s)|,$$

for  $j = 1, \dots, w$ . From  $(N_1)$ , (4), (3) and (1) there follows that

$$\begin{aligned} |\Delta^l x(g_j(n))| &\geq \frac{1}{(m-l-1)!} \times \\ &\times \sum_{s=g_j(n)}^{\infty} (s-g_j(n)+m-l-1)^{(m-l-1)} p(s) \prod_{i=1}^w |x(g_i(s))|^{\alpha_i} \geq \\ &\geq \frac{1}{(m-l-1)!} \left\{ \sum_{s=g_j(n)}^{\infty} (s-g_j(n)+m-l-1)^{(m-l-1)} p(s) \times \right. \\ &\quad \times \prod_{i=1}^w \left[ \frac{(g_i(s)-m+l)^{(l)}}{l!} \right]^{\alpha_i} |\Delta^l x(g_i(s))|^{\alpha_i} + \\ &\quad + \sum_{s=n+1}^{\infty} (s-g_j(n)+m-l-1)^{(m-l-1)} p(s) \times \\ &\quad \times \prod_{i=1}^w \left[ \frac{(g_i(s)-m+l-1)^{(l-1)}}{l!} \right]^{\alpha_i} |\Delta^{l-1} x(g_i(s))|^{\alpha_i} \left. \right\} \geq \\ &\geq \frac{1}{(m-1)!} \left\{ \prod_{i=1}^w |\Delta^l x(g_i(n))|^{\alpha_i} \sum_{s=g_j(n)}^n (s-g_j(n)+m-l-1)^{(m-l-1)} \times \right. \\ &\quad \times p(s) \prod_{i=1}^w [(g_i(s)-m+l)^l]^{\alpha_i} + \prod_{i=1}^w |\Delta^{l-1} x(g_i(n))|^{\alpha_i} \times \\ &\quad \times \sum_{s=n+1}^{\infty} (s-g_j(n)+m-l-1)^{(m-l-1)} p(s) \prod_{i=1}^w [(g_i(s)-m+l-1)^{(l-1)}]^{\alpha_i} \left. \right\}, \quad (11) \end{aligned}$$

for  $n \geq n_1$ . Now by Lemma 2 we obtain

$$\begin{aligned} \prod_{i=1}^w |\Delta^{l-1} x(g_i(n))|^{\alpha_i} &\geq \\ &\geq \prod_{i=1}^w \left[ (g_i(n)-m+l) |\Delta^l x(g_i(n))| + \frac{1}{(m-l)!} \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} |\Delta^m x(s)| \right]^{\alpha_i} \end{aligned}$$

for  $n \geq n_1$  and, similarly as above,

$$\begin{aligned} \prod_{i=1}^w |\Delta^{l-1}x(g_i(n))|^{\alpha_i} &\geq \prod_{i=1}^w [(g_i(n) - m + l) |\Delta^l x(g_i(n))| + \\ &+ \frac{1}{(m-l)!} \prod_{k=1}^w |\Delta^l x(g_k(g_i(n)))|^{\alpha_k} \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} p(s) \times \\ &\times \prod_{k=1}^w \left[ \frac{[(g_k(s) - m + l)^{(l)}]^{\alpha_k}}{l!} \right]^{\alpha_i} \geq \prod_{i=1}^w |\Delta^l x(g_i(n))|^{\alpha_i} [(g_i(n) - m + l) + \\ &+ \frac{1}{(m-1)!} \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} p(s) \prod_{k=1}^w [(g_k(s) - m + l)^{(l)}]^{\alpha_k}]^{\alpha_i}. \end{aligned} \tag{12}$$

Using (12) in (11) we get

$$\begin{aligned} (m-1)! |\Delta^l x(g_i(n))| &\geq \prod_{i=1}^w |\Delta^l x(g_i(n))|^{\alpha_i} \times \\ &\times \left\{ \sum_{s=g_j(n)}^n (s - g_j(n) + m - l - 1)^{(m-l-1)} p(s) \times \right. \\ &\times \prod_{i=1}^w [(g_i(s) - m + l)^{(l)}]^{\alpha_i} + \prod_{i=1}^w \left[ (g_i(n) - m + l) + \frac{1}{(m-1)!} \times \right. \\ &\times \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} p(s) \prod_{k=1}^w [(g_k(s) - m + l)^{(l)}]^{\alpha_k} \left. \right]^{\alpha_i} \times \\ &\times \sum_{s=n+1}^{\infty} (s - g_j(n) + m - l - 1)^{(m-l-1)} \times \\ &\left. \times p(s) \prod_{i=1}^w [(g_i(s) - m + l - 1)^{(l-1)}]^{\alpha_i} \right\}. \end{aligned}$$

Raising both sides of the above inequality to the power  $\alpha_j$  and then multiplying the resulting inequalities we obtain

$$\begin{aligned} (m-1)! \prod_{j=1}^w |\Delta^l x(g_j(n))|^{\alpha_j} &\geq \prod_{i=1}^w |\Delta^l x(g_i(n))|^{\alpha_i} \times \\ &\times \prod_{j=1}^w \left\{ \sum_{s=g_j(n)}^n (s - g_j(n) + m - l - 1)^{(m-l-1)} p(s) \prod_{i=1}^w [(g_i(s) - m + l)^{(l)}]^{\alpha_i} + \right. \end{aligned}$$

$$\begin{aligned}
 & + \prod_{i=1}^w \left[ (g_i(n) - m + l) + \frac{1}{(m-1)!} \sum_{s=n_1}^{g_i(n)-1} s^{(m-l)} p(s) \times \right. \\
 & \times \left. \prod_{k=1}^w [(g_k(s) - m + l)^{(l)}]^{\alpha_k} \right]^{\alpha_i} \sum_{s=n+1}^{\infty} (s - g_j(n) + m - l - 1)^{(m-l-1)} \times \\
 & \times p(s) \prod_{i=1}^w [(g_i(s) - m + l - 1)^{(l-1)}]^{\alpha_i} \Big\}^{\alpha_j},
 \end{aligned}$$

which contradicts (10). Therefore every solution of  $(N_1)$  is oscillatory for  $m$  even.

Consider the case of  $m$  odd. Then  $x$  satisfies (1) for some  $l \in \{0, 2, \dots, m - 1\}$ . The case  $l \in \{2, \dots, m - 1\}$  is impossible (analogously as above). We shall prove that  $\lim_{n \rightarrow \infty} |x(n)| = 0$  with  $l = 0$ . Suppose to the contrary that  $\lim_{n \rightarrow \infty} |x(n)| = C > 0$ . Then  $|x(g_i(n))| \geq C$  for  $i = 0, \dots, w$  and  $n \geq n_2 \geq n_1$ . From (8) with  $i = 0$ :

$$x(n) = \sum_{j=0}^{m-1} \frac{(n-s)^{(j)}}{j!} \Delta^j x(s) + \frac{1}{(m-1)!} \sum_{r=s}^{n-m} (n-r-1)^{(m-1)} \Delta^m x(r)$$

and from (1) it follows that

$$\sum_{s=n_1}^{\infty} s^{(m-1)} |\Delta^m x(s)| < \infty.$$

Thus, by  $(N_1)$

$$s^{(m-1)} |\Delta^m x(s)| = s^{(m-1)} p(s) \prod_{i=1}^w |x(g_i(s))|^{\alpha_i} \geq C s^{(m-1)} p(s)$$

for  $s \geq n_2$ , hence

$$\sum_{s=n_2}^{\infty} s^{(m-1)} p(s) < \infty,$$

which is impossible in view of  $(C)$ . The obtained contradiction proves that  $\lim_{n \rightarrow \infty} x(n) = 0$ . Therefore  $\lim_{n \rightarrow \infty} \Delta^i x(n) = 0$  monotonically for  $i = 0, 1, \dots, m - 1$  and the proof is complete. □

From Theorems 1 and 2 we draw the following corollary.

**Corollary 2.** *Consider difference equation  $(N_1)$  subject to conditions (6) and (10). Then all solutions of  $(N_1)$  are oscillatory.*

The last theorem concerns equation  $(N_2)$ .

**Theorem 3.** *Assume that  $(C)$  and (10) are satisfied for all  $l \in \{1, 2, \dots, m - 1\}$  with  $m + l$  even. Then for an even  $m$  every solution of  $(N_2)$  is oscillatory either  $\lim_{n \rightarrow \infty} \Delta^i x(n) = 0$  or  $\lim_{n \rightarrow \infty} \Delta^i x(n) = \infty$  and for an odd  $m$  every solutions of  $(N_2)$  is either oscillatory or  $\lim_{n \rightarrow \infty} \Delta^i x(n) = \infty$  monotonically for  $i = 0, 1, \dots, m - 1$ .*



*Proof.* Assume that  $(N_2)$  has a nonoscillatory solution  $x(n) \neq 0$  for  $n \geq n_0$ . Then, by Lemma 1,  $x$  satisfies (1) for some  $l \in \{0, 1, \dots, m\}$  with  $m + l$  even.

Let  $m$  be even; then  $l \in \{0, 2, \dots, m - 2, m\}$ . The case  $l \in \{2, \dots, m - 2\}$  is impossible (see proof of Theorem 2); however, for  $l = 0$  there is  $\lim_{n \rightarrow \infty} \Delta^i x(n) = 0$ . Consider the case  $l = m$ , then from (1) we obtain inequalities

$$x(n)\Delta^k x(n) > 0 \quad \text{for } k = 0, 1, \dots, m - 1 \quad \text{and} \quad x(n)\Delta^m x(n) \geq 0$$

for  $n \geq n_1$ . This implies that there exist a constant  $C > 0$  and  $n_2 \geq n_1$  such that

$$|x(g_i(n))| \geq C [g_i(n)]^{(m-1)}, \quad (i = 0, 1, \dots, w)$$

and

$$|\Delta^{m-1} x(n)| = |\Delta^{m-1} x(n_2)| + \sum_{s=n_2}^{n-1} |\Delta^m x(s)|.$$

From the last two expressions and  $(N_2)$ , it follows that

$$\begin{aligned} |\Delta^{m-1} x(n)| &\geq \sum_{s=n_2}^{n-1} |\Delta^m x(s)| \geq \sum_{s=n_2}^{n-1} p(s) \prod_{i=1}^w |x(g_i(s))|^{\alpha_i} \geq \\ &\geq C \sum_{s=n_2}^{n-1} p(s) \prod_{i=1}^w \left[ (g_i(s))^{(m-1)} \right]^{\alpha_i}. \end{aligned}$$

It is clear that above and (C) gives  $\lim_{n \rightarrow \infty} |\Delta^i x(n)| = \infty$  for  $i = 0, 1, \dots, m - 1$ .

Consider the case of  $m$  odd. Then  $x$  satisfies (1) for some  $l \in \{1, 3, \dots, m\}$ . From (10) it follows that  $l \notin \{1, 3, \dots, m - 2\}$ . In case  $l = m$  (similarly as for  $m$  even), there is that  $\lim_{n \rightarrow \infty} |\Delta^i x(n)| = \infty$  ( $i = 0, 1, \dots, m - 1$ ). The proof is complete.  $\square$

From Theorems 1 and 3 there follows:

**Corollary 3.** Consider difference equation  $(N_2)$  subject to conditions (6) and (10). Then all solutions of  $(N_2)$  are oscillatory or  $\lim_{n \rightarrow \infty} |\Delta^i x(n)| = \infty$  monotonically for  $i = 0, 1, \dots, m - 1$ .

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