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A DYNAMICAL INVERSE PROBLEM FOR A PARABOLIC EQUATION

Abstract. A problem of dynamical reconstruction of unknown distributed or boundary disturbances acting upon nonlinear parabolic equations is discussed. A regularized algorithm which allows us to reconstruct disturbances synchro with the process under consideration is designed. This algorithm is stable with respect to informational noises and computational errors.

Keywords: nonlinear parabolic equations, inverse problem.

Mathematics Subject Classification: 35K90, 35R30.

1. INTRODUCTION

A problem of dynamical reconstruction of unknown inputs of parabolic systems is considered. Both distributed and boundary disturbances may play a role of these inputs. A system described by a parabolic equation is supposed to be functioning on a given interval $T = [0, \vartheta]$. Evolution of its phase state $x(t)$, $t \in T$ (system trajectory) is determined by some input (disturbance) $u(\cdot)$ belonging to a given functional set $P(\cdot)$. This input is unknown, as is the phase trajectory $x(\cdot)$. However, there are some sensors which allow us to perform inaccurate measurements of outputs $x(\tau_i)$ at discrete sufficiently frequent time moments $\tau_i \in T$, $\tau_i < \tau_{i+1}$. It is required to reconstruct a disturbance $u_*(\cdot)$ generating $x(\cdot)$: $u_*(\cdot) = u_*(\cdot; x(\cdot))$. Since precise reconstruction of $u_*(\cdot)$ is impossible (due to the measurement error), it is necessary to design an algorithm for calculating an approximation to $u_*(\cdot)$. The smaller is the value of measurement error and the denser is the partition of the interval T, the better the approximation must be. The problem under discussion is treated within the framework of the class of inverse problems. A posteriori formulations of inverse problems for distributed equations were investigated by numerous authors [1–3].

A method of dynamical reconstruction of input for a finite-dimensional dynamical system affine in disturbance was suggested in [4]. This method was effectively conformed in [5, 6] to solving different inverse problems for systems described by partial differential equations. It is based on the theory of positional control [7] in the combination with regularization methods of smoothing functional and discrepancy [3], well-known in the theory of ill-posed problems.

The aim of this paper (continuing the results of [5, 6, 8, 9]) is to demonstrate (based on the theory of boundary control developed in [10–14]) the possibilities of applying the method of auxiliary positional-controlled models to investigate the problems of reconstruction of unknown distributed or boundary disturbances acting upon nonlinear parabolic equations. The structure of the paper is as follows. The general scheme of an approach to solving the problems in question is presented in Section 1. Then (Section 2) the problem of reconstruction of distributed or boundary disturbances in parabolic equations is investigated.

2. METHOD OF POSITIONAL CONTROL WITH A MODEL

Let us pass to actual formulation of the problem under consideration and describing an approach to its solution.

There is a dynamical system Σ functioning on a time interval $T = [0, \vartheta]$ and described by a parabolic equation. At every moment t, its state is characterized by an element $x(t)$ of an infinite-dimensional space X. A motion of the system $x(t) = x(t; 0, x_0, u(\cdot))$ starts from an initial state x_0 under the action of an input (disturbance) $u(\cdot) \in P(\cdot) \subset L_2(T; U)$. At discrete sufficiently frequent time moments $\tau_i \in T$, $\tau_i = \tau_{i-1} + \delta$, $i \in [1 : m]$, $\tau_0 = 0$, $\tau_m = \vartheta$, system phase states $x(\tau_i)$ are inaccurately measured. The measurement results are elements $\xi_i \in \Xi$ satisfying inequalities

$$\chi(x(\tau_i), \xi_i) \leq h. \tag{1}$$

Here U, X, and Ξ are spaces of disturbances, outputs, and measurements, respectively (U is a uniformly convex Banach space), $\Xi(x(\cdot), h)$ is a set of admissible measurements, i. e., of piecewise constant functions $t \to \xi(t) \in \Xi$ with property (1), χ is a criterion for a measurement error, $h \in (0, 1)$ is a parameter of measurement accuracy. Let the symbol $x_{a,b}(\cdot)$ denote a function $x(t)$, $t \in [a, b]$ (which is considered as an element of a functional space), the symbol $P_{\tau_i,\tau_{i+1}}$ denote the restriction of a set of functions $P(\cdot) \subset L_2(T; U)$ to the half-interval $[\tau_i, \tau_{i+1})$.

The problem consists in the construction of a family of algorithms

$$D_h : \{\tau_i, \xi_i\} \longmapsto v_{\tau_i,\tau_{i+1}}(\cdot) \in P_{\tau_i,\tau_{i+1}}$$

such that the following convergence takes place under an appropriate relation between h and $\delta = \delta(h)$

$$\|v^h(\cdot) - u_*(\cdot; x(\cdot))\|_{L_2(T;U)} \to 0 \quad \text{as} \quad h \to 0, \tag{2}$$

where

$$v^h(\cdot) = D_h\xi(\cdot),$$

$u^*(\cdot; x(\cdot))$ is an element of the set of controls $U(x(\cdot)) \in P(\cdot)$ generating the output $x(\cdot)$.

Using well-known constructions of the theory of ill-posed problems [3], we introduce the following straightforward definition.

Definition. *A family D_h, $h \in (0,1)$, of operators acting from $\Xi(x(\cdot), h)$ into $P(\cdot)$ is called* regularized *if for any output $x(\cdot)$*

$$\nu(x(\cdot)) = \lim_{h\to 0} \sup \left\{ \|D_h\xi(\cdot) - u_*(\cdot; x(\cdot))\|_{L_2(T;U)} : \xi^h(\cdot) \in \Xi(x(\cdot), h) \right\} = 0.$$

The problem discussed in this paper consists in the construction of regularized families of Volterra operators D_h, $h \in (0,1)$.

The approach to its solution (described below) is based on the well-known principle of the theory of positional control — the principle of auxiliary controlled models. Its essence consists in the following. An auxiliary dynamical system M (a model) whose motion is a solution of a specific parabolic equation is chosen. Hereinafter, this motion is denoted by

$$w^h(t) = w^h(t; 0, w_*, v^h_{t_0,t}(\cdot)), \quad t \in T, \tag{3}$$

w_* is an initial state of the model, $v^h(\cdot) \in P(\cdot)$ is a control. Once the model has been chosen, the algorithm of input reconstruction is identified with the feedback control law in the model. The initial state w_* is fixed before the control process starts. Model control laws, being called strategies according to the terminology of the theory of positional control [7], are identified with pairs $(\Delta_h, \mathcal{U}_h)$, where $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ is a partition of the interval T into half-intervals $[\tau_{h,i}, \tau_{h,i+1})$, $\tau_{h,i+1} = \tau_{h,i} + \delta$, $\delta = \delta(h)$, $\tau_{h,0} = 0$, $\tau_{h,m_h} = \vartheta$, $\mathcal{U}_h$ is a function relating an element

$$v^h_{\tau_i,\tau_{i+1}}(\cdot) = \mathcal{U}_h(\tau_i, \xi_i, \psi_i) \in P_{\tau_i,\tau_{i+1}} \tag{4}$$

to every triple (τ_i, ξ_i, ψ_i), $i \in [0 : m_h - 1]$. Here $\tau_i = \tau_{h,i}$, $\xi_i = \xi_{t_0,\tau_i}(\tau_i)$, $\xi_{t_0,\tau_i}(\cdot) \in \Xi(x(\cdot), h)_{t_0,\tau_i}$ is an admissible measurement $\xi(t)$, $t_0 \le t \le \tau_i$, $w^h(\tau_i)$ is a model phase state at moment τ_i, $x(\cdot)$ is a system phase trajectory, $\psi_i \in \Xi$ is a result of inaccurate measurement of state $w^h(\tau_i)$:

$$\chi_1(w^h(\tau_i), \psi_i) \le h,$$

χ_1 is a criterion for a measurement error for model trajectory.

Thus, for every $h \in (0,1)$ the triple $(M, \Delta_h, \mathcal{U}_h)$ determines some algorithm D_h on the set of measurements $\xi(\cdot) \in \Xi(x(\cdot), h)$, forming output $v^h(\cdot) = D_h\xi(\cdot)$ by feedback principle (3), (4). A regularized family D_h, $h \in (0,1)$, is suggested to be constructed from algorithms of such kind (we identify every algorithm D_h with the triple $(M, \Delta_h, \mathcal{U}_h)$). So the problem consists in the construction of a regularized family of algorithms $D_h = (M, \Delta_h, \mathcal{U}_h)$, $h \in (0, 1)$, of form (3), (4).

The operation of the algorithm D_h (for a fixed h) may be outlined as follows. Before the moment t_0, a partition $\Delta = \Delta_h = \{\tau_i\}_{i=0}^m$, $(\tau_i = \tau_{h,i},\ m = m_h)$ of the interval T and an auxiliary model M are chosen and fixed. The operation is decomposed into $m-1$, $m = m_h$, identical steps. At the i-th step carried out during the interval $[\tau_i, \tau_{i+1})$, the following sequence of actions is fulfilled. The output $x(\tau_i)$ is measured inaccurately, i. e., the element ξ_i with property (1) is calculated. Then the model control is determined by rule (4) and the memory correction is realized: the new part of the trajectory $w^h(t) = w(t; \tau_i, w^h(\tau_i), w^h_{\tau_i,\tau_{i+1}}(\cdot))$, $t \in (\tau_i, \tau_{i+1}]$, is formed instead of $w^h(\tau_i)$. The procedure stops at the time ϑ.

3. DISTURBANCE RECONSTRUCTION FOR PARABOLIC SYSTEMS

Let a system Σ be described by the following parabolic equation

$$\begin{gathered} x_t(t,\eta) - \Delta x(t,\eta) = f(t,\eta) + (B_1 u_1(t))(\eta) + \Phi(x(t,\eta)) \\ \text{in} \quad T \times \Omega = Q, \quad T = [0, \vartheta] \end{gathered} \tag{5}$$

with the initial condition

$$x(0,\eta) = x_0(\eta) \quad \text{in} \quad \Omega \tag{6}$$

and the boundary condition

$$x(t)|_\Gamma = B_2 u_2(t), \quad t \in T. \tag{7}$$

Here $\Omega \subset R^n$ is an open bounded domain with a sufficiently smooth boundary Γ, Δ is the Laplace operator, $f(\cdot) \in L_2(T; L_2(\Omega))$ is a given disturbance, $H = L_2(\Omega)$, $\Phi(\cdot)$ is a Lipschitz function, $B_1 \in \mathcal{L}(U_1; L_2(\Omega))$ and $B_2 \in \mathcal{L}(U_2; L_2(\Gamma))$ are continuous linear operators, U_1 and U_2 are uniformly convex Banach spaces.

Following [10, 11], we give the definition of a solution. Let σ be the Dirichlet map (harmonic extension of boundary data) defined by

$$\sigma u_2 = h \Longleftrightarrow \begin{cases} \Delta h = 0 & \text{in } \Omega, \\ h = u_2 & \text{in } \Gamma, \quad u_2 \in L_2(\Gamma), \end{cases}$$
$$\sigma : \text{ continuous } L_2(\Gamma) \to H.$$

Introduce the map

$$t \to p(t; \cdot, \cdot, \cdot) : H \times L_2(T; U) \times C(T; H) \to C(T; H),$$

$$p(t; x_0, u(\cdot), z(\cdot)) = S(t)x_0 + A \int_0^t S(t-\tau)\sigma B_2 u_2(\tau)\, d\tau + \\ + \int_0^t S(t-\tau)\{f(\tau) + B_1 u_1(\tau) + \Phi(z(\tau))\}\, d\tau, \quad t \in T.$$

Here $Ah = \Delta h$, $h \in \mathcal{D}(A) = H_1^0(\Omega) \cap H_2(\Omega)$ is the generator of a contracting semigroup of continuous linear operators $\{S(t); t \geq 0\}$ in H.

The function $x(\cdot) = x(\cdot; 0, x_0, u(\cdot)) \in C(T; H)$ satisfying the integral equation

$$x(t) = p(t; x_0, u(\cdot), x(\cdot)) \quad \forall\, t \in T$$

is called a solution of equation (5)–(7) corresponding to a control $u(\cdot) \in P(\cdot)$.

Let us formulate the problem under consideration. Unknown input disturbances $u_1(\cdot)$ and $u_2(\cdot)$ act upon the system. Let $u(t) = \{u_1(t), u_2(t)\} \in P = P_1 \times P_2$ for a. a. $t \in T$, $P_1 \subset U_1$, $P_2 \subset U_2$ are convex, bounded, and closed sets. At discrete, sufficiently frequent time moments

$$\tau_i \in T, \quad \tau_i = \tau_{i-1} + \delta, \quad i \in [1 : m-1], \quad \tau_0 = 0, \quad \tau_m = \vartheta$$

phase states of system (5)–(7) $x(\tau_i, \eta) = x(\tau_i; 0, x_0, u(\cdot)) \in H = L_2(\Omega)$ are inaccurately measured. Results of measurements (elements $\xi_i \in H$) satisfy the inequalities

$$|\xi_i - x(\tau_i)|_H \leq h, \tag{8}$$

where h is the measurement accuracy.

The operators B_1 and B_2, as well as the function Φ are assumed to be inaccurately known. Namely, we know families of continuous linear operators B_1^h and B_2^h as well as of function Φ_h such that

$$|B_1 - B_1^h|_{L(U_1, L_2(\Omega))} \leq h, \quad |B_2 - B_2^h|_{L(U_2, L_2(\Gamma))} \leq h,$$
$$|\Phi(x) - \Phi_h(x)|_H \leq h, \quad \forall x \in L_2(\Omega).$$

The problem consists in designing a family D_h, $h \in (0,1)$, of algorithms of dynamical reconstruction of an unknown input disturbance $u_*(\cdot) = \{u_1^*(\cdot), u_2^*(\cdot)\} \in P(\cdot) = \{u(\cdot) = \{u_1(\cdot), u_2(\cdot)\} \in L_2(T; U) : u_1(t) \in P_1, u_2(t) \in P_2$ for a. a. $t \in T\}$ generating an unknown output $x(\cdot)$, $x(\cdot; 0, x_0, u_*(\cdot))$. Here $U = U_1 \times U_2$ is the space of disturbances, the symbol $x(\cdot; 0, x_0, u_*(\cdot))$ denotes the solution of equation (5) with initial and boundary conditions (6)–(7) and control $u(\cdot) = u_*(\cdot)$.

In this section, we design the rule for constructing the family D_h, $h \in (0,1)$, based on the scheme outlined in Section 1.

Let us pass to solving the problem of constructing aregularized family of operators $D_h = (M, \Delta_h, \mathcal{U}_h)$, $h \in (0,1)$. Following the approach described above, it is first necessary to choose an auxiliary system (a model). As the model, we take the following linear system described by the parabolic equation

$$\begin{gathered} w_t(t,\eta) - \Delta w(t,\eta) = f(t,\eta) + (B_1^h v_1^h(t))(\eta) + v_3^h(t,\eta) \quad \text{in} \quad T \times \Omega, \\ w(0,\eta) = w_0(\eta) \quad \text{in} \quad \Omega \end{gathered} \tag{9}$$

with the Dirichlet boundary condition

$$w(t)|_\Gamma = B_2^h v_2^h(t), \quad t \in T.$$

By a solution of (9) generated by controls $\{v_1^h(\cdot),\, v_2^h(\cdot)\} \in P(\cdot)$ and $v_3^h(\cdot) \in L_2(T;H)$, we mean a function $w^h(\cdot) = w(\cdot;0,w_0,v^h(\cdot)) \in C(T;H)$, $v^h(\cdot) = \{v_1^h(\cdot), v_2^h(\cdot), v_3^h(\cdot)\}$, determined by the equality [12]

$$w^h(t) = S(t)w_0 + A\int\limits_{t_0}^{t} S(t-\tau)\sigma B_2^h v_2^h(\tau)\,d\tau + \\ + \int\limits_0^t S(t-\tau)\{f(\tau) + B_1^h v_1^h(\tau) + v_3^h(\tau)\}\,d\tau, \quad t\in T. \quad (10)$$

As is known [12, 13], such a solution exists and is unique if $v^h(\cdot) \in P(\cdot)\times L_\infty(T;H)$.

Let a family $\{\Delta_h\}$ of partitions $\Delta_h = \{\tau_i\}_{i=0}^m$, $\tau_i = \tau_{h,i}$, $m = m_h$, $\tau_0 = 0$, $t_m = \vartheta$ of the interval T with diameters $\delta = \delta(h)$ be chosen. We determine the positional strategy $(\Delta_h, \mathcal{U}_h)$ assuming that

$$\mathcal{U}^h(\tau_i,\xi_i,\psi_i) = \{v_1^h(t), v_2^h(t), v_3^h(t) : t\in\delta_i\}, \quad (11)$$

$$v_1^h(t) = v_{1i}^h, \qquad t\in\delta_i, \quad (12)$$

$$2(s_i^*, A^{-1}B_1^h v_{1i}^h)_H + \alpha(h)|v_{1i}^h|_{U_1}^2 \le \inf\left\{2(s_i^*, A^{-1}B_1^h v_1)_H + \alpha(h)|v_1|_{U_1}^2 : v_1\in P_1\right\} + h\delta,$$

$$v_2^h(t) = v_*(t-\tau_i) \quad \text{for a. a.} \quad t\in\delta_i, \quad (13)$$

$$2\int\limits_0^\delta \left(\frac{\partial}{\partial n}\Delta^{-1}S(\delta-s)s_i^*\Big|_\Gamma, B_2^h v_*(s)\right)_{L_2(\Gamma)} ds + \alpha(h)\int\limits_0^\delta |v_*(s)|_{U_2}^2\,ds \le \quad (14)$$

$$\le \inf\{2\int\limits_0^\delta \left(\frac{\partial}{\partial n}\Delta^{-1}S(\delta-s)s_i^*\Big|_\Gamma, B_2^h v(s)\right)_{L_2(\Gamma)} ds + \alpha(h)\int\limits_0^\delta |v(s)|_{U_2}^2\,ds :$$

$$v(s)\in P_2 \quad \text{for a. a.} \quad s\in[0,\delta]\} + h\delta,$$

$$s_i^* = A^{-1}(\psi_i - \xi_i), \quad |\psi_i - w^h(\tau_i)|_H \le h,$$

$$v_3^h(t) = \Phi_h(\xi_i) \quad \text{for a. a.} \quad t\in\delta_i. \quad (15)$$

Turning back to the general scheme (Section 1), we conclude that all its elements are determined; here $\Xi = H$,

$$\chi(x(\tau_i),\xi_i) = |\xi_i - x(\tau_i)|_H, \quad \chi_1(y^h(\tau_i),\psi_i) = |\psi_i - y^h(\tau_i)|_H,$$

$$y_* = w_0, \quad y^h(\cdot) = y(\cdot;0,y_*,v^h(\cdot)) = w^h(\cdot) = w(\cdot;0,w_0,v^h(\cdot)).$$

The family $D_h = (M, \Delta_h, \mathcal{U}_h)$ is given by (9)–(15).

Let $\varphi_x(\cdot)$ be the continuity modulus of a function $t\to x(t)\in H$ in T, i. e., $\varphi_x(\delta) = \sup\{|\varphi(x(t_1)) - \varphi(x(t_2))| :\ t_1,\ t_2\in T,\ |t_1 - t_2| < \delta\}$, $U(x(\cdot))$ be the set of all controls from $P(\cdot)$ which are compatible with an output $x(\cdot)$, i. e.,

$$U(x(\cdot)) = \{u(\cdot)\in P(\cdot) : x(\cdot) = x(\cdot;0,x_0,u(\cdot))\}.$$

In other words,

$$U(x(\cdot)) = \left\{ u(\cdot) = \{u_1(\cdot), u_2(\cdot)\} \in P(\cdot) : \right.$$
$$x(t) - S(t)x_0 - \int_0^t S(t-\tau)\{f(\tau) + \Phi(x(\tau))\}\,d\tau =$$
$$\left. = A\int_0^t S(t-\tau)\sigma B_2 u_2(\tau)\,d\tau + \int_0^t S(t-\tau)B_1 u_1(\tau)\,d\tau \qquad \forall\, t \in T \right\}.$$

It is easily seen that this set is convex, bounded, and closed in $L_2(T;U)$. Therefore, it consists of the single element $u_*(\cdot) = u_*(\cdot; x(\cdot)) = \{u_{1*}(\cdot), u_{2*}(\cdot)\}$ with minimal $L_2(T;U)$-norm. Let the following conditions of concordance of parameters be fulfilled.

Condition 1.

$$\delta(h) \to 0+, \quad \alpha(h) \to 0+, \quad \{\delta(h) + h + \varphi_x(\delta(h))\}\alpha^{-1}(h) \to 0 \quad as \quad h \to 0.$$

In this case, the following theorem holds.

Theorem 1. *Let the model initial state* $w_0 = w_0^h \in H$ *be such that*

$$|x_0 - w_0^h|_H \le h.$$

Then the convergence

$$\{v_1^h(\cdot), v_2^h(\cdot)\} \to u_*(\cdot; x(\cdot)) = \{u_{1*}(\cdot; x(\cdot)), u_{2*}(\cdot; x(\cdot))\} \quad in \quad L_2(T;U) \quad as \quad h \to 0$$

takes place and, consequently, the family $D_h = (M, \Delta_h, \mathcal{U}_h)$ *of form (9)–(15) is regularized.*

Before proving the theorem, we describe the sequence of actions necessary to execute the algorithm D_h. Additionally, we give auxiliary statements.

The operation of the algorithm D_h of form (9), (11)–(15) for fixed $h \in (0,1)$ is carried out according to the following scheme. Before the process, a value of $h \in (0,1)$ and a partition $\Delta_h = \{\tau_i\}_{i=0}^m$, $\tau_i = \tau_{hi}$, $m = m_h$, with a diameter $\delta = \delta(h) = \delta(\Delta_h)$ are fixed. The operation of the algorithm is decomposed into $(m-1)$ identical steps. At the i-th step carried out during the interval $\delta_i = [\tau_i, \tau_{i+1}]$, we calculate the control

$$v^h_{\tau_i,\tau_{i+1}}(\cdot) = \mathcal{U}^h(\tau_i, \xi_i, \psi_i)$$

by means of mapping $\mathcal{U}^h$ of form (11)–(15) Then the model phase trajectory is recalculated: we find $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), v^h_{\tau_i,\tau_{i+1}}(\cdot))$ instead of $w^h(\tau_i)$. The procedure stops at the time ϑ.

Let us pass to auxiliary statements. It is easily seen that the following lemmas hold.

Lemma 1. *A solution of equation (9) possesses the semigroup property, i. e., for any* $t \in (0, \vartheta)$, $\Delta t > 0$, $t + \Delta t \le \vartheta$, $v(\cdot) \in P(\cdot) \times L_\infty(T; H)$ *the equality*

$$w(t + \Delta t; 0, w_0, v(\cdot)) = w(t + \Delta t; t, w(t), v(\cdot))$$

is valid.

Lemma 2. *A bundle of solutions of equation (5)–(7)* $X_T = \{x(\cdot; 0, x_0, u(\cdot)) : u(\cdot) \in P(\cdot)\}$ *is bounded in* $C(T; H)$.

The proof of the lemma is based on boundedness of the set $P(\cdot)$, Lipschitz property of the function Φ, inequality (3.14) [13]

$$|AS(t)\sigma B_2 u_2|_H \le ct^{-7/8} |B_2 u_2|_{L_2(\Gamma)}, \quad t > 0, \quad u_2 \in U_2$$

and contractibility of the semigroup $\{S(t); t \ge 0\}$.

Let $P_3(\cdot) \subset L_\infty(T; H)$ be an arbitrary bounded set. The following lemma may be proved in a way similar to that in the proof of Lemma 2.

Lemma 3. *A bundle of solutions of equation (9)*

$$W_T = \{w(\cdot; 0, w_0, v(\cdot)) : v(\cdot) = \{v_1(\cdot), v_2(\cdot), v_3(\cdot)\} \in P(\cdot) \times P_3(\cdot)\}$$

is bounded in $C(T; H)$.

Let

$$\varepsilon(t) = |A^{-1}(w^h(t) - x(t))|_H^2 + \\ + \alpha(h) \int_0^t \left\{ |v_2^h(s)|_{U_2}^2 + |v_1^h(s)|_{U_1}^2 - |u_{2*}(s)|_{U_2}^2 - |u_{1*}(s)|_{U_1}^2 \right\} ds,$$

where the control

$$v^h(t) = \mathcal{V}^h(\tau_i, \xi_i, \psi_i), \quad t \in [\tau_i, \tau_{i+1})$$

is calculated from (11)–(15). For $x(\cdot) \in X_T$, by $\Xi_h(x(\cdot))$, we denote the set of all piecewise constant functions $\xi(t) : T \to H$ satisfying $\sup_{t \in T} |\xi(t) - x(t)| \le h$. For $\psi(\cdot) \in W_T$ the set $\Xi_h(\psi(\cdot))$ is defined by analogy.

Lemma 4. *The inequality*

$$\varepsilon_i \equiv \varepsilon(\tau_i) \le k(\delta + h + \varphi_x(\delta)), \quad i \in [1 : m],$$

holds uniformly in all $\xi(\cdot) \in \Xi_h(x(\cdot))$, $\psi(\cdot) \in \Xi_h(\psi(\cdot))$, $h \in (0, 1)$, *and partitions* $\Delta = \{\tau_i\}_{i=0}^m$ *of the interval* T *with diameters* $\delta \le 1$.

Proof. By virtue of Lemma 1,

$$
\begin{aligned}
\varepsilon_{i+1} = \Bigg| A^{-1} \Bigg\{ & S(\delta)(w^h(\tau_i) - x(\tau_i)) + \\
& + A \int_0^\delta S(\delta-\tau)\sigma B_2 \big(v_2^h(\tau_i+\tau) - u_{2*}(\tau_i+\tau)\big)\, d\tau + \\
+ \int_0^\delta S(\delta-\tau) & \left[B_1(v_1^h(\tau_i+\tau) - u_{1*}(\tau_i+\tau)) + v_{3i}^h - \Phi(x(\tau_i+\tau))\right] d\tau \Bigg\} \Bigg|_H^2 + \\
& + \alpha(h) \int_0^{\tau_{i+1}} \left\{|v_2^h(s)|_{U_2}^2 + |v_1^h(s)|_{U_1}^2 - |u_{2*}(s)|_{U_2}^2 - |u_{1*}(s)|_{U_1}^2\right\} ds \le \\
\le \sum_{j=1}^3 J_{ji} + \alpha(h) & \int_0^{\tau_{i+1}} \left\{|v_2^h(s)|_{U_2}^2 + |v_1^h(s)|_{U_1}^2 - |u_{2*}(s)|_{U_2}^2 - |u_{1*}(s)|_{U_1}^2\right\} ds, \quad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{1i} &= |s_i|_H^2, \qquad s_i = A^{-1}S(\delta)(w^h(\tau_i) - x(\tau_i)), \\
J_{2i} &= 2\left(s_i, \int_0^\delta S(\delta-\tau)\sigma B_2\left(v_2^h(\tau_i+\tau) - u_{2*}(\tau_i+\tau)\right) d\tau\right)_H, \\
J_{3i} &= 2\left(s_i, A^{-1}\int_0^\delta S(\delta-\tau) B_1\left(v_1^h(\tau_i+\tau) - u_{1*}(\tau_i+\tau)\right) d\tau\right)_H, \\
J_{4i} &= 2\left(s_i, A^{-1}\int_0^\delta S(\delta-\tau) \left\{v_{3i}^h - \Phi(x(\tau_i+\tau))\right\} d\tau\right)_H,
\end{aligned}
$$

$$
\begin{aligned}
J_{5i} = 3\Bigg\{ & \left|\int_0^\delta S(\delta-\tau)\sigma B_2\left(v_2^h(\tau_i+\tau) - u_{2*}(\tau_i+\tau)\right) d\tau)\right|_H^2 + \\
& + \left|A^{-1}\int_0^\delta S(\delta-\tau) B_1\left(v_1^h(\tau_i+\tau) - u_{1*}(\tau_i+\tau)\right) d\tau\right|_H^2 + \\
& + \left|A^{-1}\int_0^\delta S(\delta-\tau)\left\{v_{3i}^h - \Phi(x(\tau_i+\tau))\right\} d\tau\right|_H^2 \Bigg\}.
\end{aligned}
$$

Since the semigroup $\{S(t); t \ge 0\}$ is contracting, the operator A^{-1} commutates with $S(\delta)$, the function $\Phi(\cdot)$ is Lipschitz, and the following inequalities hold:

$$|\Phi_h(\xi_i) - \Phi(x(\tau_i + \tau))|_H \le h + Lh + L\varphi_x(\delta), \quad \tau_i \in [0, \delta],$$
$$J_{1i} \le |A^{-1}(w^h(\tau_i) - x(\tau_i))|_H^2, \tag{17}$$
$$J_{4i} \le 2\delta(1+L)L|A^{-1}\left(w^h(\tau_i) - x(\tau_i)\right)|_H |A^{-1}|_{\mathcal{L}(H;H)}(h + \varphi_x(\delta)\} \le k_0\delta(h + \varphi_x(\delta)). \tag{18}$$

Here L is a Lipschitz constant for the function $\Phi(\cdot)$. Moreover by virtue of the boundedness of sets P_1 and P_2 and relations $A^{-1} \in \mathcal{L}(H;H)$, $\sigma \in \mathcal{L}(L_2(\Gamma);H)$, there is

$$J_{5i} \le k_1\delta^2, \tag{19}$$

where k_1 is a constant in $[0, +\infty)$. Notice that

$$A\int_0^t S(t-s)x\, ds = S(t)x - x \quad \forall\, x \in H.$$

Therefore,

$$|A^{-1}\{S(t)x - x\}|_H \le t|x|_H. \tag{20}$$

From (8), (13), (20) and Lemmas 2, 3, there follows

$$|s_i - s_i^*|_H = |A^{-1}\{S(\delta)(w^h(\tau_i) - x(\tau_i)) - (\psi_i - \varphi_i)\}|_H \le k_2(h + \delta). \tag{21}$$

Consequently,

$$J_{2i} \le 2\left(s_i^*, \int_0^\delta S(\delta - \tau)\sigma\Big(B_2^h v_2^h(\tau_i + \tau) - B_2 u_{2*}(\tau_i + \tau)\Big)\, d\tau\right)_H + k_3\delta\{h + \delta\}. \tag{22}$$

It is known [13] that

$$\sigma^* S(t)x = \frac{\partial}{\partial n}\Delta^{-1}S(t)x|_\Gamma \qquad \forall\, x \in H. \tag{23}$$

Thus, by virtue of (14), (22), (23),

$$J_{2i} + \alpha(h)\int_0^{\tau_{i+1}} \{|v_2^h(s)|_{U_2}^2 - |u_{2*}(s)|_{U_2}^2\}\, ds \le$$
$$\le (1 + k_3)\delta(h + \delta) + \alpha(h)\int_0^{\tau_i} \left\{|v_2^h(s)|_{U_2}^2 - |u_{2*}(s)|_{U_2}^2\right\} ds. \tag{24}$$

Analogously, using (20), Lemmas 2, 3, and the boundedness of the set $P(\cdot)$ in $L_\infty(T;U)$, we derive

$$J_{3i} + \alpha(h) \int\limits_0^{\tau_{i+1}} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds \le$$

$$\le 2\left(s_i, A^{-1}\int\limits_0^{\delta} B_1^h\Big(v_1^h(\tau_i+\tau) - u_{1*}(\tau_i+\tau)\Big)\, d\tau\right)_H +$$

$$+ \alpha(h) \int\limits_0^{\tau_{i+1}} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds + k_4(\delta^2 + h) \le$$

$$\le 2\left(s_i^*, A^{-1}\int\limits_0^{\delta} B_1^h\Big(v_1^h(\tau_i+\tau) - u_{1*}(\tau_i+\tau)\Big)\, d\tau\right)_H +$$

$$+ \alpha(h) \int\limits_{\tau_i}^{\tau_{i+1}} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds + k_5\delta(\delta+h) +$$

$$+ \alpha(h) \int\limits_0^{\tau_i} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds.$$

Hence, using (12), it follows that

$$J_{3i} + \alpha(h) \int\limits_0^{\tau_{i+1}} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds \le$$

$$\le \alpha(h) \int\limits_0^{\tau_i} \{|v_1^h(s)|^2_{U_1} - |u_{1*}(s)|^2_{U_1}\}\, ds + k_5\delta(\delta+h). \quad (25)$$

Taking into account estimates (16), (17)–(19), (24) and (25), and Lemmas 2, 3, we conclude that

$$\varepsilon_{i+1} \le \varepsilon_i + k_6(h + \varphi_x(\delta)) + k_7\delta(\delta+h) \le \varepsilon_0 + k_8\delta(\delta + h + \varphi_x(\delta)).$$

Thus,

$$\varepsilon_i \le |A^{-1}(x_0 - w_0)|^2_H + k_8\vartheta(\delta + h + \varphi_x(\delta)) \le k(\delta + h + \varphi_x(\delta)), \qquad i \in [1:m]. \quad (26)$$

Lemma is proved. □

Introduce the set

$$U_1(x(\cdot)) = \Bigg\{ u(\cdot) = \Big\{ u_1(\cdot), u_2(\cdot)\} \in P(\cdot) : A^{-1}\{x(t) - S(t)x_0 - \\ - \int_0^t S(t-\tau)\big\{f(\tau) + \Phi(x(\tau))\}\, d\tau\big\} = \\ = \int_0^t S(t-\tau)\sigma B_2 u_2(\tau)\, d\tau + A^{-1}\int_0^t S(t-\tau) B_1 u_1(\tau)\, d\tau \qquad \forall\, t \in T \Bigg\}. \tag{27}$$

Lemma 5. *The equality $U(x(\cdot)) = U_1(x(\cdot))$ is true.*

We give a proof for the case $B_1 \equiv 0$. Let $u(\cdot) = u_1(\cdot) \in U_1(x(\cdot))$. Then, by virtue of (3.13) [14] and the relation $\sigma \in L(L_2(\Gamma); L_2(\Omega))$, there is

$$t \to A \int_0^t S(t-\tau)\sigma B_2 u_2(\tau)\, d\tau \in L_2(T; H). \tag{28}$$

Applying the operator A to the right and left-hand sides of equality (27) and using (28), we obtain $u_1(\cdot) \in U(x(\cdot))$. The inverse statement may be proved in the same manner.

Lemma 6. *Let $x_{0i} \to x_0$ in H, $\gamma_i \to 0+$, $v_i(\cdot) = \{v^h_{1i}(\cdot), v^h_{2i}(\cdot)\} \to u_0(\cdot) = \{u_1(\cdot), u_2(\cdot)\}$ weakly in $L_2(T;U)$, $v_i(\cdot) \in P(\cdot)$, $v^h_{3i}(\cdot) \to \Phi(x(\cdot))$ in $L_2(T;H)$ as $i \to \infty$,*

$$\sup_{t\in T} |A^{-1}x(t) - A^{-1}w_i(t)|_H \le \gamma_i, \tag{29}$$

where $w_i(\cdot)$ is the solution of equation (9) for $v^h_j(\cdot) = v^h_{ji}(\cdot)$, $j \in [1:3]$. Then the inclusion

$$u_0(\cdot) \in U(x(\cdot))$$

is valid.

Proof. Introduce the function

$$x_*(t) = A^{-1}p(t; x_0, u_0(\cdot), x(\cdot)).$$

There is

$$\sup_{t\in T} |x_*(t) - A^{-1}x(t)|_H \le \sup_{t\in T} |x_*(t) - y_i(t)|_H + \sup_{t\in T} |y_i(t) - A^{-1}x(t)|_H, \\ y_i(t) = A^{-1}w_i(t). \tag{30}$$

By virtue of the assumption of the lemma, the second term on the right-hand side of inequality (30) tends to zero as $i \to \infty$. We shall show that this property is also valid for the first term.

Assuming the contrary, we conclude that for some subsequence $\{y_{i_j}(\cdot)\} \in \{y_i(\cdot)\}$, $t_j \to t_* \in T$ (as $j \to \infty$), the following inequality is true:

$$
\begin{aligned}
0 < \varepsilon < |x_*(t_j) - y_{i_j}(t_j)|_H^2 = \\
= \Bigg(A^{-1}\{S(t_j)(x_0 - x_{0i_j}) + A \int_0^{t_j} S(t_j - \tau)\sigma B_2\{u_2(\tau) - u_{2i_j}(\tau)\}\, d\tau + \\
+ \int_0^{t_j} S(t_j - \tau)\left\{ B_1\{u_1(\tau) - u_{1i_j}(\tau)) + \Phi(x(\tau)) - v_{3i_j}^h(\tau)\}\, d\tau \right\}, x_*(t_j) - y_{i_j}(t_j) \Bigg)_H \le \\
\le \Bigg(A^{-1}\{S(t_j)(x_0 - x_{0i_j}) + A \int_0^{t_j} S(t_j - \tau)\sigma B_2\{u_2(\tau) - u_{2i_j}(\tau)\}\, d\tau + \\
+ \int_0^{t_j} S(t_j - \tau)\Big\{ B_1\{u_1(\tau) - u_{1i_j}(\tau)) + \Phi(x(\tau)) - v_{3i_j}^h(\tau)\}\, d\tau \Big\}, \\
x_*(t_j) - A^{-1}x(t_j) \Bigg)_H + k_1\gamma_j. \quad (31)
\end{aligned}
$$

Let $K = \sup\limits_{t\in T, i} |x_*(t) - y_i(t)|_H$, $K_1 = \sup\limits_{t\in T} |S(t)|_{\mathcal{L}(H;H)}$. From (31) we derive

$$
\begin{aligned}
0 < \varepsilon < |x_*(t_j) - y_{i_j}(t_j)|_H^2 \le KK_1|A^{-1}|_{\mathcal{L}(H;H)}|x_0 - x_{0i_j}|_H + \\
+ \int_0^{t_j} \Big((S(t_j - \tau)\sigma B_2)^*(x_*(t_j) - y_{i_j}(t_j)), (u_2(\tau) - u_{2i_j}(\tau) \Big)_{U_2} d\tau + \\
+ \int_0^{t_j} \Big((A^{-1}S(t_j - \tau)B_1)^*(x_*(t_j) - A^{-1}x(t_j)), (u_1(\tau) - u_{1i_j}(\tau) \Big)_{U_1} d\tau + \\
+ K|A^{-1}|_{\mathcal{L}(H;H)} \int_0^{t_j} \Big| \Phi(x(\tau)) - v_{3i_j}^h(\tau) \Big|_H d\tau. \quad (32)
\end{aligned}
$$

However, taking into account the conditions of the lemma and continuity of the semigroup $\{S(t); t \ge 0\}$ and functions $x_*(t)$, $A^{-1}x(t)$ in H, we deduce that the right-hand side of inequality (32) tends to zero as $j \to \infty$. The contradiction obtained allows us to conclude that

$$\sup_{t\in T} |x_*(t) - A^{-1}x(t)|_H = 0,$$

i. e.,

$$p(\cdot; x_0, u_0(\cdot), x(\cdot)) = x(\cdot).$$

Thus, $u_0(\cdot) \in U(x(\cdot))$. Lemma is proved. □

Proof of Theorem. The proof is similar to the proofs of analogous statements (see [4, 6]). We shall show that for an arbitrary sequence $h_j \to 0+$ as $j \to \infty$, any family $\{\Delta_{h_j}\}$ of partitions of the interval T with diameters $\delta(h_j)$, $\{\delta(h_j) + h_j + \varphi_x(\delta(h_j))\}\alpha^{-1}(h_j) \to 0$ as $j \to \infty$, and any $\xi^{h_j}(\cdot) \in \Xi_{h_j}(x(\cdot))$ the convergence

$$\{v^h_{1j}(\cdot), v^h_{2j}(\cdot)\} \to u_*(\cdot; x(\cdot)) \quad \text{in} \quad L_2(T; U) \quad \text{as} \quad j \to \infty \tag{33}$$

takes place. Here controls $v^h_{1j}(\cdot)$ and $v^h_{2j}(\cdot)$ are determined according to (12), (14) for $h = h_j$, $\xi(\cdot) = \xi^{h_j}(\cdot)$. Assuming the contrary, we conclude that there exists a subsequence of the sequence $v^h_j(\cdot) = \{v^h_{1j}(\cdot), v^h_{2j}(\cdot)\}$ (for simplicity we denote it with the same symbol $v^h_j(\cdot)$) such that

$$\begin{gathered} v^h_j(\cdot) = \{v^h_{1j}(\cdot), v^h_{2j}(\cdot)\} \to v_0(\cdot) = \{v_{10}(\cdot), v_{20}(\cdot)\} \neq u_*(\cdot; x(\cdot)) \\ \text{weakly in} \quad L_2(T; U) \quad \text{as} \quad j \to \infty. \end{gathered} \tag{34}$$

It is easily seen from Lemma 4 that under the conditions of the theorem one can indicate a sequence $\{\gamma_j\}$, $\gamma_j \to 0$ as $j \to 0$ such that inequality (29) holds. Therefore, from Lemma 6 it follows that

$$|v_0(\cdot)|_{L_2(T;U)} \geq |u_*(\cdot; x(\cdot))|_{L_2(T;U)}. \tag{35}$$

Further, owing to the known properties of weak limit, we obtain

$$\varliminf_{j\to\infty} |v^h_j(\cdot)|_{L_2(T;U)} \geq |v_0(\cdot; x(\cdot))|_{L_2(T;U)}. \tag{36}$$

From Lemma 4 we derive

$$|v^h_j(\cdot)|^2_{L_2(T;U)} \leq |u_*(\cdot; x(\cdot))|^2_{L_2(T;U)} + k\{\delta(h_j) + h_j + \varphi_x(\delta(h_j)\}/\alpha(h_j),$$

where

$$|u_*(\cdot; x(\cdot))|^2_{L_2(T;U)} = |u_{1*}(\cdot; x(\cdot))|^2_{L_2(T;U_1)} + |u_{2*}(\cdot; x(\cdot))|^2_{L_2(T;U_2)}.$$

Consequently,

$$\varlimsup_{j\to\infty} |v^h_j(\cdot)|_{L_2(T;U)} \leq |u_*(\cdot; x(\cdot))|_{L_2(T;U)}. \tag{37}$$

From (35)–(37) we deduce that

$$\lim_{j\to\infty} |v^h_j(\cdot)|_{L_2(T;U)} = |u_*(\cdot; x(\cdot))|_{L_2(T;U)}. \tag{38}$$

However, in a uniformly convex Banach space strong convergence is a consequence of the weak convergence of functions and convergence of their norms. Therefore, taking into account (34), (38), we conclude that convergence (33) takes place. The theorem is proved. □

Acknowledgements

The work was partially supported by the Russian Foundation for Basic Research (Project 04-01-00059), the Program on Basic Research of the Presidium of the Russian Acad. Sci., the Ural-Siberian Integration Project, and the Program for support of leading scientific schools of Russia.

REFERENCES

[1] Banks H. T., Kunisch K., *Estimation techniques for distributed parameter systems*, Birkhäuser, Boston, 1989.

[2] Engl H. W., Hanke M., Neubauer A., *Regularization of inverse problems*, Kluwer, The Netherlands, 1996.

[3] Tikhonov A. N., Arsenin V., *Solutions of ill-posed problems*, Wiley, New York, 1977.

[4] Osipov Yu. S., Kryazhimskii A. V., *Inverse problems for ordinary differential equations: dynamical solutions*, Gordon and Breach, London, 1995.

[5] Osipov Yu. S., Kryazhimskii A. V., Maksimov V. I., *Dynamical inverse problems for parabolic systems*, Differential Equations 36 (2000) 5, 579–597 (in Russian).

[6] Maksimov V. I., *Dynamical inverse problems of distributed systems*, VSP, The Netherlands, 2002.

[7] Krasovskii N. N., Subbotin A. I., *Game-theoretical control problems*, Springer, Berlin, 1988.

[8] Maksimov V. I., Pandolfi L., *Dynamical reconstruction of inputs for constraction semigroup systems: the boundary input case*, J. Optimization Theory and Applications 103 (1999) 2, 117–129.

[9] Maksimov V. I., Pandolfi L., *On reconstruction of controls in nonlinear distributed systems*, Prikl. Mat. Mekh. 63 (1999) 2, 37–41 (in Russian).

[10] Lasiecka I., Triggiani R., *Differential and algebraic Riccati equations with applications to boundary/point control problems: continuous theory and approximation theory*, Lecture Notes in Control and Information Sciences, Springer-Verlag 164 (1991).

[11] Lasiecka I., Triggiani R., *Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems*, Appl. Math. and Optim. 23 (1991) 2, 109–154.

[12] Lasiecka I., *Boundary control of parabolic systems: regularity of optimal solutions*, Appl. Math. and Optim. 4 (1978) 4, 301–328.

[13] Barbu V., *Boundary control problems with convex cost criterion*, SIAM J. Control Optim. 18 (1980) 2, p. 227–243.

[14] Lasiecka I., *Unified theory for abstract parabolic boundary problems — a semigroup approach*, Appl. Math. and Optim. 6 (1980) 6, p. 287–334.

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Received: September 26, 2005.