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SPECIAL FUNCTIONS IN FUZZY ANALYSIS

Abstract. In the treatment of Fuzzy Logic an useful tool appears: the *membership function*, with the information about the degree of completion of a condition which defines the respective Fuzzy Set or Fuzzy Relation. With their introduction, it is possible to prove some results on the foundations of Fuzzy Logic and open new ways in Fuzzy Analysis.

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1. INTRODUCTION TO FUZZY THEORY

When we solve problems in A. I., their representation will be through the Fuzzy Logic techniques, a very useful procedure. For instance, in the problems related to the "real world". As you know, it is one of the "possible worlds" only. We define the *"world"* as *"a complete and consistent description of how the things are or how they could have been"*.

In solving questions of this type the Monotonic Logic often does not work, whereas such type of Logic, is classical in formal worlds, such as Mathematics.

Other Non-Monotonic Logics must also be introduced, where now the extension of the set of sentences can modify the conclusion. This happens frequently in the real world: for instance, in the medical sciences, or in the common sense reasoning, with partial information, giving temporal, revisable and provisional conclusions.

For their part, we need Fuzzy Sets, Fuzzy Relations and so on, to describe the gradation of certainty in our world. It is shown through a new function, the aforementioned *membership function* μ, which describes the degree of fulfillment for each element of the property defining the set. Or the "degree of relation" between two determined elements. Such "membership degree" value can be assigned by the corresponding μ, the *"membership function"*, whose range is the closed unit interval $[0, 1]$. Thus

$$\mu : C \to [0, 1] \,.$$

According to this, a fuzzy set can be defined as:

$$C = \{x \mid \mu(x), \forall x \in U\}$$

where the vertical symbol | does not mean "such that", but it adjoins the information on the "membership degree" of such element to the set C.

Given $\{U_i\}_{i=1}^n$, n universes of discourse, we define a *fuzzy relation* R through a membership function that to each n-tuple, $(x_i)_{i=1}^n$, where $x_i \in U_i$, $i = 1, 2, \ldots, n$, associates a value in the closed unit interval $[0, 1]$:

$$(x_i)_{i=1}^n \in \prod_{i=1}^n U_i \to r \in [0,1],$$

$$(x_1,\ x_2,\ \ldots,\ x_n) \to \mu(x_1,\ x_2,\ \ldots,\ x_n) = r : 0 \le r \le 1.$$

The fuzzy relation R can be defined through such "membership function", μ. In this manner, we will have gradation of the relationship:

$$\rceil R, \ldots, \frac{1}{3}R, \ldots, \frac{1}{2}R, \ldots, \frac{2}{3}R, \ldots, R.$$

The Cartesian product of two fuzzy sets, F (in the universe U_1) and G (in the universe U_2), is the following fuzzy binary relation:

$$F \times G = \left\{(x,\ y) \mid \mu_{F\ \times\ G}(x, y) = \min\left[\mu_F(x),\ \mu_G(y)\right], \forall x \in U_1, \forall y \in U_2\right\}.$$

As you know, we can consider this fuzzy relation as a subset of the adequate Cartesian product: $R \subset F \times G$.

The composition of fuzzy relations can be defined by:

$$R_1(U_1,\ U_2) \circ R_2(U_2,\ U_3) = R_3(U_1,\ U_3)$$

where:

$$R_3(U_1,\ U_3) = \left\{(x,\ z) \mid \mu_{R_1 \circ R_2}(x,\ z), \forall x \in U_1, \forall z \in U_3\right\},$$

being:

$$\mu_{R_1 \circ R_2}(x, z) = \max\left\{\forall y \in U_2 : \min\ \left(\mu_{R_1}(x, y),\ \mu_{R_2}(y, z)\right)\right\} =$$
$$= \max_{y \in U_2}\left\{\ \min\ \left(\mu_{R_1}(x, y),\ \mu_{R_2}(y, z)\right)\right\}.$$

There exists a clear analogy between the composition of fuzzy relations and the matricial product. For this reason, the composition ($\circ$) of fuzzy relations can be also denominated as the: *"max-min matricial product"*. As a particular case of the composition of fuzzy relations, we can introduce the composition of a fuzzy set with a fuzzy relation. Obviously, in such case, the fuzzy set may be represented by a row or column matrix. These can be very useful in *"Fuzzy Inference"*.

2. THE NON-BOOLEAN ALGEBRA OF FUZZY SETS

We can introduce *new generalized versions of the Classical Logic*: the *Modus Ponens Generalized*, or the *Modus Tollens Generalized*, and also the *Hypothetic Syllogism.*

To each *Fuzzy Predicate*, we can associate *a Fuzzy Set*, defined by such property, that is, composed of the such elements of the universe of discourse which totally or partially verify such condition.

So, we can prove that *the class of Fuzzy Sets* with the operations: $\cup$, $\cap$ *and* c (c being passing to the complement) *does not constitute a Boolean Algebra*, because neither the Contradiction Law nor the Third Excluded Principle work in it. Both proofs can be expressed easily, in algebraic or geometrical way.

In the first case, the algebraic proof consists in seeing how the equality:

$$X \cup c\,(X) = U$$

may not hold. Because, if we restate the problem by means of membership functions:

$$\mu_{X \,\cup\, c(X)} = \mu_U,$$

we see that:

$$\mu_{X \,\cup\, c(X)}\,(x) = \mu_U\,(x)\,, \quad \forall x \in U$$

is wrong, for some $x \in U$.

We know that

$$\mu_U\,(x) = 1, \forall x$$

because $x \in U$ necessarily: all our elements are in the Universe.

In the first member:

$$\mu_{X \cup c(X)} = \left(\max\ \left\{\mu_X, \mu_{c\ (X)}\right\}\right)(x) = (\max\ \{\mu_X,\ 1 - \mu_X\})\,(x)$$

through the definition of the membership function for the union of fuzzy sets.

But if we take: $x \mid \mu_X\,(x)$, such that: $0 < \mu_X\,(x) < 1$, strictly into the unit interval, it fails clearly.

Because, if for instance: $\mu_X\,(x) = 0.2$, then $\mu_{c(X)}\,(x) = 0.8$, and we will obtain:

$$\mu_{X \,\cup\, c(X)}\,(x) = \max\ \{0.2,\ 0.8\} = 0.8 \neq 1 = \mu_U\,(x)\,.$$

This clearly fails too.

Therefore, *the family of Fuzzy Sets does not verify the Third Excluded Law.* Through a geometrical procedure, it can be shown with an easy diagram.

For the *Contradiction Law*, in Fuzzy Sets, we must prove the possibility of:

$$F \cap c\,(F) \neq \emptyset$$

or equivalently, the existence of an $x \in U$ such that:

$$\mu_{F \,\cap\, c(F)}(x) = \min\,\left(\mu_F(x),\ \mu_{c(F)}(x)\right) = \min\,\left(\mu_F(x),\ 1 - \mu_F(x)\right) \neq 0 = \mu_{\emptyset}(x)\,.$$

As in the previous case, it is enough to take x with membership degree between 0 and 1, both excluded.

For instance, if: $\mu_F(x) = 0.3$, then $\mu_{c(F)}(x) = 0.7$. So:

$$\mu_{F \,\cap\, c(F)}(x) = \min\,\left(\mu_F(x),\ \mu_{c(F)}(x)\right) = \min\,(0.3,\ 0.7) = 0.3 \neq 0 = \mu_{\emptyset}(x)\,.$$

Also here, through geometrical procedures, it can be shown with an diagram (Fig. 1).

Therefore, we conclude that, in general, *the Contradiction Law does not work in the class of Fuzzy Sets.*

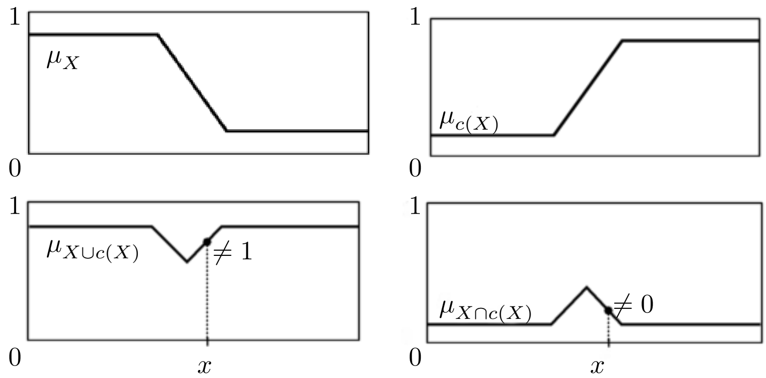

Fig. 1

Note that, for the union and intersection of fuzzy sets, the membership function can be defined by:

$$\mu_{F\cup G}(x) = \max\left\{\mu_F(x), \mu_G(x)\right\},$$
$$\mu_{F\cap G}(x) = \min\left\{\mu_F(x), \mu_G(x)\right\}.$$

The fuzzy relations can be composed in this way: Let R_1 and R_2 be two fuzzy relations. Then, their *composition* R_3 is defined by: $R_3 = R_2 \circ R_1$, where

$$R_3(i,\ j) = \max\ \left[\min\ \left\{R_1(i,\ k), R_2(k,\ j)\right\}\right].$$

Observe the possibility of immediate translation into matricial language.

3. DIFFERENCE OF FUZZY SETS

If we take two sets, A and B, the *difference* is given by:

$$A - B = A \cap c\,(B)\,.$$

There exist two means of obtaining the difference between fuzzy sets:

Simple method: For instance, if we take:

$$A = \{a \mid 0.1,\ b \mid 0.3,\ c \mid 0.6,\ d \mid 0.9\}\,,$$
$$B = \{a \mid 0.2,\ b \mid 0.5,\ c \mid 0.8,\ d \mid 1\}\,,$$

then:

$$c\,(B) = \{a \mid 0.8,\ b \mid 0.5,\ c \mid 0.2,\ d \mid 0\}\,.$$

Therefore:

$$A - B = A \cap c\,(B) = \{a \mid 0.1,\ b \mid 0.3,\ c \mid 0.2,\ d \mid 0\}\,.$$

Bounded Difference is defined through a new operator θ, by means of the membership function:

$$\mu_{A\ \theta\ B}\,(x) = \max\left\{\mu_A\,(x) - \mu_B\,(x)\,,\ 0\right\}.$$

It is clear that it does *not* verify the *commutative property*, because:

$$B\ \theta\ A = \{a \mid 0.1,\ b \mid 0.2,\ c \mid 0.2,\ d \mid 0.1\} \neq A\ \theta\ B.$$

To introduce the *distance between fuzzy sets* A and B, we can consider different *possibilities,* now based in the values of the membership functions at the point $x \in U$:

i) the well known *Euclidean distance:*

$$e\,(A,\ B) = \left[\sum \{\mu_A\,(x) - \mu_B\,(x)\}^2\right]^{1/2};$$

ii) the *Hamming distance:*

$$d\,(A,\ B) = \sum |\mu_A\,(x_i) - \mu_B\,(x_i)|$$

with $i \in \{1, 2, \ldots n\}$ and $x_i \in U$, universe of discourse.

We can easily prove the four conditions required of any distance.

One can also define the *relative Hamming distance* (δ), when the universal set U is finite, for instance, with n elements:

$$\text{if } \sharp (U) = n \Rightarrow \delta (A, B) = \frac{1}{n} d (A, B) .$$

For instance, let A and B as in the aforementioned example. Then:

$$\begin{aligned} e (A, B) &= 0.316, \\ d (A, B) &= 0.6, \\ \delta (A, B) &= \frac{1}{n} d (A, B) . \end{aligned}$$

So, if $n = 4$, then:

$$\delta (A, B) = \frac{1}{4}\ d (A, B) = 0.15.$$

And generalizing, we can also define the *Minkowskian distance:*

$$d_w (A, B) = \left[\sum \left| \mu_A (x) - \mu_B (x) \right|^w \right]^{1/w}$$

where $w \in [1, +\infty]$.

Observe that when $w = 1$, we obtain the *Hamming distance*.

And when $w = 2$, we find the *Euclidean distance*.

Both are therefore, particular cases of *Minkowskian distances*.

4. FUZZY DISTANCE BETWEEN FUZZY SETS

We need to introduce the *Extension Principle*, according to which: If we start from the Cartesian product of universal sets:

$$U = \prod U_i, \quad i = 1, 2, \ldots , r$$

and a collection of fuzzy sets, each one in the corresponding universal set:

$$A_i \subseteq U_i, \quad i = 1, 2, \ldots r,$$

then we define the *cartesian product of fuzzy sets:*

$$\prod A_i, \quad i = 1, 2, \ldots r$$

through their membership function:

$$\mu_{\prod A_i} (x_1,\ x_2,\ \ldots,\ x_r) \equiv \min\ \left\{ \mu_{A_1} (x_1) ,\ \mu_{A_2} (x_2) ,\ \ldots,\ \mu_{A_r} (x_r) \right\} .$$

Let F be the function from the universe U to the universe V. Then the fuzzy set:

$$B \subseteq V$$

can be obtained using F and the collection of fuzzy sets, $\{A_i\}_{i=1}^{r}$, in the following way:

$$\mu_B(y) = 0, \text{ if } F^{-1}(y) = \emptyset$$

and

$$\mu_B(y) = \max_{y=F(x_1,x_2,\ldots,x_r)} \left[\min \left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_r}(x_r)\right\}\right], \text{ if } F^{-1}(y) \neq \emptyset.$$

If the function F is *one-to-one,* then:

$$\mu_B(y) = \mu_A\left(F^{-1}[y]\right), \text{ when } F^{-1}(y) \neq \emptyset.$$

Let $(U,\ d)$ be a *pseudometric space.* Therefore, with:

$$d : U^2 \to R_+ \cup \{0\}$$

such that verifies:

1) $d(x,x) = 0,\ \forall x \in U$;

2) $d(x,y) = d(y,x), \forall x,y \in U$;

3) $d(x,z) \leq d(x,y) + d(y,z), \forall x,y,z \in U$.

Remember also that the additional condition:

4) *if* $d(x,y) = 0$, *then* $x = y$

converts d in a distance, and then $(U,\ d)$ is a *metric space.*

In our pseudometric space $(U,\ d)$, if we take two fuzzy subsets A and B, it is possible by the extension principle to introduce the *pseudometric distance between* A *and* B:

$$\forall \rho \in R_+,\ \mu_{d(A,B)}(\rho) = \max_{\rho=d(a,b)} \left[\min \{\mu_A(a), \mu_B(b)\}\right].$$

And it is also a *fuzzy set.*

5. FINAL REMARK

The considerations above, concerning some special functions in A. I., allow us to approach various problems in A. I.

Obviously, it will be necessary to examine which results remain true and which fail in the Fuzzy Analysis, in comparison with the Classical Analysis results.

REFERENCES

[1] W.J. Clancey: *Notes on heuristic classification*, Artificial Intelligence 59 (1993), 191–196.

[2] A. Garrido: *Matrix Theory and Artificial Iinteligence*, International Conference on Matrix Theory, Technion, Haifa, 2005.

[3] A. Garrido: *Logics in A. I.*, Logic in Hungary, Univ. Corvinus, Budapest, Janos Bolyai Math. Society, 2005.

[4] M. Ginsberg: *Essentials of artificial intelligence*, Morgan Kaufmann Publ., San Francisco, Cal., 1993.

[5] A. Kandel: *Fuzzy Mathematical Techniques with Applications.* Addison-Wesley Publ. Co., Mass., 1986.

[6] J. Mira *et al.*: *Aspectos Básicos de la I. A.* Sanz y Torres. Madrid, 1995.

[7] N.J. Nilsson: *Principles of Artificial Intelligence.* Tioga Publ. Co., Palo Alto, California, 1980

[8] W. Pedrycz: *Fuzzy Control and Fuzzy Systems. 2nd extended edition.*, John Wiley & Sons, New York, 1993.

[9] L.-X. Wang: *A Course in Fuzzy Systems and Control*, Prentice-Hall, 1997.

[10] L.A. Zadeh, J. Kacprzyk: *Fuzzy Logic for the Management of Uncertainty*, John Wiley & Sons, New York, 1992.

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