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ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS

Abstract. In this paper we study asymptotic behavior of solutions of a higher order neutral difference equation of the form

$$\Delta^m(x_n + p_n x_{n-\tau}) + f(n, x_{\sigma(n)}) = h_n.$$

We present conditions under which all nonoscillatory solutions of the above equation have the property $x_n = cn^{m-1} + o(n^{m-1})$ for some $c \in R$.

Keywords: neutral difference equation, asymptotic behavior, nonoscillatory solution.

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1. INTRODUCTION

In this paper we consider a higher order neutral type difference equation of the form

$$\Delta^m(x_n + p_n x_{n-\tau}) + f(n, x_{\sigma(n)}) = h_n, \quad n = 1, 2, \dots \tag{E}$$

where $m \geq 2$, $(p_n), (h_n)$ are sequences of real numbers, τ is a nonnegative integer, $(\sigma(n))$ is a sequence of integers with $\sigma(n) \leq n$ and $\lim\limits_{n \longrightarrow \infty} \sigma(n) = \infty$, $f : N \times R \longrightarrow R$. For all $k \in N$ we use the usual factorial notation

$$n^{\underline{k}} = n(n-1)\dots(n-k+1) \quad \text{with} \quad n^{\underline{0}} = 1.$$

By a solution of equation (E) we mean a real sequence (x_n) which is defined for $n \geq \min\limits_{i \geq 1}\{i - \tau, \sigma(i)\}$ and which satisfies equation (E) for all $n = 1, 2 \dots$. A nontrivial solution (x_n) of equation (E) is said to be nonoscillatory if it is eventually positive or eventually negative.

Recently there has been an increasing interest in the study of the qualitative behavior of solutions of higher order neutral difference equations, see for example [3, 4, 7–14] and the references cited therein.

The purpose of this paper is to establish conditions under which each nonoscillatory solution of equation (E) has the property $x_n = cn^{m-1} + o(n^{m-1})$ for some $c \in R$. Similar problem for non-neutral difference equations was considered in [1, 2, 5, 13]. For a less general neutral difference equations (with $m = 2$, $p_n = p$, $\sigma(n) = n$ and $h_n \equiv 0$), we refer to [6] .

2. MAIN RESULTS

In the proof of the main result we will need the following lemmas.

Lemma 1. (See [7]) *Let (a_n) be any real sequence. Then*

$$\sum_{i_m=N_0}^{n-1} \sum_{i_{m-1}=N_0}^{i_m-1} \cdots \sum_{i_1=N_0}^{i_2-1} a_{i_1} = \sum_{j=N_0}^{n-1} \frac{(n-j-1)^{\underline{m-1}}}{(m-1)!} a_j.$$

Lemma 2. (See [6]) *Let $x, u : N \longrightarrow R$ be sequences and define*

$$z_n = x_n + u_n x_{n+k}, \quad n \geq \max\{0, -k\},$$

where k is an integer. Assume that (x_n) is bounded, $\lim_{n\longrightarrow\infty} z_n = l \in R$ and $\lim_{n\longrightarrow\infty} u_n = p \in R$. Then the following statements hold:

(i) *if $p = -1$, then $l = 0$;*
(ii) *if $|p| \neq 1$, then (x_n) is convergent and $\lim_{n\longrightarrow\infty} x_n = \frac{l}{1+p}$.*

Theorem 1. *Suppose that $p_n \geq 0$, $\lim_{n\longrightarrow\infty} p_n = p \neq 1$ and:*

(i) *$f(n, u)$ is continuous in u;*
(ii) *there exists a continuous function $g : R_+ \longrightarrow R_+$ and a sequence $\phi : N \longrightarrow R_+$ such that*

$$|f(n, u)| \leq \phi_n g\left(\frac{|u|}{n^{\underline{m-1}}}\right) \quad for \quad n = 1, 2, \ldots,$$

where

$$\sum_{n=1}^{\infty} \phi_n < \infty$$

and the function g is nondecreasing and

$$G(x) = \int_1^x \frac{ds}{g(s)} \longrightarrow \infty \quad as \quad x \longrightarrow \infty; \tag{1}$$

(iii) $\sum_{n=1}^{\infty} |h_n| < \infty$.

Then for every nonoscillatory solution (x_n) *of equation* (E) *there exists a real constant* c *such that*

$$\lim_{n\longrightarrow\infty} \frac{\Delta^i x_n}{n^{m-i-1}} = \frac{c}{(m-i-1)!}, \quad i = 0, 1, \ldots, m-1.$$

Proof. Let (x_n) be a nonoscillatory solution of equation (E). Then there exists an integer $n_0 \geq 1$, such that $x_n > 0$ or $x_n < 0$ for all $n \geq n_0$. Set

$$z_n = x_n + p_n x_{n-\tau}. \tag{2}$$

Then $|z_n| > |x_n|$ for $n \geq n_1 = n_0 + \tau$. It follows from (E) that

$$\Delta^m z_n = h_n - f(n, x_{\sigma(n)}).$$

Let us denote $\Delta^i z_{n_1} = c_i$ for $i = 0, 1, \ldots m-1$. Summing the above equation from n_1 to $n-1$, we obtain

$$\Delta^{m-1} z_n = c_{m-1} + \sum_{j=n_1}^{n-1} h_j - \sum_{j=n_1}^{n-1} f(j, x_{\sigma(j)}). \tag{3}$$

Summing again, we get

$$\Delta^{m-2} z_n = c_{m-1}(n-n_1) + c_{m-2} + \sum_{j=n_1}^{n-1} \sum_{i=n_1}^{j-1} h_i - \sum_{j=n_1}^{n-1} \sum_{i=n_1}^{j-1} f(i, x_{\sigma(i)}),$$

and after m steps, we obtain

$$|z_n| \leq |c_0| + |c_1|n + |c_2|\frac{n^2}{2!} + \ldots + |c_{m-1}|\frac{n^{m-1}}{(m-1)!} +$$
$$+ \sum_{i_m=n_1}^{n-1} \sum_{i_{m-1}=n_1}^{i_m-1} \cdots \sum_{i_1=n_1}^{i_2-1} |h_i| + \sum_{i_m=n_1}^{n-1} \sum_{i_{m-1}=n_1}^{i_m-1} \cdots \sum_{i_1=n_1}^{i_2-1} |f(i, x_{\sigma(i)})|.$$

Using Lemma 1, we get

$$|z_n| \leq \sum_{i=0}^{m-1} \frac{n^i}{i!}|c_i| + \sum_{j=n_1}^{n-1} \frac{(n-j-1)^{m-1}}{(m-1)!}|h_i)| + \sum_{j=n_1}^{n-1} \frac{(n-j-1)^{m-1}}{(m-1)!}|f(i, x_{\sigma(i)})| \leq$$
$$\leq n^{m-1} \sum_{i=0}^{m-1} |c_i| + n^{m-1} \sum_{j=n_1}^{n-1} |h_i| + n^{m-1} \sum_{j=n_1}^{n-1} |f(i, x_{\sigma(i)})|.$$

Hence, by (iii), we obtain

$$\frac{|z_n|}{n^{m-1}} \leq \sum_{i=0}^{m-1} |c_i| + \sum_{j=n_1}^{n-1} |h_j| + \sum_{j=n_1}^{n-1} |f(j, x_{\sigma(j)})| \leq A + \sum_{j=n_1}^{n-1} |f(j, x_{\sigma(j)})|, \tag{4}$$

where A is an appropriate constant.

By (ii) it is obvious that

$$|f(n, x_{\sigma(n)})| \leq \phi_n g\left(\frac{|x_{\sigma(n)}|}{n^{m-1}}\right) \leq \phi_n g\left(\frac{|z_{\sigma(n)}|}{n^{m-1}}\right). \tag{5}$$

Now, let us denote

$$b_n = A + \sum_{j=n_1}^{n-1} |f(j, x_{\sigma(j)})|, \qquad n \geq n_1. \tag{6}$$

Then, by (4) and (6)

$$\frac{|z_n|}{n^{m-1}} \leq b_n, \qquad n \geq n_1. \tag{7}$$

Let $n_2 \geq n_1$ be large enough for $\sigma(n) \geq n_1$ if $n \geq n_2$. Then, since (b_n) is nondecreasing, by (7) we get

$$\frac{|z_{\sigma(n)}|}{n^{m-1}} \leq \frac{|z_{\sigma(n)}|}{(\sigma(n))^{m-1}} \leq b_{\sigma(n)} \leq b_n, \qquad \text{for } n \geq n_2. \tag{8}$$

Hence, by (5), (6), we obtain

$$\Delta b_i \leq \phi_i g\left(\frac{|z_{\sigma(i)})|}{i^{m-1}}\right) \leq \phi_i g(b_i)$$

and therefore we get

$$\int_{b_i}^{b_{i+1}} \frac{ds}{g(s)} \leq \frac{\Delta b_i}{g\,(b_i)} \leq \phi_i.$$

Summing both sides of the above inequality from n_2 to $n-1$, we obtain

$$\int_{b_{n_2}}^{b_n} \frac{ds}{g(s)} \leq \sum_{i=n_2}^{n-1} \phi_i.$$

In view of the definition of the function G, this implies that

$$G(b_n) \leq G(b_{n_2}) + \sum_{i=n_2}^{n-1} \phi_i.$$

From (1) and the properties of the function g, the function G^{-1} exists, is positive and nondecreasing. So, we get $b_n \leq G^{-1}(G(b_{n_2}) + \sum_{i=n_2}^{n-1} \phi_i)$. Therefore, using (8) we obtain

$$\frac{|z_{\sigma(n)}|}{n^{m-1}} \leq G^{-1}(G(b_{n_2}) + \sum_{i=n_2}^{n-1} \phi_i) \leq G^{-1}(G(b_{n_2}) + \sum_{i=n_2}^{\infty} \phi_i). \tag{9}$$

We put

$$k_1 = G(b_{n_2}) + \sum_{i=n_2}^{\infty} \phi_i.$$

Then, from (9), we get

$$\frac{|z_{\sigma(n)}|}{n^{m-1}} \le G^{-1}(k_1) = k_2, \quad \text{for every} \quad n \ge n_2.$$

On the other hand, by (ii) there is

$$\sum_{i=n_2}^{n-1} |f(i, x_{\sigma(i)})| \le \sum_{i=n_2}^{n-1} \phi_i g\left(\frac{|x_{\sigma(i)}|}{i^{m-1}}\right) \le \sum_{i=n_2}^{n-1} \phi_i g\left(\frac{|z_{\sigma(i)}|}{i^{m-1}}\right) \le$$
$$\le g(k_2) \sum_{i=n_2}^{n-1} \phi_i.$$

Therefore, the series $\sum_{i=1}^{\infty} f(i, x_{\sigma(i)})$ is absolutely convergent and from (3) and (iii) we see that there exists $a \in R$, such that

$$\lim_{n \longrightarrow \infty} \Delta^{m-1} z_n = a.$$

Then, by Stolz Theorem, we obtain

$$\lim_{n \longrightarrow \infty} \frac{(m-1)! z_n}{n^{m-1}} = \ldots = \lim_{n \longrightarrow \infty} \frac{\Delta^{m-2} z_n}{n} = \lim_{n \longrightarrow \infty} \Delta^{m-1} z_n = a.$$

Hence, $\lim_{n \longrightarrow \infty} \frac{z_n}{n^{m-1}} = \frac{a}{(m-1)!}$.

Now we put $w_n = \frac{z_n}{n^{m-1}}$ and $y_n = \frac{x_n}{n^{m-1}}$. Then (2) implies

$$w_n = y_n + u_n y_{n-\tau},$$

where $u_n = p_n \frac{(n-\tau)^{m-1}}{n^{m-1}}$. Note that $|y_n| = \frac{|x_n|}{n^{m-1}} \le \frac{|z_n|}{n^{m-1}} \le k_2$, so (y_n) is bounded. By Lemma 2, there is

$$\lim_{n \longrightarrow \infty} \frac{x_n}{n^{m-1}} = \lim_{n \longrightarrow \infty} y_n = \frac{c}{(m-1)!},$$

where $c = \frac{a}{1+p}$. This completes the proof. □

Theorem 1 improves Theorem 1 in [6].

Remark 1. *If in the proof of Theorem 1 we choose c_{m-1} sufficiently large, then $\lim_{n \longrightarrow \infty} \Delta^{m-1} z_n \neq 0$ and the corresponding solution (x_n) of equation* (E) *has the property*

$$x_n = c n^{m-1} + o(n^{m-1}),$$

where $c \neq 0$.

Theorem 1 applied to the linear equation

$$\Delta^m(x_n + p_n x_{n-\tau}) + a_n x_{\sigma(n)} = 0, \tag{E1}$$

where $m \geq 2$, (p_n) is a sequence of real numbers, $p_n \geq 0$, $\lim_{n\longrightarrow\infty} p_n = p \neq 1$, τ is a nonnegative integer, (a_n) is a sequences of real numbers, $(\sigma(n))$ is a sequences of integers with $\sigma(n) \leq n$ and $\lim_{n\longrightarrow\infty} \sigma(n) = \infty$, leads to the following corollary.

Corollary 1. *Assume that*

$$\sum_{j=1}^{\infty} j^{m-1}|a_j| < \infty.$$

Then every nonoscillatory solution (x_n) *of equation* (E1) *has the asymptotic property*

$$x_n = cn^{m-1} + o(n^{m-1}), \tag{10}$$

where c *is a real constant.*

Proof. The conclusion of Corollary 1 follows from Theorem 1 with

$$\phi_n = n^{m-1}|a_n| \qquad \text{and} \qquad g(u) = u.$$

□

Example 1. *Consider the difference equation*

$$\Delta^3\left(x_n + \frac{1}{2}x_{n-1}\right) + \frac{1}{4}\frac{1}{2^n n^2 + 1}x_n = 0, \quad n \geq 1. \tag{11}$$

From Corollary 1 it follows that every nonoscillatory solution of equation (11) *has the asymptotic property* (10). *In particular,* $x_n = n^2 - n + \frac{1}{2^n}$ *is one such solution.*

Corollary 2. *Consider the difference equation*

$$\Delta^m(x_n + p_n x_{n-k}) + a_n x_n^{\alpha} = 0, \quad 0 < \alpha < 1 \tag{12}$$

where (p_n), (a_n) *are sequences of real numbers;* $p_n \geq 0$, $\lim_{n\longrightarrow\infty} p_n = p \neq 1$ *and* τ *is nonnegative integer. If*

$$\sum_{j=1}^{\infty} j^{\alpha(m-1)}|a_j| < \infty,$$

then every nonoscillatory solution (x_n) *of equation* (12) *has the property* $x_n = cn^{m-1} + o(n^{m-1})$, *where* c *is a real number.*

Proof. Apply Theorem 1 with $\phi_n = n^{\alpha(m-1)}|a_n|$ and $g(u) = u^{\alpha}$.

□

Example 2. *Consider the difference equation*

$$\Delta^4(x_n + 2x_{n-1}) - \frac{36n+84}{(n+4)^{\frac{5}{2}}} x_n^{\frac{1}{2}} = 0, \quad n \geq 1, \tag{13}$$

All conditions of Corollary 2 are satisfied. It is easy to check that $x_n = \frac{1}{n}$ *is a solution of equation* (13) *with the required property.*

For $p_n \equiv 0$, equation (E) takes the form

$$\Delta^m x_n + f(n, x_{\sigma(n)}) = h_n, \quad n = 1, 2, \ldots. \tag{E2}$$

From the proof of Theorem 1 for the non-neutral equation (E2) we obtain following result.

Theorem 2. *Let assumptions* (i),(ii), (iii) *of Theorem 1 be satisfied. Then for every solution* (x_n) *of equation* (E2) *there exists a real constant* c *such that*

$$\lim_{n \longrightarrow \infty} \frac{\Delta^i x_n}{n^{m-i-1}} = \frac{c}{(m-i-1)!}, \quad i = 0, 1, \ldots, m-1.$$

Note, that Theorem 1 holds for all nonoscillatory solutions of equation (E), while Theorem 2 is true for all solution of equation (E2).
Compare this result with Theorem 4 in [13].

Corollary 3. *Let assumptions* (i)–(iii) *of Theorem 1 be satisfied. Then every solution* (x_n) *of equation* (E2) *has the asymptotic property*

$$x_n = cn^{m-1} + o(n^{m-1}),$$

where c *is a real constant.*

REFERENCES

[1] A. Drozdowicz, M. Migda, *Asymptotic behavior of solutions of the mth-order nonhomogeneous difference equations,* J. Comput. Appl. Math. 81 (1997) 1, 75–81.

[2] A. Gleska, J. Werbowski, *Comparison theorems for the asymptotic behavior of solutions of nonlinear difference equations,* J. Math. Anal. Appl. 226 (1998) 2, 456–465.

[3] W.T. Li, S.S. Cheng, *Asymptotic trichotomy for positive solutions of a class of odd order nonlinear neutral difference equations*, Computers Math. Applic. 35 (1998), 101–108.

[4] J.R. Graef, A. Miciano, P. Spikes, P. Sundaram, E. Thandapani, *Oscilatory and asymptotic behavior of solutions of nonlinear neutral-type difference equations*, J. Austral. Math. Soc. 38 (1996), 163–171.

[5] M. Migda, J. Migda, *On the asymptotic behavior of solutions of higher order nonlinear difference equations,* Nonlinear Anal. 47 (2001) 7, 4687–4695.

[6] M. Migda, J. Migda, *Asymptotic properties of solutions of second-order neutral difference equations,* Nonlinear Anal. 63 (2005), e789–e799.

[7] M. Migda, *On unstable neutral difference equations of higher order,* Indian J. Pure Appl. Math. 36(10) (2005), 557–567.

[8] N. Parhi, A.K. Tripathy, *Oscillation of a class of nonlinear neutral difference equations of higher order*, J. Math. Anal. Appl. 284 (2003), 756-774.

[9] B. Szmanda, *Note on the behaviour of solutions of difference equations of arbitrary order*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 8 (1997), 52–59.

[10] E. Thandapani, P. Sundaram, J.R. Graef, A. Miciano, P. Spikes, *Classification of nonoscillatory solutions of higher order neutral type difference equations*, Arch. Math. (Brno) 31 (1995), 263–277.

[11] E. Thandapani, P. Sundaram, J.R. Graef, P.W. Spikes, *Asymptotic behaviour and oscillation of solutions of neutral delay difference equations of arbitrary order,* Math. Slovaca 47 (1997), 539–551.

[12] A. Zafer, Y. Yalin, Y.S. Yilmaz, *Positive solutions and oscillation of higher order neutral difference equations,* Arch. Math. (Brno) 36 (2000), 623–636.

[13] A. Zafer, *Oscillatory and asymptotic behavior of higher order difference equations,* Math. Comput. Modelling 21, 4(1995), 43–50.

[14] Y. Zhou, B.G. Zhang, *Existence of nonoscillatory solutions of higher-order neutral delay difference equations with variable coefficients,* Computers Math. Applic. 45 (2003), 991–1000.

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