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## NEARLY PERFECT SETS IN THE $n$ -FOLD PRODUCTS OF GRAPHS

**Abstract.** The study of nearly perfect sets in graphs was initiated in [2]. Let  $S \subseteq V(G)$ . We say that  $S$  is a *nearly perfect* set (or is *nearly perfect*) in  $G$  if every vertex in  $V(G) - S$  is adjacent to at most one vertex in  $S$ . A nearly perfect set  $S$  in  $G$  is called *1-maximal* if for every vertex  $u \in V(G) - S$ ,  $S \cup \{u\}$  is not nearly perfect in  $G$ . We denote the minimum cardinality of a 1-maximal nearly perfect set in  $G$  by  $n_p(G)$ . We will call the 1-maximal nearly perfect set of the cardinality  $n_p(G)$  an  $n_p(G)$  - set. In this paper, we evaluate the parameter  $n_p(G)$  for some  $n$ -fold products of graphs. To this effect, we determine 1-maximal nearly perfect sets in the  $n$ -fold Cartesian product of graphs and in the  $n$ -fold strong product of graphs.

**Keywords:** dominating sets, product of graphs.

**Mathematics Subject Classification:** 05C69, 05C70.

### 1. INTRODUCTION

Let  $G$  be a simple graph and  $u \in V(G)$ . By  $N_G(u)$  we denote the *open neighbourhood* of  $u$ ; i.e.,  $N_G(u) = \{v: uv \in E(G)\}$ . We say that the subset  $A \subseteq V(G)$  is called *independent* in  $G$  if  $N_G(x) \cap A = \emptyset$  for every vertex  $x \in A$ . We denote the cardinality of the maximum independent set in  $G$  by  $\alpha(G)$ . The subset  $A \subseteq V(G)$  is called a *perfect dominating* set in  $G$  if every vertex outside  $A$  has exactly one neighbour in  $A$ . We say that the subset  $S \subseteq V(G)$  is a *nearly perfect* set (or is *nearly perfect*) in  $G$  if every vertex in  $V(G) - S$  is adjacent to at most one vertex in  $S$ . Note that the set  $S = \emptyset$  is a nearly perfect set in an arbitrary graph. A nearly perfect set  $S$  in  $G$  is called *1-maximal* if for every vertex  $u \in V(G) - S$ ,  $S \cup \{u\}$  is not nearly perfect in  $G$ . Observe that  $V(G)$  is a 1-maximal nearly perfect set in  $G$ . We denote the minimum cardinality of a 1-maximal nearly perfect set in  $G$  by  $n_p(G)$ . We will call the 1-maximal nearly perfect set of the cardinality  $n_p(G)$  an  $n_p(G)$  - set. Let  $n \geq 2$ . The  *$n$ -fold Cartesian product of graphs*  $G_1, \dots, G_n$  is the graph  $\mathbf{X}_{i=1}^n G_i$  with  $V(\mathbf{X}_{i=1}^n G_i) = V(G_1) \times \dots \times V(G_n) = X_{i=1}^n V(G_i)$ , where  $(x_1, \dots, x_n)(y_1, \dots, y_n) \in E(\mathbf{X}_{i=1}^n G_i)$  if

there exists  $j \in \{1, 2, \dots, n\}$  such that  $x_j y_j \in E(G_j)$  and  $x_i = y_i$  for  $i \neq j$ . This is equivalent to the inductive definition  $\mathbf{X}_{i=1}^n G_i = (\mathbf{X}_{i=1}^{n-1} G_i) \times G_n$ . The  $n$ -fold strong product of graphs  $G_1, \dots, G_n$  ( $n \geq 2$ ) is the graph  $\bigotimes_{i=1}^n G_i$  with  $V(\bigotimes_{i=1}^n G_i) = V(G_1) \times \dots \times V(G_n) = X_{i=1}^n V(G_i)$ , where  $(x_1, \dots, x_n)(y_1, \dots, y_n) \in E(\bigotimes_{i=1}^n G_i)$  if  $(x_1, \dots, x_n)(y_1, \dots, y_n) \in E(\mathbf{X}_{i=1}^n G_i)$  or if  $x_i y_i \in E(G_i)$  for  $i = 1, 2, \dots, n$ . This is equivalent to the inductive definition  $\bigotimes_{i=1}^n G_i = (\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$ . For concepts not defined here, see [1].

## 2. MAIN RESULTS

In [3], the following proposition is proved.

**Proposition 1.** *If  $S_i$  is a nearly perfect set in  $G_i$  for  $i = 1, 2$ , then  $S_1 \times S_2$  is a nearly perfect set in  $G_1 \times G_2$ .*

In our further investigations, a nearly perfect set  $S_i$  in  $G_i$  is not empty and  $S_i \neq V(G_i)$  for  $i = 1, 2, \dots, n$ .

**Theorem 1.** *Let  $S_i \subseteq V(G_i)$  for  $i = 1, 2, \dots, n$ . The subset  $S = X_{i=1}^n S_i \subseteq V(\mathbf{X}_{i=1}^n G_i)$  is a nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$  if and only if a subset  $S_i$  is a nearly perfect set in  $G_i$  for each  $i = 1, 2, \dots, n$ .*

*Proof.* Let  $S_i$  be an arbitrary nonempty subset of  $V(G_i)$  for  $i = 1, 2, \dots, n$ . Suppose that there exists  $j \in \{1, 2, \dots, n\}$  such that  $S_j$  is not nearly perfect in  $G_j$ . We will show that  $S = X_{i=1}^n S_i$  is not nearly perfect in  $\mathbf{X}_{i=1}^n G_i$ . If  $S_j$  is not nearly perfect in  $G_j$ , then there exists a vertex  $w_j \in V(G_j) - S_j$  such that  $|N_{G_j}(w_j) \cap S_j| \geq 2$ . Hence, say  $u_j, v_j \in N_{G_j}(w_j) \cap S_j$ . By  $S_i \neq \emptyset$ , we can take a vertex  $w_i \in S_i$  for each  $i \neq j$ . Therefore, the vertex  $\mathbf{w} = (w_1, \dots, w_n) \in V(\mathbf{X}_{i=1}^n G_i) - S$  is adjacent to at least two different vertices  $(w_1, \dots, w_{j-1}, u_j, w_{j+1}, \dots, w_n)$  and  $(w_1, \dots, w_{j-1}, v_j, w_{j+1}, \dots, w_n)$  in  $S$ . Thus, the set  $S = X_{i=1}^n S_i$  is not nearly perfect in  $\mathbf{X}_{i=1}^n G_i$ .

The ‘‘if’’ part is proved by induction on  $n$ . Proposition 1 guarantees that the result holds for  $n = 2$ . Suppose  $n > 3$  and assume that if  $S_i$  is nearly perfect in  $G_i$  for  $i = 1, 2, \dots, n - 1$ , then  $X_{i=1}^{n-1} S_i$  is a nearly perfect set in  $\mathbf{X}_{i=1}^{n-1} G_i$ . The rest of the proof runs as in the case  $n = 2$ , with  $G_1 = \mathbf{X}_{i=1}^{n-1} G_i$  and  $G_2 = G_n$ . Thus,  $(X_{i=1}^{n-1} S_i) \times S_n = X_{i=1}^n S_i$  is a nearly perfect set in  $(\mathbf{X}_{i=1}^{n-1} G_i) \times G_n = \mathbf{X}_{i=1}^n G_i$ . We conclude from the inductive definition of the  $n$ -fold Cartesian product that  $S = X_{i=1}^n S_i$  is a nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$ , which completes the proof.  $\square$

**Proposition 2.** *Let  $S_i \subseteq V(G_i)$  and let  $S_i \neq \emptyset$  for  $i = 1, 2, \dots, n$ . The subset  $S = X_{i=1}^n S_i \subseteq V(\mathbf{X}_{i=1}^n G_i)$  is independent in  $\mathbf{X}_{i=1}^n G_i$  if and only if the subset  $S_i$  is independent in  $G_i$  for each  $i = 1, 2, \dots, n$ .*

*Proof.* Let  $S_i$  be an arbitrary nonempty subset of  $V(G_i)$  for  $i = 1, 2, \dots, n$  and let  $S = X_{i=1}^n S_i \subseteq V(\mathbf{X}_{i=1}^n G_i)$  be independent in  $\mathbf{X}_{i=1}^n G_i$ . Suppose that there exists  $j \in \{1, 2, \dots, n\}$  such that  $S_j$  is not independent in  $G_j$  ( $S_j \neq \emptyset$ ). This means that there exist vertices  $u_j, v_j \in S_j$  such that  $u_j v_j \in E(G_j)$ . Therefore, the vertices  $(w_1, \dots, w_{j-1}, u_j, w_{j+1}, \dots, w_n) = \mathbf{w}_u$  and  $(w_1, \dots, w_{j-1}, v_j, w_{j+1}, \dots, w_n) = \mathbf{w}_v$ ,

where  $w_i \in S_i$  for  $i = 1, 2, \dots, n$ , are neighbouring in  $\mathbf{X}_{i=1}^n G_i$ . Moreover, vertices  $\mathbf{w}_u, \mathbf{w}_v \in S$ . Thus,  $S$  is not independent in  $\mathbf{X}_{i=1}^n G_i$ .

Now, let  $S_i$  be an arbitrary independent subset of  $V(G_i)$  for  $i = 1, 2, \dots, n$ . By the definition of  $n$ -fold Cartesian product, it follows that the set  $S = X_{i=1}^n S_i$  is independent in  $\mathbf{X}_{i=1}^n G_i$ , as required.  $\square$

**Theorem 2.** *Let  $S_i \subseteq V(G_i)$  for  $i = 1, 2, \dots, n$ . The subset  $S = X_{i=1}^n S_i \subseteq V(\mathbf{X}_{i=1}^n G_i)$  is a 1-maximal nearly perfect independent set in  $\mathbf{X}_{i=1}^n G_i$  if and only if the subset  $S_i$  is a 1-maximal nearly perfect independent set and a perfect dominating set in  $G_i$  for each  $i = 1, 2, \dots, n$ .*

*Proof.* Let  $S_i \subseteq V(G_i)$  be an arbitrary nonempty set for  $i = 1, 2, \dots, n$ . Suppose that  $S = X_{i=1}^n S_i$  is a 1-maximal nearly perfect independent set in  $\mathbf{X}_{i=1}^n G_i$ . By Theorem 1 and Proposition 2,  $S_i$  is independent and nearly perfect in  $G_i$  for each  $i = 1, 2, \dots, n$ . We just have to show that the nearly perfect set  $S_i$  in  $G_i$  is 1-maximal and perfect dominating in  $G_i$  for each  $i = 1, 2, \dots, n$ .

First we prove that the nearly perfect set  $S_i$  in  $G_i$  is 1-maximal in  $G_i$  for  $i = 1, 2, \dots, n$ . On the contrary, suppose that there exists  $j \in \{1, 2, \dots, n\}$  such that  $S_j$  is not 1-maximal in  $G_j$ . From this it follows that there exists a vertex  $w_j \in V(G_j) - S_j$  such that  $S_j \cup \{w_j\}$  is a nearly perfect set in  $G_j$ . Therefore, by Theorem 1, the set  $S_{w_j} = S_1 \times \dots \times S_{j-1} \times (S_j \cup \{w_j\}) \times S_{j+1} \times \dots \times S_n$  is nearly perfect in  $\mathbf{X}_{i=1}^n G_i$ . So, every vertex outside  $S_{w_j}$  has at most one neighbour in  $S_{w_j}$ . Thus, it has at most one neighbour in  $S \cup \{\mathbf{x}_{w_j}\} \subseteq S_{w_j}$ , where  $\mathbf{x}_{w_j} = (x_1, \dots, x_{j-1}, w_j, x_{j+1}, \dots, x_n)$ . Moreover, every vertex from  $S_{w_j} - S$  has at most one neighbour in  $S$ , since  $S$  is nearly perfect in  $\mathbf{X}_{i=1}^n G_i$ . By the assumption that  $S_i$  is independent in  $G_i$  for  $i = 1, 2, \dots, n$ , we see that for every vertex  $x_i \in S_i$ ,  $N_{G_i}(x_i) \cap S_i = \emptyset$  for  $i = 1, 2, \dots, n$ . For this reason the vertex  $\mathbf{x}_{w_j} = (x_1, \dots, x_{j-1}, w_j, x_{j+1}, \dots, x_n) \in S_{w_j} - S$  does not have neighbours in  $S_{w_j} - S$ . From the above, it follows that  $S \cup \{\mathbf{x}_{w_j}\}$  is a nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$ . This contradicts the 1-maximality of  $S$ .

Now, we prove that  $S_i$  is a perfect dominating set in  $G_i$  for  $i = 1, 2, \dots, n$ . On the contrary, suppose that there exists  $j \in \{1, 2, \dots, n\}$  such that  $S_j$  is not perfect dominating in  $G_j$ , i.e., there is a vertex  $u_j \in V(G_j) - S_j$  such that  $|N_{G_j}(u_j) \cap S_j| \neq 1$ . Since  $S_j$  is a nearly perfect set in  $G_j$ , we see that there must be  $|N_{G_j}(u_j) \cap S_j| = 0$ . Let  $u_i \in V(G_i) - S_i$  for every  $i \neq j$ . In  $\mathbf{X}_{i=1}^n G_i$ ,  $\mathbf{u} = (u_1, u_2, \dots, u_n) \in X_{i=1}^n (V(G_i) - S_i)$  does not have neighbours in  $S$ . From the definition of the  $n$ -fold Cartesian product, it follows that the vertex  $\mathbf{u}$  can have neighbours only in a set  $A_1 \cup A_2$ , where  $A_1 = V(G_1) \times \dots \times V(G_{j-1}) \times \{u_j\} \times V(G_{j+1}) \times \dots \times V(G_n)$  and  $A_2 = \{u_1\} \times \dots \times \{u_{j-1}\} \times (V(G_j) - S_j) \times \{u_{j+1}\} \times \dots \times \{u_n\}$ . But vertices of  $A_1$  do not have neighbours in  $S$ , because  $|N_{G_j}(u_j) \cap S_j| = 0$ , and vertices of  $A_2$  do not have neighbours in  $S$  either, because  $A_2$  is a subset of  $X_{i=1}^n (V(G_i) - S_i)$ . Thus,  $S \cup \{\mathbf{u}\}$  is a nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$ , which contradicts the 1-maximality of  $S$ .

Now, let  $S_i$  be a 1-maximal nearly perfect independent set in  $G_i$ . From Theorem 1 and Proposition 2, it follows that  $S$  is a nearly perfect independent set in  $\mathbf{X}_{i=1}^n G_i$ . It only remains to show that the nearly perfect set  $S$  in  $\mathbf{X}_{i=1}^n G_i$  is 1-maximal in  $\mathbf{X}_{i=1}^n G_i$ . The proof is by induction on  $n$ . Let  $n = 2$  and let  $(y_1, y_2)$  be an arbitrary vertex outside  $S_1 \times S_2$ . Without loss of generality, we have two cases to consider.

1. Let  $y_2 \in S_2$ . From the 1-maximality of  $S_1$  (as a subset of  $G_1$ ), it follows that  $S_1 \cup \{y_1\}$  is not nearly perfect in  $G_1$ . Hence, we can find a vertex  $v \in V(G_1) - (S_1 \cup \{y_1\})$  having at least two neighbours in  $\{(y_1, y_2)\} \cup \{(s_1, y_2) | s_1 \in S_1\}$  which is obviously a subset of  $S_1 \times S_2 \cup \{(y_1, y_2)\}$ . Consequently,  $S_1 \times S_2 \cup \{(y_1, y_2)\}$  is not nearly perfect in  $G_1 \times G_2$ .
2. Let  $y_1 \in V(G_1) - S_1$  and  $y_2 \in V(G_2) - S_2$ . By assumption, there are vertices  $s_1 \in N_{G_1}(y_1) \cap S_1$  and  $s_2 \in N_{G_2}(y_2) \cap S_2$ . In  $G_1 \times G_2$ ,  $(y_1, s_2)$  has at least two neighbours in  $S_1 \times S_2 \cup \{(y_1, y_2)\}$ , i.e.,  $(y_1, y_2)$  and  $(s_1, s_2)$ . Thus,  $S_1 \times S_2 \cup \{(y_1, y_2)\}$  is not nearly perfect in  $G_1 \times G_2$ .

Thus,  $S_1 \times S_2$  is a 1-maximal nearly perfect set in  $G_1 \times G_2$ .

Suppose  $n > 2$  and assume the result holds for all sequences of graphs with less than  $n$  elements. Hence,  $X_{i=1}^{n-1} S_i$  is a 1-maximal nearly perfect set in  $\mathbf{X}_{i=1}^{n-1} G_i$ . We can apply the reasoning used in the case  $n = 2$  with  $G'_1 = \mathbf{X}_{i=1}^{n-1} G_i$  and  $G'_2 = G_n$ . Since  $S_n$  is a 1-maximal nearly perfect set in  $G_n$ ,  $(X_{i=1}^{n-1} S_i) \times S_n$  is a 1-maximal nearly perfect set in  $(\mathbf{X}_{i=1}^{n-1} G_i) \times G_n$ . By the inductive definition of the  $n$ -fold Cartesian product,  $S = X_{i=1}^n S_i$  is a 1-maximal nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$ , which proves the theorem.  $\square$

**Corollary 1.** *If  $S_i$  is an  $\alpha(G_i)$ -set for each  $i = 1, 2, \dots, n$ , then  $n_p(\mathbf{X}_{i=1}^n G_i) \leq |X_{i=1}^n S_i| = \prod_{i=1}^n |S_i| = \prod_{i=1}^n \alpha(G_i)$ .*

**Corollary 2.** *If  $S_i$  is an independent  $n_p(G_i)$ -set and a perfect dominating set in  $G_i$  for each  $i = 1, 2, \dots, n$ , then  $n_p(\mathbf{X}_{i=1}^n G_i) \leq |X_{i=1}^n S_i| = \prod_{i=1}^n |S_i| = \prod_{i=1}^n n_p(G_i)$ .*

It has been proved in [2] that the  $n_p(P_{3k+1})$ -sets and  $n_p(C_{3k})$ -sets, for  $k \geq 1$ , are independent and perfect dominating in  $P_{3k+1}$  and  $C_{3k}$ , respectively.

Theorem 2 leads immediately to the following conclusion.

**Corollary 3.** *If  $n_p(\mathbf{X}_{i=1}^n G_i) = 1$ , then  $n_p(G_i) = 1$  for each  $i = 1, 2, \dots, n$ .*

It is easily seen that  $n_p(G_i) = 1$  for each  $i = 1, 2, \dots, n$  does not always imply  $n_p(\mathbf{X}_{i=1}^n G_i) = 1$ . For example, let  $G_1$  be a graph such that  $V(G_1) = \{x_0, x_1, x_2, x_3\}$  and  $E(G_1) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_1\}$  and let  $G_2$  be a cycle on three vertices. Then  $n_p(G_i) = 1$  for each  $i = 1, 2$  but  $n_p(G_1 \times G_2) = 2$ .

**Theorem 3.** *Let  $S_i \subseteq V(G_i)$ , for  $i = 1, 2, \dots, n$ . If  $S = X_{i=1}^n S_i \subseteq V(\otimes_{i=1}^n G_i)$  is a nearly perfect set in  $\otimes_{i=1}^n G_i$ , then  $S_i$  is a nearly perfect set in  $G_i$  for each  $i = 1, \dots, n$ .*

*Proof.* Our proof starts with the observation that  $\otimes_{i=1}^n G_i$  is an overgraph of  $\mathbf{X}_{i=1}^n G_i$  and  $V(\otimes_{i=1}^n G_i) = V(\mathbf{X}_{i=1}^n G_i)$ . Consequently, if  $S \subseteq V(\otimes_{i=1}^n G_i) = V(\mathbf{X}_{i=1}^n G_i)$  is a nearly perfect set in  $\otimes_{i=1}^n G_i$ , then  $S$  is also a nearly perfect set in  $\mathbf{X}_{i=1}^n G_i$ . By Theorem 1, we obtain the desired conclusion.  $\square$

**Theorem 4.** *If  $S_i \subseteq V(G_i)$  is an independent nearly perfect set in  $G_i$  for each  $i = 1, 2, \dots, n$ , then  $S = X_{i=1}^n S_i \subseteq V(\otimes_{i=1}^n G_i)$  is a nearly perfect set in  $\otimes_{i=1}^n G_i$ .*

*Proof.* The proof is by induction on  $n$ . The proof for the case  $n = 2$  was given in [3]. Assume that  $X_{i=1}^{n-1} S_i$  is nearly perfect in  $\otimes_{i=1}^{n-1} G_i$ . We can apply the reasoning used

in the case  $n = 2$  with  $G'_1 = \bigotimes_{i=1}^{n-1} G_i$  and  $G'_2 = G_n$ . Since  $S_n$  is a nearly perfect set in  $G_n$ ,  $(X_{i=1}^{n-1} S_i) \times S_n$  is a nearly perfect set in  $(\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$ . We conclude from the inductive definition of the  $n$ -fold strong product that  $S = X_{i=1}^n S_i$  is a nearly perfect set in  $\bigotimes_{i=1}^n G_i$ , as required.  $\square$

**Theorem 5.** *If  $S_i \subseteq V(G_i)$  is a 1-maximal nearly perfect independent set in  $G_i$  for each  $i = 1, \dots, n$ , then  $S = X_{i=1}^n S_i \subseteq V(\bigotimes_{i=1}^n G_i)$  is a 1-maximal nearly perfect set in  $\bigotimes_{i=1}^n G_i$ .*

*Proof.* By Theorem 4,  $S$  is a nearly perfect set in  $\bigotimes_{i=1}^n G_i$ . The proof is completed by showing that the nearly perfect set  $S$  in  $\bigotimes_{i=1}^n G_i$  is 1-maximal in  $\bigotimes_{i=1}^n G_i$ . The proof is by induction on  $n$ . The result for the case  $n = 2$  was given in [3]. Assume that  $X_{i=1}^{n-1} S_i$  is a 1-maximal nearly perfect set in  $\bigotimes_{i=1}^{n-1} G_i$ . We can apply the reasoning used in the case  $n = 2$  with  $G'_1 = \bigotimes_{i=1}^{n-1} G_i$  and  $G'_2 = G_n$ . Since  $S_n$  is a nearly perfect set in  $G_n$ ,  $(X_{i=1}^{n-1} S_i) \times S_n$  is a 1-maximal nearly perfect set in  $(\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$ . We conclude from the inductive definition of the  $n$ -fold strong product that  $S = X_{i=1}^n S_i$  is a 1-maximal nearly perfect set in  $\bigotimes_{i=1}^n G_i$ , and the proof is complete.  $\square$

**Corollary 4.** *If  $S_i$  is an  $\alpha(G_i)$ -set for each  $i = 1, 2, \dots, n$ , then  $n_p(\bigotimes_{i=1}^n G_i) \leq |X_{i=1}^n S_i| = \prod_{i=1}^n |S_i| = \prod_{i=1}^n \alpha(G_i)$ .*

**Corollary 5.** *If  $S_i$  is an independent  $n_p(G_i)$ -set for each  $i = 1, 2, \dots, n$ , then  $n_p(\bigotimes_{i=1}^n G_i) \leq |X_{i=1}^n S_i| = \prod_{i=1}^n |S_i| = \prod_{i=1}^n n_p(G_i)$ .*

The results from [2] and Corollary 5 give  $n_p(P_{3k+1} \otimes C_{3l}) \leq n_p(P_{3k+1}) \cdot n_p(C_{3l}) = (k+1) \cdot l$ .

Theorem 5 leads evidently to the following conclusion.

**Corollary 6.** *If  $n_p(G_i) = 1$  for  $i = 1, 2, \dots, n$ , then  $n_p(\bigotimes_{i=1}^n G_i) = 1$ .*

It is easy to check that  $n_p(\bigotimes_{i=1}^n G_i) = 1$  does not always imply  $n_p(G_i) = 1$ , for  $i = 1, 2, \dots, n$ . For example,  $n_p(K_2 \otimes K_2) = 1$  but  $n_p(K_2) = 2$ .

**Theorem 6.** *Let  $S_i \subseteq V(G_i)$  for  $i = 1, 2, \dots, n$ . If  $S = X_{i=1}^n S_i \subseteq V(\bigotimes_{i=1}^n G_i)$  is a 1-maximal nearly perfect set in  $\bigotimes_{i=1}^n G_i$  and  $S$  contains an isolated vertex of  $\bigotimes_{i=1}^n G_i$ , then  $S_i$  is a 1-maximal nearly perfect set in  $G_i$  for each  $i = 1, 2, \dots, n$ .*

*Proof.* Our proof starts with the observation that if  $S$  is a 1-maximal nearly perfect set in  $\bigotimes_{i=1}^n G_i$ , then  $S \neq \emptyset$  and, in consequence,  $S_i \neq \emptyset$  for every  $i = 1, 2, \dots, n$ . Therefore, by Theorem 3,  $S_i$  is a nearly perfect set in  $G_i$  for each  $i = 1, 2, \dots, n$ . It suffices to prove that the nearly perfect set  $S_i$  in  $G_i$  is 1-maximal for each  $i = 1, 2, \dots, n$ . Suppose, contrary to our claim, that there exists  $j \in \{1, 2, \dots, n\}$  such that  $S_j$  is not 1-maximal in  $G_j$ . Thus, there exists a vertex  $w_j \in V(G_j) - S_j$  such that  $S_j \cup \{w_j\}$  is a nearly perfect set in  $G_j$ . Since  $S$  contains an isolated vertex of  $\bigotimes_{i=1}^n G_i$ , say  $\mathbf{x} = (x_1, \dots, x_n) \in S$ , we conclude that  $x_i$  (as a vertex of  $G_i$ ) does not have neighbours in  $S_i$ , for  $i = 1, 2, \dots, n$ . Therefore, the vertex  $\mathbf{x}_{w_j} = (x_1, \dots, x_{j-1}, w_j, x_{j+1}, \dots, x_n) \in S_1 \times \dots \times \{w_j\} \times S_{j+1} \times \dots \times S_n = S_{w_j}$  does not have neighbours in  $S_{w_j}$ . By the definition of the  $n$ -fold strong product, it follows that the vertex  $\mathbf{x}_{w_j}$  has neighbours in a set  $\{x_1\} \times \dots \times \{x_{j-1}\} \times N_{G_j}(w_j) \times \{x_{j+1}\} \times \dots \times \{x_n\}$

only. But  $\{x_1\} \times \cdots \times \{x_{j-1}\} \times (S_j \cup \{w_j\}) \times \{x_{j+1}\} \times \cdots \times \{x_n\}$  is a nearly perfect set in the induced subgraph  $(\{x_1\} \times \cdots \times \{x_{j-1}\} \times V(G_j \times \{x_{j+1}\} \times \cdots \times \{x_n\}))_{\otimes_{i=1}^n G_i}$  since  $S_j \cup \{w_j\}$  is a nearly perfect in  $G_j$ . From the above it follows that for every vertex  $\mathbf{y} = (y_1, \dots, y_n)$  outside  $S \cup \{\mathbf{x}_{w_j}\}$  the inequality  $|N_{\otimes_{i=1}^n G_i}(\mathbf{y}) \cap (S \cup \{\mathbf{x}_{w_j}\})| \leq 1$  holds. Hence  $S \cup \{\mathbf{x}_{w_j}\}$  is a nearly perfect set in  $\otimes_{i=1}^n G_i$ , which contradicts the 1-maximality of  $S$ .  $\square$

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