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The Authors dedicate their article to academician **Prof. Yuriy A. Mitropolski**
on occasion of his 90-years Birthday with compliments and gratitude
to his great talent and impact to modern nonlinear analysis and mathematical physics.

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ON THE GEOMETRIC STRUCTURE OF CHARACTERISTIC VECTOR FIELDS RELATED WITH NONLINEAR EQUATIONS OF THE HAMILTON-JACOBI TYPE

Abstract. The Cartan-Monge geometric approach to the characteristics method for Hamilton-Jacobi type equations and nonlinear partial differential equations of higher orders is analyzed. The Hamiltonian structure of characteristic vector fields related with nonlinear partial differential equations of first order is analyzed, the tensor fields of special structure are constructed for defining characteristic vector fields naturally related with nonlinear partial differential equations of higher orders. The generalized characteristics method is developed in the framework of the symplectic theory within geometric Monge and Cartan pictures. The related characteristic vector fields are constructed making use of specially introduced tensor fields, carrying the symplectic structure. Based on their inherited geometric properties, the related functional-analytic Hopf-Lax type solutions to a wide class of boundary and Cauchy problems for nonlinear partial differential equations of Hamilton-Jacobi type are studied. For the non-canonical Hamilton-Jacobi equations there is stated a relationship between their solutions and a good specified functional-analytic fixed point problem, related with Hopf-Lax type solutions to specially constructed dual canonical Hamilton-Jacobi equations.

Keywords: Hamilton-Jacobi equations, the Cartan-Monge geometric approach, Hopf-Lax type representation.

Mathematics Subject Classification: 34A30, 34B05, 34B15.

1. INTRODUCTION: GEOMETRIC BACKGROUNDS OF THE CLASSICAL CHARACTERISTICS METHOD

Solutions to linear partial differential equations, as is well known [16, 22, 27], may be studied effectively enough using many classical approaches, such as Fourier method, spectral theory and Green function method. Nevertheless, none of them, regrettably, may be applied to analyzing solution manifolds of general nonlinear partial differential equations, even of the first and second orders. Since the classical Cauchy works on the problem to the date, few approaches to treating such equations have only been developed [1, 17, 22, 25–27], among which the famous characteristics method appears to be the most effective and fruitful. During the last century this method was further developed by many mathematicians, including P. Lax, H. Hopf, O. A. Oleinik, S. Kruzhkov, V. Maslov, P. Lions, L. Evans, D. Blackmore [11–15, 22, 25, 27, 30] and others. It was long ago, too, the deep connection of the characteristics method with Hamiltonian analysis was observed, reducing the problem to studying some systems of ordinary differential equations. This aspect was prevailing in works of H. Hopf, P. Lax and O. Oleinik (see [3, 14, 22]), who in this way described a wide class of so called generalized solutions to first order nonlinear partial differential equations. The most known result within this field is attributed to H. Hopf and P. Lax, who first to find a very interesting variational representation for solutions of first order nonlinear partial differential equations known as the Hopf–Lax type representation. As these results were strongly based on some geometric notions, it was natural to analyze the Cauchy characteristics method from the differential-geometric point of view, initiated still in the classical works of G. Monge and E. Cartan [28]. Within the framework of the Monge geometric approach to studying solutions of partial differential equations in [18] we proposed a generalization of the classical Cauchy characteristic method for equations of first and higher orders, making use of certain purposefully designed tensor fields, closely related with them. These tensor fields appear very naturally within an extended Monge approach as some geometric objects, generalizing the classical Hamilton type equations for characteristic vector fields. Moreover, this geometric approach together with some Cartan's compatibility considerations [1,23,28] is naturally extended to a wide class of nonlinear partial differential equations of first and higher orders. And even more, if the introduced tensor field is chosen in such a way that it carries an associated symplectic structure, the corresponding solutions to generalized Hamilton-Jacobi equations may be found, in general, effectively in the implicit functional-analytic Hopf-Lax type form, which is equivalent [19] to some well-posed fixed point problem.

The characteristics method [1, 22, 27, 30], proposed in XIX century by A. Cauchy, was later very nontrivially developed by G. Monge, who had introduced the geometric notion of characteristic surface, related with partial differential equations of first order. The latter, being augmented with a very important notion of characteristic vector fields, appeared to be fundamental [25, 30] for the characteristics method, whose main essence consists in bringing about the problem of studying solutions to our partial differential equation to an equivalent one of studying some set of ordinary differential equations. This way of reasoning succeeded later in development of the Hamilton-Jacobi theory, making it possible to describe a wide class of solutions to partial differential equations of first order of the form

$$H(x; u, u_x) = 0, \tag{1.1}$$

where $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n; \mathbb{R})$, $||H_x|| \neq 0$, is called a Hamiltonian function and $u \in C^2(\mathbb{R}^n; \mathbb{R})$ is unknown function under search. The equation (1.1) is endowed still with a boundary value condition

$$u|_{\Gamma_\varphi} = u_0, \tag{1.2}$$

with $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$, defined on some smooth almost everywhere hypersurface

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0, ||\varphi_x|| \neq 0\}, \tag{1.3}$$

where $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ is some smooth function on $\mathbb{R}^n$.

Following to the Monge's ideas, let us introduce the characteristic surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ as

$$S_H := \{(x; u, p) \in \mathbb{R}^{n+1} \times \mathbb{R}^n : H(x; u, p) = 0\}, \tag{1.4}$$

where we put, by definition, $p := u_x \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. The characteristic surface (1.4) was effectively described by Monge within his geometric approach by means of the so called Monge cones $K \subset T(\mathbb{R}^{n+1})$ and their duals $K^* \subset T^*(\mathbb{R}^{n+1})$ [29, 30]. The corresponding differential-geometric analysis of this Monge scenario was later done by E. Cartan, who reformulated [28,30] the geometric picture, drown by Monge, by means of the related compatibility conditions for dual Monge cones and the notion of integral submanifold $\Sigma_H \subset S_H$, naturally assigned to special vector fields on the characteristic surface S_H. In particular, Cartan had introduced on S_H the differential 1-form

$$\alpha^{(1)} := du - \langle p, dx \rangle, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, and demanded its vanishing along the dual Monge cones $K^* \subset T^*(\mathbb{R}^{n+1})$, concerning the corresponding integral submanifold imbedding mapping

$$\pi : \Sigma_H :\to S_H. \tag{1.6}$$

This means that the 1-form

$$\pi^* \alpha_1^{(1)} := du - \langle p, dx \rangle|_{\Sigma_H} \Rightarrow 0 \tag{1.7}$$

for all points $(x; u, p) \in \Sigma_H$ of a solution surface Σ_H, defined in such a way that $K^* = T^*(\Sigma_H)$. The obvious corollary from the condition (1.7) is the second Cartan condition

$$d\pi^* \alpha_1^{(1)} = \pi^* d\alpha_1^{(1)} = \langle dp, \wedge dx \rangle|_{\Sigma_H} \Rightarrow 0. \tag{1.8}$$

These two Cartan's conditions (1.7) and (1.8) should be still augmented with the characteristic surface S_H invariance condition for the differential 1-form $\alpha_2^{(1)} \in \Lambda^1(S_H)$ as

$$\alpha_2^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{1.9}$$

The conditions (1.7), (1.8) and (1.9), when imposed on the characteristic surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$, make it possible to construct the proper characteristic vector fields on S_H, whose suitable characteristic strips [29,30] generate the searched solution surface Σ_H. Thereby, having solved the corresponding Cauchy problem related with boundary value conditions (1.2) and (1.3) for these characteristic vector fields, considered as ordinary differential equations on S_H, one may construct a solution to our partial differential equation (1.1). And what is interesting, this solution in many cases may be represented [19,22] in exact functional-analytic Hopf-Lax type form. The latter is a natural consequence from the related Hamilton-Jacobi theory, whose main ingredient consists in proving the fact that the solution to our equation (1.1)) is exactly the extremal value of some Lagrangian functional, naturally associated [1, 23, 24] with a given Hamiltonian function.

Below we will construct the proper characteristic vector fields for partial differential equations of first order (1.1) on the characteristic surface S_H, generating the solution surface Σ_H as suitable characteristic strips related with boundary conditions (1.2) and (1.3), and next generalize the Cartan-Monge geometric approach for partial differential equations of second and higher orders.

2. THE CHARACTERISTIC VECTOR FIELDS METHOD: FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Consider on the surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ a characteristic vector field $K_H : S_H \to T(S_H)$ in the form

$$\left.\begin{array}{l} dx/d\tau = a_H(x;u,p) \\ dp/d\tau = b_H(x;u,p) \\ du/d\tau = c_H(x;u,p) \end{array}\right\} := K_H(x;u,p), \tag{2.1}$$

where $\tau \in \mathbb{R}$ is a suitable evolution parameter and $(x;u,p) \in S_H$. Since, owing to the Cartan-Monge geometric approach, there hold conditions (1.7), (1.8) and (1.9) along the solution surface Σ_H, we may satisfy them, applying the interior anti-differentiation operation $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ of the Grassmann algebra $\Lambda(S_H)$ of differential forms [21,23,31] on S_H to the corresponding differential forms $\alpha_1^{(1)}$ and $d\alpha_1^{(1)} \in \Lambda(S_H)$:

$$i_{K_H}\alpha_1^{(1)} = 0,\ i_{K_H}d\alpha_1^{(1)} = 0. \tag{2.2}$$

As a result of simple calculations one finds that

$$\begin{gathered} c_H = \langle p, a_H \rangle, \\ \beta^{(1)} := \langle b_H, dx \rangle - \langle a_H, dp \rangle |_{S_H} = 0 \end{gathered} \tag{2.3}$$

for all points $(x;u,p) \in S_H$. The obtained 1-form $\beta^{(1)} \in \Lambda^1(S_H)$ must be, evidently, compatible with the defining invariance condition (1.9) on S_H. This means that there exists a scalar function $\mu \in C^1(S_H;\mathbb{R})$, such that the condition

$$\mu\alpha_2^{(1)} = \beta^{(1)} \tag{2.4}$$

holds on S_H. This gives rise to such final relationships:

$$a_H = \mu\partial H/\partial p,\ b_H = -\mu(\partial H/\partial x + p\partial H/\partial u), \tag{2.5}$$

which together with the first equality of (2.3) complete the search for the structure of the characteristic vector fields $K_H : S_H \to T(S_H)$:

$$K_H = (\mu\partial H/\partial p; \langle p, \mu\partial H/\partial p\rangle, -\mu(\partial H/\partial x + p\partial H/\partial u))^{\intercal}. \tag{2.6}$$

Now we may pose a suitable Cauchy problem for the equivalent set of ordinary differential equations (2.1) on S_H as follows:

$$\begin{aligned} &dx/d\tau = \mu\partial H/\partial p : x|_{\tau=0} \stackrel{?}{=} x_0(x) \in \Gamma_\varphi,\ x|_{\tau=t(x)} = x \in \mathbb{R}^n\backslash\Gamma_\varphi; \\ &du/d\tau = \langle p, \mu\partial H/\partial p\rangle : u|_{\tau=0} = u_0(x_0(x)), u|_{\tau=t(x)} \stackrel{?}{=} u(x); \\ &dp/d\tau = -\mu(\partial H/\partial x + p\partial H/\partial u) : p|_{\tau=0} = \partial u_0(x_0(x))/\partial x_0, \end{aligned} \tag{2.7}$$

where $x_0(x) \in \Gamma_\varphi$ is the intersection point of the corresponding vector field orbit, starting at a fixed point $x \in \mathbb{R}^n\backslash\Gamma_\varphi$, with the boundary hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ (see Fig. 1) at the moment of "time" $\tau = t(x) \in \mathbb{R}$.

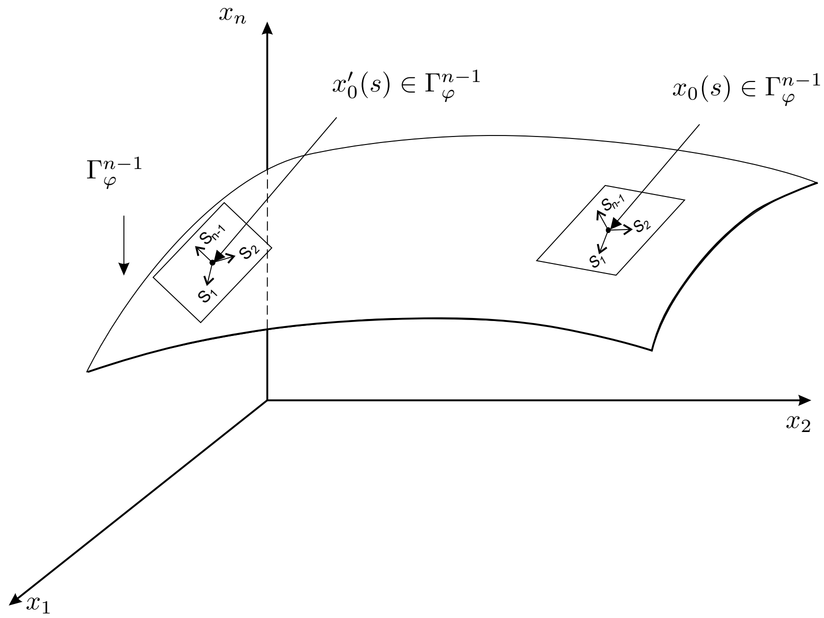

Fig. 1. The boundary $\Gamma_\varphi^{n-1} = \{x_0 \in \mathbb{R}^n : \quad \varphi(x_0) = 0\}$. $x_0(s) \in \Gamma_\varphi^{n-1}$, $s \in \mathbb{R}^{n-1}$ – local coordinates

As a result of solving the corresponding "inverse" Cauchy problem (2.7) one finds the following exact functional-analytic expression for a solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ to the boundary value problem (1.2) and (1.3):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \bar{\mathcal{L}}(x; u, p) d\tau, \tag{2.8}$$

where, by definition,

$$\bar{\mathcal{L}}(x;u,p) := \langle p, \mu\partial H/\partial p\rangle \tag{2.9}$$

for all $(x;u,p) \in S_H$. If the Hamiltonian function $H : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$ is nondegenerate, that is $HessH := \det(\partial^2 H/\partial p\partial p\} \neq 0$ for all $(x;u,p) \in S_H$, then the first equation of (2.7) may be solved with respect to the variable $p \in \mathbb{R}^n$ as

$$p = \psi(x, \dot{x}; u) \tag{2.10}$$

for $(x, \dot{x}) \in T(\mathbb{R}^n)$, where $\psi : T(\mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is some smooth mapping. This gives rise to the following canonical Lagrangian function expression:

$$\mathcal{L}(x, \dot{x}; u) := \bar{\mathcal{L}}(x; u, p)|_{p=\psi(x,\dot{x};u)} \tag{2.11}$$

and to the resulting solution (2.8):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \mathcal{L}(x, \dot{x}; u) d\tau. \tag{2.12}$$

The functional-analytic form (2.12) is already proper for constructing its equivalent Hopf-Lax type form, being very important for finding so called generalized solutions [3, 22, 25] to the partial differential equation (1.1). This aspect of the Cartan-Monge geometric approach we suppose to analyze in detail elsewhere.

3. THE CHARACTERISTIC VECTOR FIELDS METHOD: SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Assume we are given a second order partial differential equation

$$H(x; u, u_x, u_{xx}) = 0, \tag{3.1}$$

where solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ and the generalized "Hamiltonian" function $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n); \mathbb{R})$. Putting $p^{(1)} := u_x$, $p^{(2)} := u_{xx}$, $x \in \mathbb{R}^n$, one can construct within the Cartan-Monge generalized geometric approach the characteristic surface

$$S_H := \{(x; u, p^{(1)}, p^{(2)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n) : H(x; u, p^{(1)}, p^{(2)}) = 0\} \tag{3.2}$$

and a suitable Cartan's set of differential one- and two-forms:

$$\begin{aligned} \alpha_1^{(1)} &:= du - \langle p^{(1)}, dx\rangle|_{\Sigma_H} \Rightarrow 0, \\ d\alpha_1^{(1)} &:= \langle dx, \wedge dp^{(1)}\rangle|_{\Sigma_H} \Rightarrow 0, \\ \alpha_2^{(1)} &:= dp^{(1)} - \langle p^{(2)}, dx\rangle|_{\Sigma_H} \Rightarrow 0, \\ d\alpha_2^{(1)} &:= \langle dx, \wedge dp^{(2)}\rangle|_{\Sigma_H} \Rightarrow 0, \end{aligned} \tag{3.3}$$

vanishing upon the corresponding solution submanifold $\Sigma_H \subset S_H$. The set of differential forms (3.3) should be augmented with the characteristic surface S_H invariance differential 1-form

$$\alpha_3^{(1)} := dH|_{S_H} \Rightarrow 0, \tag{3.4}$$

vanishing, respectively, upon the characteristic surface S_H.

Let the characteristic vector field $K_H : S_H \to T(S_H)$ on S_H is given by expressions

$$\left.\begin{array}{r} dx/d\tau = a_H(x;u,p^{(1)},p^{(2)}) \\ du/d\tau = c_H(x;u,p^{(1)},p^{(2)}) \\ dp^{(1)}/d\tau = b_H^{(1)}(x;u,p^{(1)},p^{(2)}) \\ dp^{(2)}/d\tau = b_H^{(2)}(x;u,p^{(1)},p^{(2)}) \end{array}\right\} := K_H(x;u,p^{(1)},p^{(2)}), \tag{3.5}$$

for all $(x;u,p^{(1)},p^{(2)}) \in S_H$. To find the vector field (3.5) it is necessary to satisfy the Cartan compatibility conditions in the following geometric form:

$$\begin{aligned} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} &\Rightarrow 0, \ i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} &\Rightarrow 0, \ i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \end{aligned} \tag{3.6}$$

where, as above, $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ is the internal differentiation of differential forms along the vector field $K_H : S_H \to T(S_H)$. As a result of conditions (3.6) one finds that

$$\begin{aligned} c_H = \langle p^{(1)}, a_H\rangle,\ b_H^{(1)} &= \langle p^{(2)}, a_H\rangle, \\ \beta_1^{(1)} &:= \langle a_H, dp^{(1)}\rangle - \langle b_H^{(1)}, dx\rangle|_{S_H} \Rightarrow 0, \\ \beta_2^{(1)} &:= \langle a_H, dp^{(2)}\rangle - \langle b_H^{(2)}, dx\rangle|_{S_H} \Rightarrow 0, \end{aligned} \tag{3.7}$$

being satisfied upon S_H identically. The conditions (3.7) must be augmented still with the characteristic surface invariance condition (3.4). Notice now that 1-form $\beta_1^{(1)} = 0$ owing to the second condition of (3.7) and the third condition of (3.3). Thus, we need now to make compatible the basic scalar 1-form (3.4)) with the vector-valued 1-form $\beta_2^{(1)} \in \Lambda(S_H) \otimes \mathbb{R}^n$. To do this let us construct, making use of the $\beta_2^{(1)}$, the following parametrized set of, respectively, scalar 1-forms:

$$\beta_2^{(1)}[\mu] := \langle \bar{\mu}^{(1|0)} \otimes a_H, dp^{(2)}\rangle - \langle b_H^{(2)}, \bar{\mu}^{(1|0)} \otimes dx\rangle|_{S_H} \Rightarrow 0, \tag{3.8}$$

where $\bar{\mu}^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ is any smooth vector-valued function on S_H. The compatibility condition for (3.8) and (3.4) gives rise to the next relationships:

$$\begin{aligned} \bar{\mu}^{(1|0)} \otimes a_H &= \partial H/\partial p^{(2)}, \\ \langle \bar{\mu}^{(1|0)}, b_H^{(2)}\rangle &= -(\partial H/\partial x + p^{(1)}\partial H/\partial u + \langle \partial H/\partial p^{(1)}, p^{(2)}\rangle), \end{aligned} \tag{3.9}$$

holding on S_H. Take now such a dual vector function $\mu^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ that $\langle \mu^{(1|0)}, \bar{\mu}^{(1|0)}\rangle = 1$ for all points of S_H. Then from (3.9) one finds easily that

$$\begin{aligned} a_H &= \langle \mu^{(1|0)}, \partial H/\partial p^{(2)}\rangle, \\ b_H^{(2)} &= -\mu^{(1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u + \langle \partial H/\partial p^{(1)}, p^{(2)}\rangle). \end{aligned} \tag{3.10}$$

Combining now the first two relationships of (3.7) with the found above relationships (3.10) we get the final form for the characteristic vector field (3.5):

$$\begin{aligned} K_H = (a_H; \langle p^{(1)}, a_H\rangle, \langle p^{(2)}, a_H\rangle, -\mu^{(1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u \\ + \langle \partial H/\partial p^{(1)}, p^{(2)}\rangle))^{\intercal}, \end{aligned} \tag{3.11}$$

where $a_H = \langle \mu^{(1|0)}, \partial H/\partial p^{(2)}\rangle$ and $\mu^{(1|0)} \in C^1(S_H; \mathbb{R}^n)$ is some smooth vector-valued function on S_H. Thereby, we may construct as earlier solutions to our partial differential equation of second order (3.1) by means of solving the equivalent Cauchy problem for the set of ordinary differential equations (3.5) on the characteristic surface S_H.

4. THE CHARACTERISTIC VECTOR FIELDS METHOD: PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

Consider a general nonlinear partial differential equation of higher order $m \in \mathbb{Z}_+$ as

$$H(x; u, u_x, u_{xx}, \ldots, u_{mx}) = 0, \tag{4.1}$$

where there is assumed that $H \in C^2(\mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2}; \mathbb{R})$. Within the generalized Cartan-Monge geometric characteristics method we need to construct the related characteristic surface S_H as

$$\begin{aligned} S_H := \{(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2} : \\ H(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}) = 0\}, \end{aligned} \tag{4.2}$$

where we put $p^{(1)} := u_x \in \mathbb{R}^n$, $p^{(2)} := u_{xx} \in \mathbb{R}^n \otimes \mathbb{R}^n$, ..., $p^{(m)} \in (\mathbb{R}^n)^{\otimes m}$ for $x \in \mathbb{R}^n$. The corresponding solution manifold $\Sigma_H \subset S_H$ is defined naturally as the integral submanifold of the following set of one- and two-forms on S_H :

$$\begin{aligned} \alpha_1^{(1)} &:= du - \langle p^{(1)}, dx\rangle|_{\Sigma_H} \Rightarrow 0, \\ d\alpha_1^{(1)} &:= \langle dx, \wedge dp^{(1)}\rangle|_{\Sigma_H} \Rightarrow 0, \\ \alpha_2^{(1)} &:= dp^{(1)} - \langle p^{(2)}, dx\rangle|_{\Sigma_H} \Rightarrow 0, \\ d\alpha_2^{(1)} &:= \langle dx, \wedge dp^{(2)}\rangle|_{\Sigma_H} \Rightarrow 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_m^{(1)} &:= dp^{(m-1)} - \langle p^{(m)}, dx\rangle|_{\Sigma_H} \Rightarrow 0, \\ d\alpha_m^{(1)} &:= \langle dx, \wedge dp^{(m)}\rangle|_{\Sigma_H} \Rightarrow 0, \end{aligned} \tag{4.3}$$

vanishing upon Σ_H. The set of differential forms (4.3) is augmented with the determining characteristic surface S_H invariance condition

$$\alpha_{m+1}^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{4.4}$$

Proceed now to constructing the characteristic vector field $K_H : S_H \to T(S_H)$ on the hypersurface S_H within the developed above generalized characteristics method. Take the expressions

$$\left.\begin{array}{r} dx/d\tau = a_H(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \\ du/d\tau = c_H(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)} \\ dp^{(1)}/d\tau = b_H^{(1)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \\ dp^{(2)}/d\tau = b_H^{(2)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dp^{(m)}/d\tau = b_H^{(m)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}), \end{array}\right\} := K_H(x;u,p^{(1)},p^{(2)}), \tag{4.5}$$

for $(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \in S_H$ and satisfy the corresponding Cartan compatibility conditions in the following geometric form:

$$\begin{array}{rcl} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} & \Rightarrow & 0,\ i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} & \Rightarrow & 0,\ i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ & & \ldots\ldots\ldots\ldots.. \\ i_{K_H}\alpha_m^{(1)}|_{\Sigma_H} & \Rightarrow & 0,\ i_{K_H}d\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0. \end{array} \tag{4.6}$$

As a result of suitable calculations in (4.6) one gets the following expressions:

$$\begin{array}{rcl} c_H & = & \langle p^{(1)}, a_H\rangle,\ b_H^{(1)} = \langle p^{(2)}, a_H\rangle, \\ \beta_1^{(1)} & : & = \langle a_H, dp^{(1)}\rangle - \langle b_H^{(1)}, dx\rangle|_{S_H} \Rightarrow 0, \\ \beta_2^{(1)} & : & = \langle a_H, dp^{(2)}\rangle - \langle b_H^{(2)}, dx\rangle|_{S_H} \Rightarrow 0, \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \beta_m^{(1)} & : & = \langle a_H, dp^{(m)}\rangle - \langle b_H^{(m)}, dx\rangle|_{S_H} \Rightarrow 0, \end{array} \tag{4.7}$$

being satisfied upon S_H identically.

It is now easy to see that all of 1-forms $\beta_j^{(1)} \in \Lambda^1(S_H) \otimes (\mathbb{R}^n)^{\otimes j}$, $j = \overline{1, m-1}$ are vanishing identically upon S_H owing to the relationships (4.3). Thus, as a result we obtain the only relationship

$$\beta_m^{(1)} := \langle a_H, dp^{(m)}\rangle - \langle b_H^{(m)}, dx\rangle|_{S_H} \Rightarrow 0, \tag{4.8}$$

which should be compatibly combined with that of (4.4). To do this suitably with the tensor structure of the 1-forms (4.8), we take a smooth tensor function $\bar{\mu}^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H and construct the parametrized set of scalar 1-forms

$$\beta_m^{(1)}[\mu] := \langle \bar{\mu}^{(m-1|0)} \otimes a_H, dp^{(m)}\rangle - \langle b_H^{(m)}, \bar{\mu}^{(m-1|0)} \otimes dx\rangle|_{S_H} \Rightarrow 0, \tag{4.9}$$

which may be now identified with the 1-form (4.4). This gives rise right away to the relationships

$$\begin{aligned} &\bar{\mu}^{(m-1|0)} \otimes a_H = \partial H/\partial p^{(m)}, \\ &\langle \bar{\mu}^{(m-1|0)}, b_H^{(m)}\rangle = -(\partial H/\partial x + p^{(1)}\partial H/\partial u + \langle \partial H/\partial p^{(1)}, p^{(2)}\rangle + \ldots \\ &\qquad + \langle \partial H/\partial p^{(m-1)}, p^{(m)}\rangle), \end{aligned} \tag{4.10}$$

holding on S_H.

Now we may take such a dual tensor-valued function $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H that $\langle \mu^{(m-1|0)}, \bar{\mu}^{(m-1|0)} \rangle = 1$ for all points of S_H. Then from (4.10) we easily get the searched unknown expressions

$$\begin{aligned} a_H &= \langle \mu^{(m-1|0)}, \partial H/\partial p^{(m)} \rangle, \\ b_H^{(m)} &= -\mu^{(1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u + \langle \partial H/\partial p^{(1)}, p^{(2)} \rangle + \ldots \\ &\quad \ldots + \langle \partial H/\partial p^{(m-1)}, p^{(m)} \rangle). \end{aligned} \tag{4.11}$$

The obtained above result (4.11) combined with suitable expressions from (4.7) give rise to the following final form for the characteristic vector field (4.5):

$$\begin{aligned} K_H = (a_H; &\langle p^{(1)}, a_H \rangle, \langle p^{(2)}, a_H \rangle, \ldots, \langle p^{(m)}, a_H \rangle, \\ &- \mu^{(m-1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u \\ &+ \langle \partial H/\partial p^{(1)}, p^{(2)} \rangle + \ldots + \langle \partial H/\partial p^{(m-1)}, p^{(m)} \rangle))^{\intercal}, \end{aligned} \tag{4.12}$$

where $a_H = \langle \mu^{(m-1|0)}, \partial H/\partial p^{(m)} \rangle$ and $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ is some smooth tensor-valued function on S_H. The resulting set (4.5) of ordinary differential equations on S_H makes it possible to construct exact solutions to our partial differential equation (4.1) in a suitable functional-analytic form, being often very useful for analyzing its properties important for applications. On these and related questions we plan to stop in detail elsewhere later.

Namely, if for instance a first order differential equation is given as

$$H(x; u, u_x) = 0, \tag{4.13}$$

where $x \in \mathbb{R}^n$, $H \in C^1(\mathbb{R}^{2n+1}; \mathbb{R})$, $||H_{u_x}|| \neq 0$, the characteristics vector fields on the related Monge hypersurface

$$S_H := \{(x; u, p) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : \bar{H}(x; u, p) := H(x; u, \pi)|_{\pi=\psi(x;u,p)} = 0\} \tag{4.14}$$

are represented [18] as follows:

$$\frac{dx}{d\tau} = \mu^{(1|1)}\frac{\partial \bar{H}}{\partial p}, \frac{dp}{d\tau} = -\mu^{(1|1),*}(\frac{\partial \bar{H}}{\partial x} + \psi\frac{\partial \bar{H}}{\partial u}), \frac{du}{d\tau} = \langle \psi, \mu^{(1|1)}\frac{\partial \bar{H}}{\partial p} \rangle. \tag{4.15}$$

Here $\mu^{(1|1)} := (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ is a nondegenerate smooth tensor field on the hipersurface S_H, related to its parametrization $\pi := \psi(x; u, p) \in \mathbb{R}^n$, and $\tau \in \mathbb{R}$ is an evolution parameter.

Vector field (4.15) ensures [18] the tangency to the hyper-surface $S_H \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$ and the projection compatibility condition with the dual Monge cone K^* upon the corresponding solution hypersurface $\bar{S}_H \subset \mathbb{R}^{n+1}$ (see Fig. 2), generated by the characteristic strips $\Sigma_H \subset S_H$ through smoothly embedded sets $\Sigma \subset S_H$, consisting of points carrying the solutions to our problem (4.13). Similar results were also obtained in [18] for both partial differential equations of higher orders and systems.

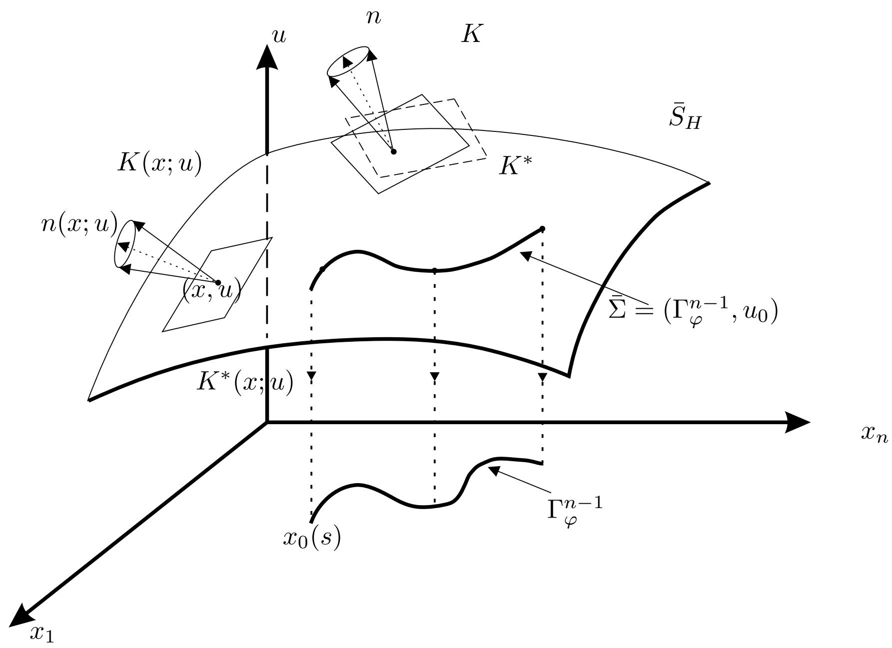

Fig. 2. Geometric Monge method. The boundary conditions: $\bar{\Sigma} = \big(\Gamma_\varphi^{n-1}, u_0\big) \subset \bar{S}_H$, $u_0 \in C^1(\Gamma_\varphi^{n-1};\mathbb{R})$, $\bar{S}_H := \big\{(x,u) \in \mathbb{R}^{n+1} : \quad u = \psi(x)\big\}$ – the boundary problem solution hypersurface

In general, the problem (4.13) is endowed with some boundary condition on a smooth hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ as

$$u|_{\Gamma_\varphi} = u_0, \tag{4.16}$$

where $u_0 \in C^1(\Gamma_\varphi;\mathbb{R})$ is a given function. The hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ may be, for simplicity, defined as

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0\}, \tag{4.17}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth mapping endowed with some local coordinates $s(x) \in \mathbb{R}^{n-1}$ in the corresponding open neighborhoods $O_\varepsilon(x) \subset \Gamma_\varphi$ of all points $x \in \Gamma_\varphi$ at some $\varepsilon > 0$. Thus, we are interested in constructing analytical solutions to the boundary problem (4.13), (4.16) and (4.17) and studying their properties. This and related aspects of this problem will be discussed in detail below.

5. BOUNDARY PROBLEM ANALYSIS

Consider the set of characteristic equations (4.15) on the hypersurface $S_H \subset \mathbb{R} \times \mathbb{R}^{n+1}$, which start at points $(x_0; u_0, p_0) \in \Sigma$ under the additional condition that the corresponding projection $\Sigma \to \bar{\Sigma}$ upon the subspace $\mathbb{R}^{n+1}$ (see Fig. 2) coincides with the boundary set $(\Gamma_\varphi; u_0) \subset \mathbb{R}^{n+1}$, that is

$$\bar{\Sigma} := (\Gamma_\varphi; u_0), \tag{5.1}$$

where $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is our boundary condition.

Condition (5.1) evidently assumes that the set $\Sigma \subset S_H$ may be defined as follows:

$$\Sigma = (\bar{\Sigma}; p_0) \tag{5.2}$$

with some $p_0 \in C^1(\Gamma_\varphi; \mathbb{R}^n)$ being yet an unknown smooth mapping. For it to be determined we need to ensure, for all points $\Sigma \subset S_H$, the above mentioned Cartan compatibility conditions, that is the conditions

$$du|_\Sigma = \langle p, dx\rangle|_\Sigma, \ \langle d\psi, \wedge dx\rangle|_\Sigma = 0, \tag{5.3}$$

where $\Sigma \subset S_H$ is given by (5.2). As a result of (5.3), one easily finds that

$$\begin{cases} \partial u_0(s)/\partial s - \langle \psi(x_0(s); u_0(x_0(s)), p_0(s)), \partial x_0(s)/\partial s\rangle = 0, \\ \bar{H}(x_0(s); u_0(x_0(s)), p_0(s)) = 0 \end{cases} \tag{5.4}$$

for all points $x_0 := x_0(s) \in \Gamma_\varphi$, $s \in \mathbb{R}^{n-1}$. Here we took into account that any point $x \in \Gamma_\varphi$ is parametrized by means of the corresponding local coordinates $s = s(x_0) \in \mathbb{R}^{n-1}$, defined in the corresponding ε-vicinities $O_\varepsilon(x) \subset \Gamma_\varphi, \varepsilon > 0$.

The system of relationships (5.4) must be solvable for a mapping $p_0 : \Gamma_\varphi \to \mathbb{R}^n$ at all points $x_0 \in \Gamma_\varphi$, which gives rise to the determinant condition

$$\det \left[\left(\frac{\partial \psi}{\partial p} \right)^* \frac{\partial x_0}{\partial s}; \left(\frac{\partial \bar{H}}{\partial p} \right)^\intercal \right] \Bigg|_{(x_0; u_0, p_0)} \neq 0 \tag{5.5}$$

owing to the implicit function theorem [2]. If condition (5.5) is satisfied at points $(x_0; u_0, p_0^{(j)}) \in S_H$, where $j = \overline{1, N}$ for some $N \in \mathbb{Z}_+$ and all points $(x_0; u_0) \in \bar{\Sigma}$, the system of equation (5.4) possesses exactly $N \in \mathbb{Z}_+$ different smooth solution $p_0^{(j)} \in C^1(\Gamma_\varphi; \mathbb{R}^n)$, $j = \overline{1, N}$, thereby determining corresponding Cauchy data (5.2) for characteristic vector fields (4.15). It is clear enough that our boundary problem (4.13), (4.16) and (4.17) possesses, in general, many solutions of different functional classes, depending on the kind of the boundary conditions chosen. For instance, as it was studied and analyzed in [19, 22, 27], this boundary problem may also possess so-called generalized solutions, which under some additional conditions allow the so called Hopf-Lax inf-type extremality form, being often very useful for studying their asymptotic and other qualitative properties.

The important problem of constructing functional-analytic solutions to our equation (4.13) under boundary conditions (4.16) and (4.17) will be discussed in detail below.

6. THE HOPF-LAX TYPE INF-TYPE FUNCTIONAL-ANALYTIC REPRESENTATION

Assume now that $p_0 \in C^1(\Gamma_\varphi;\mathbb{R}^n)$ is a smooth solution to system (5.4), thereby defining completely the required Cauchy data $\Sigma \subset S_H$ for characteristic vector fields (4.15). Thus, making use of suitable classical methods for solving these ordinary differential equations, one can, in particular, find that the function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ for each attained point $x = x(t) \in \mathbb{R}^n$ may be represented in the analytical form

$$u(x(t)) = u(x(0)) + \int_0^t \langle \psi(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau)\rangle d\tau \tag{6.1}$$

at any "time" $t \in \mathbb{R}$. Since, by definition, $x(0) := x_0(s) \in \Gamma_\varphi$ and $u(x(0)) := u_0(x_0(s))$, $s \in \mathbb{R}^{n-1}$, solution (6.1) is rewritten as

$$u(x(t)) = u_0(x_0(s)) + \int_0^t \langle \psi(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau)\rangle d\tau \tag{6.2}$$

for any $t \in \mathbb{R}$, where the integrand function in (6.2) is assumed to be found analytically.

Now, for vector field equations (4.15), pose the following "inverse" Cauchy problem

$$x|_{\tau=t(x)} = x \in \mathbb{R}^n,\ x|_{\tau=0} = x_0(s[x_0;x]) \in \Gamma_\varphi \tag{6.3}$$

for some local parameter $s[x_0;x] \in \mathbb{R}^{n-1}$ at the moment of "time" $t(x) \in \mathbb{R}$ corresponding to an arbitrary reachable point $x \in \mathbb{R}^n$ as shown on Figure 3. Here we, in particular, assumed that the evolution mapping $(\Gamma_\varphi, \mathbb{R}) \ni (x_0, \tau) \rightarrow x(\tau;\ x_0) := x \in \mathbb{R}^n$ is invertible for almost all reachable points $x \in \mathbb{R}^n$ and, accordingly, for each so found point $x_0(s[x_0;x]) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$ one may suitably determine the unique point $p_0(s[x_0;x]) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. As a result, owing to conditions (6.3), one may write down the following expression:

$$u(x) = u_0(x_0([x_0;x])) + \int_{\tau=0}^{\tau=t(x)} \mathcal{L}(\tau|x_0(s[x_0;x]);x)d\tau, \tag{6.4}$$

where $\mathcal{L} : \mathbb{R} \times (\Gamma_\varphi \times \mathbb{R}^n) \rightarrow \mathbb{R}$ is the so called "quasi-Lagrangian" function:

$$\mathcal{L}(\tau|x_0([x_0;x]);x) := \langle p(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau)\rangle, \tag{6.5}$$

which is defined by solutions to characteristic vector field equations (4.15) under conditions (6.3). The expression (6.4), on integrating it with respect to parameter $\tau \in [0, t(x)] \subset \mathbb{R}$, reduces to the analytical form

$$u(x) = u_0(x_0(s[x_0;x])) + \mathcal{P}(x_0([x_0;x]);x), \tag{6.6}$$

where points $x_0(s(x)) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$, and, by definition, the "kernel" function is as follows:

$$\mathcal{P}(x_0(s[x_0;x]);x) := \int_{\tau=0}^{\tau=t(x)} \mathcal{L}(\tau|x_0([x_0;x]);x)d\tau. \tag{6.7}$$

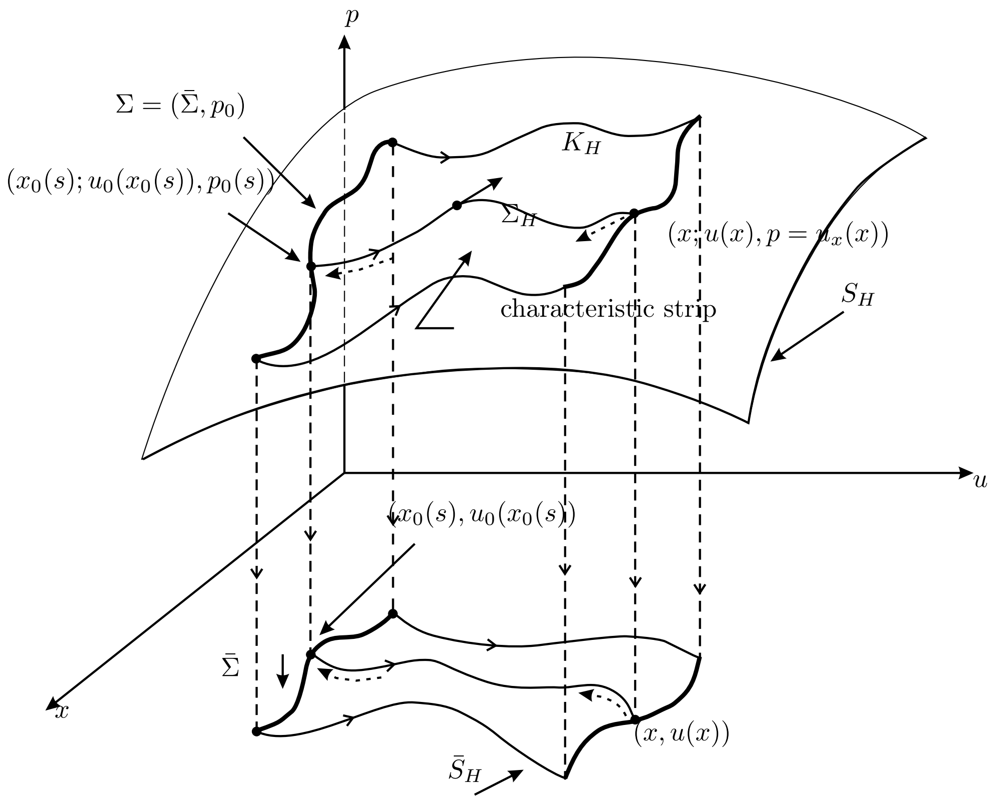

Fig. 3. Geometric Monge method. The characteristic surface: $S_H = \{(x; u, p) \in \mathbb{R}^{2n+1} : \quad H(x; u, p) = 0\}$ and initial conditions for the vector field $K_H : S_H \to T(S_H)$, satisfying the Cartan's compatibility conditions: $du - \langle p, dx \rangle_{|K_H, \Gamma_\varphi^{n-1}} = 0$ iff $\bar{S}_H || K^*$ and there exist data $\Sigma = (\bar{\Sigma}, p_0)$ defining the characteristic strip Σ_H

Expression (6.6) does solve equation (4.13) under boundary conditions (4.16) and may be effective enough for applications, if kernel-function (6.7) is constructed analytically. But, in general, if $\partial \bar{H}/\partial u \neq 0$ identically on S_H, quasi-Lagrangian function (6.5) depends effectively on the yet unknown solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$, which makes expressions (6.7) and (6.6) senseless. Since the latter expressions, obviously, strongly depend on a choice of the parametrisation $\pi := \psi(x; u, p) \in \mathbb{R}^n$, $(x; u, p) \in S_H$, at which the tensor field $\mu^{(1|1)} = (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ is under our disposition, one may propose a partial remedy to this problem.

Namely, to make optimum use of this possibility, let us additionally assume that our tensor field $\mu^{(1|1)} = (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ carries the associated symplectic structure. This means, speaking more generally, the existence of such a "symplectic" element $\Psi := (\psi_1, \psi_2)^\intercal \in C^2(\mathbb{R}^n \times \mathbb{R}^{n+1}; T^*(T^*(\mathbb{R}^n)))$ that for all

$(x,p) \in T^*(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathbb{R}^n$ the following equality holds

$$\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dp}{d\tau} \end{pmatrix} = -\vartheta \begin{pmatrix} \frac{\partial \bar{H}}{\partial x} + \psi \frac{\partial \bar{H}}{\partial u} \\ \frac{\partial \bar{H}}{\partial p} \end{pmatrix}, \tag{6.8}$$

where the co-symplectic operator $\vartheta : T^*(T^*(\mathbb{R}^n)) \to T(T^*(\mathbb{R}^n))$ of the form

$$\vartheta := \begin{pmatrix} 0 & -\mu^{(1|1)} \\ \mu^{(1|1),*} & 0 \end{pmatrix} \tag{6.9}$$

is defined as $\vartheta = \Omega^{-1}$ under the condition that the symplectic matrix

$$\Omega := \Psi' - \Psi'^{,*} = \begin{pmatrix} \frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1^*}{\partial x} & \frac{\partial \psi_1}{\partial p} - \frac{\partial \psi_2^*}{\partial x} \\ \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1^*}{\partial p} & \frac{\partial \psi_2}{\partial p} - \frac{\partial \psi_2^*}{\partial p} \end{pmatrix} \tag{6.10}$$

is nondegenerate. This, in particular, gives rise to the next important corollary: charactersitic vector field system (4.15) is Hamiltonian, allowing the natural Lagrangian extremality interpretation:

$$\frac{\delta}{\delta x} \int\limits_{\tau=0}^{\tau=t(x)} \tilde{\mathcal{L}}(x,\dot{x};u)d\tau = 0, \tag{6.11}$$

holding over the set of all smooth curves $x \in C^2([0,t(x)];\mathbb{R}^n)$, $x(0) = x_0 \in \Gamma_\varphi$, $x(t(x)) = x \in \mathbb{R}^n \backslash \Gamma_\varphi$. Here, by definition, we put $\dot{x} := dx/d\tau$ for $\tau \in [0,t(x)]$,

$$\tilde{\mathcal{L}}(x,\dot{x};u) := \langle \psi_1, \dot{x} \rangle + \langle \psi_2, \dot{p} \rangle - \bar{H}(x;u,p)\big|_{p=\alpha(x,\dot{x};u)}, \tag{6.12}$$

$\dot{p} := dp/d\tau$, where the vector $p := \alpha(x,\dot{x};u) \in \mathbb{R}^n$ solves the following system of equations, equivalent to (6.8):

$$\begin{array}{lll} (\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1^*}{\partial x})\dot{x} + & (\frac{\partial \psi_1}{\partial p} - \frac{\partial \psi_2^*}{\partial x})\dot{p} = & \frac{\partial \bar{H}}{\partial x} + \psi \frac{\partial \bar{H}}{\partial u}, \\ (\frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1^*}{\partial p})\dot{x} + & (\frac{\partial \psi_2}{\partial p} - \frac{\partial \psi_2^*}{\partial p})\dot{p} = & \frac{\partial \bar{H}}{\partial p} \end{array} \tag{6.13}$$

at points $(x,\dot{x};u) \in \mathbb{R}^{2n} \times \mathbb{R}$.

Lagrangian extremality condition (6.11) makes it possible to introduce a new "momentum" variable $\tilde{p} \in \mathbb{R}^n$, canonically conjugated with the variable $x \in \mathbb{R}^n$ as follows:

$$\tilde{p} := \partial \tilde{\mathcal{L}} / \partial \dot{x}. \tag{6.14}$$

This gives rise to a new canonical Hamiltonian system for conjugated variables $(x,\tilde{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ and a new Hamiltonian function $\tilde{H} : \mathbb{R}^{2n+1} \to \mathbb{R}$, completely equivalent to system (6.8)

$$\frac{dx}{d\tau} = \frac{\partial \tilde{H}}{\partial \tilde{p}}, \quad \frac{d\tilde{p}}{d\tau} = -\left(\frac{\partial \tilde{H}}{\partial x} + \psi \frac{\partial \tilde{H}}{\partial u}\right)\Big|_{p=\tilde{\alpha}(x,;u,\tilde{p})} \tag{6.15}$$

together with the compatibility equation

$$du/d\tau = \langle \tilde{\alpha}(x;u,\tilde{p}), \tilde{\beta}(x;u,\tilde{p})\rangle, \tag{6.16}$$

where, by definition, we put

$$\begin{aligned} &\tilde{\alpha}(x;u,\tilde{p}) := \alpha(x,\dot{x};u)|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})}, \quad p := \tilde{\alpha}(x;u,\tilde{p}) := \alpha(x,\dot{x};u)|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})}, \\ &\tilde{H}(x;\tilde{p}|u) := H(x;u,p) + \langle \tilde{p}-\psi_1, \tilde{\beta}\rangle - \langle \psi_2, d\tilde{\alpha}/d\tau\rangle|_{p:=\tilde{\alpha}(x;u,\tilde{p})}, \end{aligned} \tag{6.17}$$

based on the following relationships

$$\tilde{p} = \partial\tilde{\mathcal{L}}(x,\dot{x};u)/\partial\dot{x}\Big|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})}, \tag{6.18}$$

owing to the implicit function theorem, applied to (6.14) with respect to the variable $\dot{x} \in \mathbb{R}^n$.

Now we are in a position to write down the following Hamilton-Jacobi equation on the canonical transformations "generating" function $\tilde{u} \in C^2(\mathbb{R}^n \times \mathbb{R};\mathbb{R})$, corresponding to expressions (6.15) and (6.17):

$$\frac{\partial\tilde{u}}{\partial\tau} + \tilde{H}(x;\frac{\partial\tilde{u}}{\partial x}|u) = 0, \tag{6.19}$$

where the sought-for function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ satisfies equation (6.16).

Assume now for a while that the function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ is constant along vector field (6.8), that is

$$du/d\tau = \langle \tilde{\alpha}(x;u,\tilde{p}), \tilde{\beta}(x;u,\tilde{p})\rangle = 0 \tag{6.20}$$

for all $(x;u,\tilde{p}) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$. The condition (6.20) involves some constraints on the "symplectic" vector $\Psi = (\psi_1,\psi_2)^\intercal \in C^2(\mathbb{R}^n \times \mathbb{R}^{n+1};\mathbb{R}^n \times \mathbb{R}^n)$, which may be satisfied by means of choosing a suitable parametrization $\pi := \psi(x;u,p) \in \mathbb{R}^n$, $(x;u,p) \in S_H$, of the characterstic hypersurface S_H. Proceed now to solving the canonical Hamilton-Jacobi equation (6.19) under some Cauchy data $\tilde{u}|_{t=0} = \tilde{u}_0 \in C^2(\Gamma_\varphi;\mathbb{R})$. This problem may be solved easily enough via the standard Hopf-Lax type [19, 22] scheme. Namely, consider inverse Cauchy problem (6.3) for canonical Hamilton equations (6.15) of the form

$$\frac{dx}{d\tau} = \frac{\partial\tilde{H}}{\partial\tilde{p}}, \quad \frac{d\tilde{p}}{d\tau} = -(\frac{\partial\tilde{H}}{\partial\tilde{p}} + \tilde{\alpha}\frac{\partial\tilde{H}}{\partial u}), \tag{6.21}$$

where the parameter $\tau \in [0,t(x)] \in \mathbb{R}$. Then the corresponding solution to Hamilton-Jacobi equation (6.19) possesses the functional-analytical Hopf-Lax type form

$$\tilde{u}(x,t|u) = \inf_{y\in\Gamma_\varphi}\{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t,x;y|u)\}, \tag{6.22}$$

following right away from the expression analogous to (6.6), where, by definition, the "kernel"

$$\tilde{\mathcal{P}}(t,x;y|u) := \int_0^t \tilde{\mathcal{L}}(\tau;y,x|u)d\tau \tag{6.23}$$

is obtained from the Lagrangian function

$$\tilde{\mathcal{L}}(\tau; y, x|u) = \tilde{\mathcal{L}}(x, \dot{x}; u)\Big|_{x=\tilde{x}(\tau;y,x|u)}, \ \tilde{\mathcal{L}}(x, \dot{x}; u) := \langle \tilde{p}, \frac{\partial \tilde{H}}{\partial \tilde{p}} \rangle - \tilde{H}(x; u, \tilde{p}), \tag{6.24}$$

calculated on solutions to equations (6.21) under conditions (6.3). Then, owing to conditions (6.24), (6.17) and (6.16), the equality

$$\tilde{H}(x; \tilde{p}|u) = H(x; u, p) + \langle \tilde{p} - \psi_1, \tilde{\beta} \rangle - \langle \psi_2, d\tilde{\alpha}/d\tau \rangle|_{\tau=t(x)}, \tag{6.25}$$

holds for suitable $x_0 = x_0(x) \in \Gamma_\varphi$ and $\tau = t(x) \in \mathbb{R}$. Moreover, as $H(x; u, p) = 0$ for all points $(x; p|u) \in T^*(\mathbb{R}^n) \times \mathbb{R}$, equality (6.25) reduces to

$$\tilde{H}(x; u, \tilde{p}) = \langle \tilde{u}_x - \psi_1, \tilde{\beta} \rangle - \langle \psi_2, d\tilde{\alpha}/d\tau \rangle|_{\tau=t(x)}, \tag{6.26}$$

which will be later used for determining the sought-for solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ in an implicit form. To do this much more effectively, we consider expression (6.22) at $t = t(x) \in \mathbb{R}$ obtained in the following functional analytic form:

$$\tilde{u}(x, t(x)|u) = \inf_{y \in \Gamma_\varphi} \{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t(x), x; y|u)\}, \tag{6.27}$$

taking into account boundary condition (4.16) for the corresponding solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ of the Hamilton-Jacobi equation (4.13) at $x_0 = x_0(x|u) \in \Gamma_\varphi$ for all reachable points $x \in \mathbb{R} \backslash \Gamma_\varphi$. The Cauchy data $\tilde{u}_0 \in C^2(\Gamma_\varphi; \mathbb{R})$ may be taken, in general, arbitrarily, but so that infimum (6.27) exists and the conditions

$$\tilde{p}_0(x_0) = \partial \tilde{u}(x_0, \tau|u)/\partial x|_{\tau=0} = \partial \tilde{\mathcal{L}}(x, \dot{x}; u)/\partial \dot{x}\Big|_{\tau=0} \tag{6.28}$$

hold, if equations (6.15) and (6.20) are satisfied. Therefore, if the point $\bar{y} := y(x|u) \in \Gamma_\varphi$ is such that

$$\inf_{y \in \Gamma_\varphi} \{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t(x), x; y|u)\} = \tilde{u}_0(\bar{y}) + \tilde{\mathcal{P}}(t(x), x; \bar{y}|u), \tag{6.29}$$

then the trajectory of the point $y(x|u) = x_0(x|u) \in \Gamma_\varphi$ along vector field (6.21) will necessary satisfy condition (6.20), which makes it possible to write down the following implicit expression for the sought-for solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$:

$$u(x) = u_0(y(x|u(x))), \tag{6.30}$$

where $\bar{y} := y(x|u) \in \Gamma_\varphi$ satisfies the following determining relationship

$$\tilde{p}_0(\bar{y}) + \partial \tilde{\mathcal{P}}(t(x), x; \bar{y}|u)/\partial y = 0, \tag{6.31}$$

stemming from condition (6.28). Thereby, we may formulate the obtained result as the following theorem.

Theorem 1. *Implicit expression* (6.30) *gives rise to a functional-analytic solution to boundary problem* (4.13) *and* (4.16), *depending on the given boundary data* $u_0 \in C^2(\Gamma_\varphi;\mathbb{R})$.

Based on the derivation of the above result, we may conclude that the statement of the above theorem holds, in general, for any nontrivial smooth Hamiltonian function $H \in C^2(\mathbb{R}^{2n+1};\mathbb{R})$, for which the following two conditions

$$rank\Omega = 2n,\ rank(\frac{\partial}{\partial p}[\vartheta_{11}(\frac{\partial \bar{H}}{\partial x} + \psi\frac{\partial \bar{H}}{\partial u}) + \vartheta_{12}\frac{\partial \bar{H}}{\partial p}]) = n \tag{6.32}$$

are satisfied almost everywhere on $T^*(\mathbb{R}^n)\times\mathbb{R}$. Conditions (6.32) should hold simultaneously with that of (6.20), giving rise to implicit solution (6.29) to Hamilton-Jacobi equation (4.13) under boundary condition (4.14).

It is now easy to see that expression (6.30) is equivalent to some fixed point problem $P(u) = u$, $u \in C^2(\mathbb{R}^n;\mathbb{R})$, for the associated nonlinear mapping $P : C^2(\mathbb{R}^n;\mathbb{R}) \to C^2(\mathbb{R}^n;\mathbb{R})$, where, by definition,

$$P(u)(x) := u_0(y(x|u(x))) \tag{6.33}$$

for all reachable points $x \in \mathbb{R}^n$. This observation may be formulated as the following important theorem.

Theorem 2. *A solution to functional-analytic fixed point problem* (6.33) *solves simultaneously boundary problem* (4.16) *to generalized Hamilton-Jacobi equation* (4.13).

7. THE STRUCTURE OF HOPF-LAX TYPE FUNCTIONAL-ANALYTIC SOLUTIONS TO GENERALIZED HAMILTON-JACOBI EQUATIONS

Consider the following generalized nonlinear Hamilton-Jacobi equation

$$\partial u/\partial t + H(x,t;u,u_x) = 0 \tag{7.1}$$

with a Hamiltonian function $H \in C^2(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1};\mathbb{R})$ and pose the Cauchy problem

$$u|_{t=0} = u_0, \tag{7.2}$$

where $u_0 \in C^1(\mathbb{R}^n;\mathbb{R})$ and $t \in \mathbb{R}$ is an evolution parameter. For investigating functional-analytic solutions to Hamiton-Jacobi equation (7.1) we will apply the generalized characteristics method, proposed above. Namely, consider the following non-canonical Hamiltonian vector field on the cotangent space $T^*(\mathbb{R}^{n+1}) \ni (x,t;p,\sigma)$, generated by a non-degenerate Hamiltonian function $H \in C^2(\mathbb{R}^{2n+2};\mathbb{R})$, where the function $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$ is *a priori* assumed to solve equation (7.1) under condition (7.2), that is

$$\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dp}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & \mu^{(1|1)} \\ -\mu^{(1|1),*} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}}{\partial x} + \psi\frac{\partial \bar{H}}{\partial u} \\ \frac{\partial \bar{H}}{\partial p} \end{pmatrix}, \tag{7.3}$$
$$\frac{d\sigma}{d\tau} = -\left(\frac{\partial \bar{H}}{\partial t} + \sigma\frac{\partial \bar{H}}{\partial u}\right), \qquad \frac{dt}{d\tau} = 1,$$

where the tensor field $\mu^{(1|1)} := (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+2};\mathbb{R}^n \otimes \mathbb{R}^n)$ is chosen with respect to a suitable parametrization $\pi := \psi(x,t;u,p)$, $(x,t;u,p) \in S_H$, of the characteristic surface

$$S_H := \{(x,t;u,p,\sigma) \in \mathbb{R}^{2n+2} : \sigma + \bar{H}(x,t;u,p) = 0,$$
$$\bar{H}(x,t;u,p) := H(x,t;u,\pi)|_{\pi=\psi(x,t;u,p)},$$

compatible with the Cartan condition

$$du/d\tau = \langle\psi, \mu^{(1|1)}\frac{\partial\tilde{H}}{\partial p}\rangle - \bar{H}(x,t;u,p). \tag{7.4}$$

Since flow (7.3) is Hamiltonian, it may be represented [8, 21] dually in the related Lagrangian variatonal form:

$$\frac{\delta}{\delta x}\int_{\tau=0}^{\tau=t}\mathcal{L}(x,t;\dot{x}|u)d\tau\Bigg|_{\substack{x(0)=x_0\in\mathbb{R}^n\\ x(t)=x\in\mathbb{R}^n}} = 0 \tag{7.5}$$

for any $t \in \mathbb{R}$ and fixed points $x(0) = x_0 \in \mathbb{R}^n$ and $x(t) = x \in \mathbb{R}^n$. Here, as before, we use the notation $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, and

$$\mathcal{L}(x,t;\dot{x}|u) := \langle\psi, \mu^{(1|1)}\frac{\partial\bar{H}}{\partial p}\rangle - \bar{H}(x,t;u,p)\Big|_{p:=\alpha(x,t;\dot{x}|u)} \tag{7.6}$$

the corresponding quasi-Lagrangian function, and denoted, by definition, $p := \alpha(x,t;\dot{x}|u)$, and $\dot{p} = dp/d\tau$, solving implicitly the system of equations

$$\dot{x} - \mu^{(1|1)}\frac{\partial\tilde{H}}{\partial p} = 0, \ \dot{p} + \mu^{(1|1),*}\left(\frac{\partial\bar{H}}{\partial x} + \psi\frac{\partial\bar{H}}{\partial u}\right) = 0 \tag{7.7}$$

under the inverse Cauchy data

$$x|_{\tau=t} = x \in \mathbb{R}^n, \ x|_{\tau=0} = x_0(x,t) \in \mathbb{R}^n, p|_{\tau=0} = p_0(x,t) \in \mathbb{R}^n \tag{7.8}$$

for any fixed point $(x,t) \in \mathbb{R}^{n+1}$. Note also, that the first equation of (7.8) is always uniquely solvable with respect to the variable $p \in \mathbb{R}^n$, owing to the nondegeneracy condition

$$rank(\frac{\partial}{\partial p}[\langle\psi, \mu^{(1|1)}\frac{\partial\bar{H}}{\partial p}\rangle]) = n, \tag{7.9}$$

assumed earlier. Based now on Lagrangian variational form (7.5), one may construct the following functional-analytical Hopf-Lax type representation for the solution of Hamilton-Jacobi equation under condition (7.14):

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\{u_0(y) + \mathcal{P}(x,t;y|u)\}, \tag{7.10}$$

where, by definition, the “kernel” function

$$\mathcal{P}(x,t;y|u) := \int_{\tau=0}^{\tau=t} \mathcal{L}(x,t;\dot{x}|u)d\tau \tag{7.11}$$

is calculated on solutions to Hamiltonian equations (4.9) under conditions (7.7). In the case when $\partial H/\partial u \neq 0$ identically on S_H, we need to make the next step to skirt this problem, as expression (7.10) becomes senseless, depending on the unknown solution $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$. Assume now that the parametrization $\pi := \psi(x,t;u,p) \in \mathbb{R}^n$, $(x,t;u,p) \in S_H$, of the characteristic surface S_H is chosen so as to make expression (7.4) vanish identically:

$$du/d\tau = \langle \psi, \mu^{(1|1)}\frac{\partial \tilde{H}}{\partial p}\rangle - \bar{H}(x,t;u,p)\bigg|_{S_H} = 0. \tag{7.12}$$

Since infimum (7.10) is then attained at some point $x_0 = \bar{y} := y(x,t|u) \in \mathbb{R}^n$ for an arbitrary but fixed point $(x,t|u) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and constant value $u = u_0(\bar{y}) \in \mathbb{R}$, we may write down two important relationships:

$$\partial u_0(\bar{y})/\partial y := \psi(x(\tau),\tau;u(\tau),p(\tau))|_{\tau=0} = \psi(\bar{y},0,u_0(\bar{y}),p_0(\bar{y}|u)), \tag{7.13}$$

where the initial vector $p_0(x,t) := p_0(\bar{y}|u) \in \mathbb{R}^n$ depends on the chosen constant value $u = u_0(\bar{y}) \in \mathbb{R}$, and

$$u(x,t) = u_0(y(x,t|u(x,t)), \tag{7.14}$$

holding for all $(x,t) \in \mathbb{R}^n$. Thereby, having solved equation (7.13) with respect to the critacal point $\bar{y} := y(x,t|u) \in \mathbb{R}^n$, one can directly write down the solution to Hamilton-Jacobi equation (7.1) with Cauchy data (7.2) for all $(x,t) \in \mathbb{R}^{n+1}$ in implicit functional-analytic form (7.14). Expression (7.14) is, evidently, equivalent to the fixed point problem $P(u) = u$, $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$, for te associated nonlinear mapping $P : C^2(\mathbb{R}^n \times \mathbb{R};\mathbb{R}) \to C^2(\mathbb{R}^n \times \mathbb{R};\mathbb{R})$, where, by definition,

$$P(u)(x,t) := u_0(y(x,t|u(x,t)) \tag{7.15}$$

for all reachable points $(x,t) \in \mathbb{R}^{n+1}$. The result obtained above may be formulated as the following theorem.

Theorem 3. *A solution to functional-analytic fixed point problem* (7.15) *solves simultaneously Cauchy problem* (7.1) *to generalized Hamilton-Jacobi equation* (7.2).

Fixed point problem (7.15), in general, is solved [2] under some weak enough conditions on operator (7.15), but its solution, as is well known [2, 22, 27], is not often unique, thus more of its additional properties are to be studied. We hope to investigate such and related problems in detail elsewhere.

As an example, consider the canonical Hamilton-Jacobi equation

$$u_t + ||u_x||^2/2 = 0, \tag{7.16}$$

where $||\cdot||$ is the standard norm in the Euclidean space $\mathbb{E}^n := (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, and try to construct its exact functional-analytic [19,20,22] generalized solutions $u : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{R}$, satisfying the Cauchy condition

$$u|_{t=+0} = u_0 \tag{7.17}$$

for a given function $u_0 : \mathbb{E}^n \to \mathbb{R}$. One can easily enough to state, making use of the characteristics method [19, 22, 29, 30], that equation (7.16) possesses for smooth Cauchy data $u_0 \in C^1(\mathbb{E}^n;\mathbb{R})$ an exact functional-analytic generalized solution in the form

$$u(x,t) = u_0(y) + \frac{1}{2t}||x-y||^2, \tag{7.18}$$

where a vector $y := y(x,t) \in \mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ satisfies the following determining equation

$$\partial u_0(y)/\partial y - (x-y)/t = 0. \tag{7.19}$$

It was proved in [3, 19, 22] that in a more general case of convex and below semicontinuous Cauchy data $u_0 \in BSC_{(c)}(\mathbb{R}^n;\mathbb{R})$ the expression (7.19) allows the completely equivalent to (7.20) so called Hopf-Lax type representation

$$u(x,t) = \inf_{y\in\mathbb{E}^n}\{u_0(y) + \frac{1}{2t}||x-y||^2\}, \tag{7.20}$$

being a generalized [22] solution to the Hamilton-Jacobi equation (7.16). Solution (7.18) satisfies [19] the following natural asymptotic "viscosity" property: $\lim_{t\to\infty} u(x,t) = \inf_{y\in\mathbb{E}^n}\{u_0(y)\}$ for almost all $x \in \mathbb{E}^n$. In general, Cauchy problem (7.16) and (7.17) with functions $u_0 \in BSC(\mathbb{E}^n;\mathbb{R}) \cap C^1(\mathbb{E}^n;\mathbb{R})$ possesses a unique functional-analytic representation for its generalized solutions satisfying the standard viscosity property.

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