

Volodymyr Denysenko

**TRANSMISSION PROBLEMS  
FOR THE HELMHOLTZ EQUATION  
FOR A RECTILINEAR-CIRCULAR LUNE**

**Abstract.** The question related to the construction of the solution of plane transmission problem for the Helmholtz equation in a rectilinear-circular lune is considered. An approach is proposed based on the method of partial domains and the principle of reflection for the solutions of the Helmholtz equation through the segment.

**Keywords:** Helmholtz equation, transmission problem, infinite system of linear algebraic equations.

**Mathematics Subject Classification:** Primary 65N38; Secondary 35J25.

## 1. INTRODUCTION

It is well known that numerous acoustic situations may be analysed with use of models that lead to the solution of boundary-value problems for the Helmholtz equation (see, e.g., [1]). The method of partial domains is extensively and successfully employed in the investigation of various problems involving the emission and diffraction of acoustic waves [1, 2]. In [3] a new approach was suggested for the construction of solutions of various external and internal boundary-value problems for the Helmholtz equation in domains whose boundaries consisted of rectilinear segments and arcs of circles. This approach utilizes general ideas of the method of partial domains combined with the classical principle of reflection through straight-line segments for the solution of the Helmholtz equation. Note that the state of the problem of extension of wave fields, including applied aspects pertaining to external problems of diffraction is surveyed in [4]. In the present article the possibility of employing the principle of reflection for the construction of the solution of plane transmission problem for the Helmholtz equation [5] in a rectilinear-circular lune is analyzed. Problems of this were not considered in [3].

## 2. TRANSMISSION PROBLEM FOR THE HELMHOLTZ EQUATION

It is known (see [5], Ch. 3), that mathematical formulation of the transmission problem of acoustic waves on the domain  $\Omega$  with different acoustics properties in  $\Omega$  and  $D = \mathbb{R}^2 \setminus \bar{\Omega}$  leads to the following conjugate problem: find two functions  $u$  and  $v$  which satisfy the Helmholtz equations

$$\Delta u(x, y) + k^2 u_0(x, y) = 0, \quad (x, y) \in \Omega, \quad (1)$$

$$\Delta v(x, y) + k_1^2 v(x, y) = 0, \quad (x, y) \in D, \quad (2)$$

and conjugate conditions

$$\mu u - v = f, \quad \text{on } \Gamma, \quad (3)$$

$$\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = g, \quad \text{on } \Gamma, \quad (4)$$

where  $\Gamma = \partial\Omega$ ,  $\partial/\partial n$  is the normal derivative on  $\Gamma$ ;  $f$  and  $g$  are the functions defined on  $\Gamma$ , and  $k, k_1, \mu$  are positive numbers.

Here the boundary conditions given by (3), (4) are supplemented by the conditions of emission at infinity

$$r^{1/2} \left\{ \frac{\partial v}{\partial r} - ikv \right\} = o(1), \quad r \rightarrow \infty. \quad (5)$$

## 3. CONSTRUCTION OF A SOLUTION

We are here concerned with the formulation of an algorithm for solving boundary-value problem (1)–(5) when the domain  $\Omega$  is a rectilinear-circular lune. Note that in this situation it is impossible to find a solution by a direct application of the method of partial domains, in spite of the fact that the boundary of the domain  $\Omega$  consists of a combination of canonical coordinate curves (for details, see [3]).

Let  $(r, \theta)$  be polar coordinates on the plane:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

In what follows, for convenience, in subsequent situations it will be assumed that the angle  $\theta$  ranges within the limits  $[0, 2\pi]$  or  $\theta \in [-\pi, \pi]$ . For the specified numbers  $a > 0$  and  $b \in (0, a)$  let us define the domain  $\Omega$  as the intersection of circle  $r < a$  and half plane  $x > b$ , and let  $D = \mathbb{R}^2 \setminus \bar{\Omega}$  be the exterior of  $\Omega$ . Let  $\Omega_0$  be the intersection of the circle  $r < a$  and the domain  $D$  (or of the half plane  $x < b$ ). Let the segment  $x = b, |y| < d := \sqrt{a^2 - b^2}$  be denoted by  $\gamma$  and let the arcs be

$$\gamma_1 : r = a, |\theta| < \theta_0, \quad \gamma_0 : r = a, \theta \in (\theta_0, 2\pi - \theta_0),$$

where the angle  $\theta_0 \in (0, \pi/2)$  is defined by the equation

$$\cos \theta_0 = b/a.$$

In what follows we denote by  $\Gamma$  the boundary of the domain  $\Omega$ , and consequently  $\Gamma = \gamma_1 \cup \gamma$ .

In the following we for simplicity suppose that  $f \equiv 0$ ,  $g \equiv 0$  on the segment  $\gamma$  and that  $f, g$  are even functions in the variable  $y$  on  $\gamma_1$ . So, the unknown functions  $u, v$  will also be even functions on the variable  $y$ .

We shall represent the domain  $D$  as the closure of the union of the bounded domain  $\Omega_0$  and the unbounded domain  $r > a$ . In this connection we seek the function  $v$  in the following form

$$v = v_1, \quad |r| > a, \quad v = v_0, \quad (x, y) \in \Omega_0, \quad (6)$$

where each of the functions  $v_0, v_1$  is supposed to satisfy Helmholtz equation (2) in the corresponding domain, and emission condition (5) is true for  $v_1$ . Here the desired function  $v_1$  out of the circle  $r \leq a$  is represented in the form of series [1]

$$v_1(r, \theta) = \sum_{m=0}^{\infty} C_m \frac{H_m^{(1)}(k_1 r)}{H_m^{(1)}(k_1 a)} \cos(m\theta), \quad r > a, \quad (7)$$

where  $H_m^{(1)}(\cdot)$  is the Hankel function of the first kind and order  $m$ .

Let us consider the question of appropriate representation of the function  $v = v_0$  in the domain  $\Omega_0$ . Let us introduce into consideration the unknown even function

$$w(y) = \left. \frac{\partial v}{\partial x} \right|_{x=b}, \quad |y| < d, \quad (8)$$

with the expansion into a Fourier series

$$w(y) = \sum_{m=1}^{\infty} B_m \cos \beta_m y, \quad \beta_m = \frac{\pi(2m-1)}{2d}. \quad (9)$$

We for simplicity suppose that  $\beta_m^2 \neq k_1^2$ ,  $m = 1, 2, \dots$ . Let, according to (8), (9), the function  $w_0$  be given as

$$w_0(x, y) = \sum_{m=1}^{\infty} \frac{B_m}{\sqrt{\beta_m^2 - k_1^2}} e^{\sqrt{\beta_m^2 - k_1^2}(x-b)} \cos \beta_m y, \quad x < b, \quad -\infty < y < \infty. \quad (10)$$

The function  $w_0(x, y)$  is a  $y$ -periodic (with the period  $T = d$ ) solution of equation (2) in the half-plane  $x < b$  (in particular,  $w$  is a solution of (2) in the domain  $\Omega_0$ ) and satisfies the boundary condition

$$\frac{\partial w_0}{\partial x} = w(y), \quad |y| < d, \quad x = b. \quad (11)$$

We seek the solution  $v_0$  in the form

$$v_0 = w_0 + w_1, \quad w_1 = v_0 - w_0. \quad (12)$$

The functions  $w_0, w_1$  satisfy Helmholtz equation (2) in the domain  $\Omega_0$ . The satisfaction of boundary conditions (3), (4) on the segment  $\gamma$  yields the equations (see (11)):

$$\mu u - (w_0 + w_1) = 0, \quad x = b, |y| < d, \quad (13)$$

$$\frac{\partial w_1}{\partial x} = 0, \quad x = b, |y| < d, \quad (14)$$

$$\frac{\partial u}{\partial x} - w(y) = 0, \quad x = b, |y| < d. \quad (15)$$

Notice that for  $b > 0$  the domain  $\Omega_0$  contains the semi-circle  $r < a, \theta \in (\pi/2, 3\pi/2)$ , which makes the direct application of the method of partial domains impossible (see [3]). On the other hand, since homogeneous boundary condition (14) for the function  $w_1$  is satisfied on the boundary segment  $\gamma$ , then according to the reflection principle, the function  $w_1$  is reflected symmetrically into the domain

$$\Omega^* = \{(x, y) : (-x + 2b, y) \in \Omega\},$$

according to the rule

$$w_1(x, y) = w_1(-x + 2b, y), \quad (x, y) \in \Omega^*. \quad (16)$$

Here the closure of the combined domains  $\Omega$  and  $\Omega^*$  contains the circle  $r \leq a$  and hence the solution  $w_1$  of Helmholtz equation (2) continues into this circle as the solution of equation (2). Therefore, the function  $w_1$  can be sought as the series [1]

$$w_1 = \sum_{n=0}^{\infty} A_n \frac{J_n(k_1 r)}{J_n(k_1 a)} \cos(n\theta), \quad r < a, \quad \theta \in [0, 2\pi], \quad (17)$$

where  $J_n(\cdot)$  are Bessel functions, whereas  $A_n$  are unknown coefficients (here it is assumed that  $J_n(ka) \neq 0, n = 0, 1, \dots$ ).

Let us consider the appropriate representation for the function  $u$  in  $\Omega$ . We shall represent the domain  $\Omega$  as the intersection of the sector  $\{(r, \phi) : 0 < r < a, |\theta| < \theta_0\}$  and the half-strip  $\{(x, y) : x > b, |y| < d\}$ . This means that we seek the solution  $u$  in the form [1]:

$$u = \sum_{n=1}^{\infty} E_n \frac{J_{\alpha_n}(kr)}{J_{\alpha_n}(ka)} \cos(\alpha_n \theta) + \sum_{m=1}^{\infty} \frac{D_m}{\sqrt{\beta_m^2 - k^2}} e^{\sqrt{\beta_m^2 - k^2}(b-x)} \cos \beta_m y, \quad r < a, \quad |\theta| \leq \theta_0, \quad (18)$$

where  $\alpha_n = (2n - 1)\pi/(2\theta_0)$ , and it is assumed that  $\beta_m^2 \neq k^2, m = 1, 2, \dots$ , and  $J_{\alpha_n}(ka) \neq 0, n = 1, 2, \dots$

For the determination of the coefficients  $A_n$  in Eqs. (17), we have the conjugate boundary conditions on the arc  $\gamma_0$

$$w_1 = v_1, \quad r = a, \quad \theta \in (\theta_0, 2\pi - \theta_0), \quad (19)$$

$$\frac{\partial w_1}{\partial r} = \frac{\partial v_1}{\partial r}, \quad r = a, \quad \theta \in (\theta_0, 2\pi - \theta_0) \quad (20)$$

So, we must find the condition at the remaining arc  $|\theta| < \theta_0$ . The idea [3] consists in taking this condition from Eqs. (16) and formulating it in terms of the same unknown coefficients  $A_n$ . Proceeding in this manner and utilizing the fact that the trigonometric functions in Eqs. (17) are orthogonal, we may obtain an infinite set of linear algebraic equations for  $A_n$ . The implementation of this idea is described below.

If the point  $(r, \theta) \in \Omega_0^*$ , then its inverse image in reflection (16) is the point with polar coordinates  $\hat{r}, \hat{\theta}$  such that

$$\hat{r} \sin \hat{\theta} = r \sin \theta, \quad \hat{r} \cos \hat{\theta} = r \cos \theta + 2b.$$

Solving this equations, we get

$$\hat{r}(r, \theta) = \sqrt{r^2 - 4rb \cos \theta + 4b^2}, \quad \hat{\theta}(r, \theta) = \arcsin \left( \frac{r \sin \theta}{\hat{r}(r, \theta)} \right).$$

Thus, from Eqs. (16) and (17), we have obtained the following expression for the unknown function  $w_1$  in the domain  $\Omega_0^*$ , and hence in the domain  $\Omega \subset \Omega_0^*$ :

$$w_1(r \cos \theta, r \sin \theta) = \sum_{n=0}^{\infty} A_n \frac{J_n(k_1 \hat{r})}{J_n(k_1 a)} \cos(n\hat{\theta}), \quad (r, \theta) \in \Omega_0^*.$$

In particular,

$$w_1(a \cos \theta, a \sin \theta) = \sum_{n=0}^{\infty} A_n \frac{J_n(\hat{k}_1 r^*)}{J_n(\hat{k}_1)} \cos(n\theta^*), \quad |\theta| < \theta_0, \quad (21)$$

where

$$r^*(\theta) = a^{-1} \hat{r}(a, \theta) = \sqrt{1 + 4 \cos^2 \theta_0 - 4 \cos \theta_0 \cos \theta},$$

$$\theta^*(\theta) = \hat{\theta}(a, \theta) = \arcsin \left( \frac{\sin \theta}{r^*(\theta)} \right), \quad \hat{k}_1 = k_1 a.$$

Then, using (7), (19) and (21), we obtain the following functional equations for the unknown coefficients  $A_n$ :

$$\sum_{n=0}^{\infty} A_n \cos(n\theta) = \begin{cases} \sum_{m=0}^{\infty} A_m \frac{J_m(\hat{k}_1 r^*)}{J_m(\hat{k}_1)} \cos(m\theta^*), & \theta \in (0, \theta_0), \\ \sum_{m=0}^{\infty} C_m \cos(m\theta), & \theta \in (\theta_0, \pi). \end{cases} \quad (22)$$

On the other hand, from (13), (15) we derive the functional equations

$$\sum_{m=1}^{\infty} \left( \frac{\mu D_m}{\sqrt{\beta_m^2 - k^2}} - \frac{B_m}{\sqrt{\beta_m^2 - k_1^2}} \right) \cos(\beta_m y) =$$

$$= \left\{ \sum_{n=0}^{\infty} A_n \frac{J_n(k_1 r)}{J_n(\hat{k}_1)} \cos(n\theta) - \sum_{n=1}^{\infty} \mu E_n \frac{J_{\alpha_n}(kr)}{J_{\alpha_n}(\hat{k})} \cos(\alpha_n \theta) \right\} \Big|_{x=b}, \quad |y| < d, \quad (23)$$

$$\sum_{m=1}^{\infty} (B_m + D_m) \cos(\beta_m y) = \frac{\partial}{\partial x} \left\{ \sum_{n=1}^{\infty} E_n \frac{J_{\alpha_n}(kr)}{J_{\alpha_n}(\hat{k})} \cos(\alpha_n \theta) \right\} \Big|_{x=b}, \quad |y| < d. \quad (24)$$

Conjugate condition (20) on the arc  $\gamma_0$  and conjugate conditions (3), (4) on the arc  $\gamma_1$  lead to the equations

$$\begin{aligned} & \sum_{m=0}^{\infty} C_m \frac{k_1 H_m^{(1)' }(\hat{k}_1)}{H_m^{(1)}(\hat{k}_1)} \cos(m\theta) = \\ & = \begin{cases} k \sum_{n=1}^{\infty} E_n \frac{J'_{\alpha_n}(\hat{k})}{J_{\alpha_n}(\hat{k})} \cos(\alpha_n \theta) + \frac{\partial}{\partial r} \left\{ \sum_{m=1}^{\infty} D_m \frac{e^{\sqrt{\beta_m^2 - k^2}(b-x)}}{\sqrt{\beta_m^2 - k^2}} \cos(\beta_m y) \right\} \Big|_{r=a} - \\ \qquad \qquad \qquad - g(\theta), \quad \theta \in (0, \theta_0), \\ \sum_{n=0}^{\infty} k_1 A_n \frac{J'_n(\hat{k}_1)}{J_n(\hat{k}_1)} \cos(n\theta), \quad \theta \in (\theta_0, \pi), \end{cases} \quad (25) \\ & \mu \sum_{n=1}^{\infty} E_n \cos(\alpha_n \theta) = \sum_{m=0}^{\infty} C_m \cos(m\theta) - \\ & \qquad \qquad \qquad - \mu \sum_{m=1}^{\infty} D_m \frac{e^{\sqrt{\beta_m^2 - k^2}(b-x)}}{\sqrt{\beta_m^2 - k^2}} \cos(\beta_m y) \Big|_{r=a} + f(\theta) \quad \theta \in (0, \theta_0). \end{aligned} \quad (26)$$

From Eqs. (22)–(26) we can easily obtain an infinite set of linear algebraic equations for the determination of the unknown coefficients  $A_n, B_m, C_n, D_m, E_n, n = 0, 1, 2, \dots, m = 1, 2, \dots$  (for details, see [3]). For example, from (22), employing the orthogonality equations

$$\frac{2}{\pi} \int_0^{\pi} \cos(n\theta) \cos(m\theta) d\theta = \delta_{nm}(1 + \delta_{n0}), \quad n, m = 0, 1, \dots,$$

we obtain the set of linear algebraic equations,

$$A_n \frac{(1 + \delta_{n0})\pi}{2} = \sum_{m=0}^{\infty} A_m \int_0^{\theta_0} \frac{J_m(\hat{k}r^*)}{J_m(\hat{k})} \cos(m\theta^*) \cos(n\theta) d\theta + \sum_{m=0}^{\infty} C_m L_{m,n},$$

where  $n = 0, 1, 2, \dots$  and the coefficients

$$\begin{aligned} L_{m,n} &= \int_{\theta_0}^{\pi} \cos(m\theta) \cos(n\theta) d\theta = -\frac{\sin(n-m)\theta_0}{2(n-m)} - \frac{\sin(n+m)\theta_0}{2(n+m)}, \quad n \neq m, \\ L_{n,n} &= \int_{\theta_0}^{\pi} \cos^2(n\theta) d\theta = \frac{\pi - \theta_0}{2} - \frac{\sin(2n\theta_0)}{4n}, \quad n \geq 1, \\ L_{0,0} &= \int_{\theta_0}^{\pi} d\theta = \pi - \theta_0. \end{aligned}$$

In an analogous way we can obtain the other four sets of linear algebraic equations for the unknown coefficients in representations (7), (10), (17) and (18).

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Volodymyr Denysenko  
math@knteu.kiev.ua

Kyiv National University of Trade and Economics  
19 Kioto Street, Kyiv 02156, Ukraine

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