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POLYNOMIALS ON THE SPACE OF ω-ULTRADIFFERENTIABLE FUNCTIONS

Abstract. The space of polynomials on the the space D_ω of ω-ultradifferentiable functions is represented as the direct sum of completions of symmetric tensor powers of D'_ω.

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1. INTRODUCTION

By ultradistributions we mean the elements in the dual space of a non-quasi analytic class of infinitely differentiable functions equipped with a natural locally convex topology, as defined by Roumieu and Beurling (see, e.g. [6]). In this paper, we shall consider the space of ultradifferentiable functions introduced by Braun, Meise and Taylor in their paper [1].

We introduce the class of polynomial ultradistributions, namely the elements of the space $\mathcal{P}(D_\omega)$, where D_ω denotes the space of ultradifferentiable functions. The space $\mathcal{P}(D_\omega)$ contains the space of ultradistributions as a proper subspace and it is the smallest space in which multiplication of ultradistributions is possible. We shall describe the space $\mathcal{P}(D_\omega)$ in the terms of the direct sums of symmetric tensor powers of the space D'_ω. We shall also prove that such a direct sum is a convolution algebra.

We shall also prove that $\mathcal{P}(D_\omega)$ is topologically isomorphic to the space S' of ultradistributions for functions in infinitely many variables, which means that it can itself be treated as the space of ultradistributions.

2. POLYNOMIALS ON LOCALLY CONVEX SPACES

First we shall introduce some notation (cf. [3]). Let X be a locally convex space and $\mathcal{L}^n(X,\mathbb{C})$ denote the space of n-linear, continuous forms defined on X

$$F_n : \prod_{\imath=1}^{n} X := \underbrace{X \times \ldots \times X}_{n} \ni (x_1,\ldots,x_n) \longmapsto F_n(x_1,\ldots,x_n) \in \mathbb{C}.$$

With any n-linear continuous form $F_n \in \mathcal{L}^n(X,\mathbb{C})$, we can associate the composition

$$P_n = F_n \circ \Delta_n, \qquad \Delta_n : X \ni x \longmapsto {}^n x := (x,\ldots,x) \in \prod_{\imath=1}^{n} X,$$

which we shall call *a homogenous polynomial* of degree n on the space X. The linear space of all homogenous polynomials of degree n will be denoted by $\mathcal{P}_n(X)$.

Whenever a polynomial $P_n \in \mathcal{P}_n(X)$ is given, we can get back the linear symmetric form F_n associated with P_n, by the following polarization formula (cf., i.e, [3])

$$F_n(x_1,\ldots,x_n) = \frac{1}{2^n\, n!} \sum_{\mathfrak{e}_\imath=\pm 1} \mathfrak{e}_1 \ldots \mathfrak{e}_n\, P_n\Big(\sum_{\imath=1}^{n} \mathfrak{e}_\imath x_\imath\Big). \tag{2.1}$$

On the space $\mathcal{L}^n(X,\mathbb{C})$, we shall consider the locally convex topology of uniform convergence on bounded absolutely convex subsets of $\prod_{\imath=1}^{n} X$, this topology will be denoted by τ_β. By τ_β, we will also denote the topology of uniform convergence on bounded, absolutely convex subsets of X on the space $\mathcal{P}_n(X)$.

By the algebra of polynomials on the space X, we understand the locally convex direct sum

$$\mathcal{P}(X) := \sum_{n\in\mathbb{N}_1} \mathcal{P}_n(X) = \Big\{ P(x) = \sum_{n=1}^{m} P_n(x) : \; P_n \in \mathcal{P}_n(X); \; m \in \mathbb{N}_1 \Big\},$$

If $P \in \mathcal{P}(X)$, then we understand deg$P$ in a natural way.

It is obvious that $\mathcal{P}(X)$ is an algebra with respect to multiplication

$$\mathcal{P}(X) \times \mathcal{P}(X) \ni (P,Q) \longmapsto PQ \in \mathcal{P}(X),$$
$$P(x)\,Q(x) = \sum_{n\in\mathbb{N}_1} \sum_{m=1}^{n} P_m(x)\, Q_{n-m+1}(x), \qquad x \in X.$$

$\mathcal{P}(X)$ having been defined as above, does not contain $\mathbb{C}$.

We shall introduce the definition of the space of ω–ultradifferentiable functions in the same way as in the paper of Braun, Meise and Taylor ([1]).

Let $\omega : [0,+\infty) \to [0,+\infty)$ be a weight function, which means that ω is continuous and satisfies

$$(i)\ \omega(2t) = O(\omega(t)),$$

$$(ii)\ \int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty,$$

$$(iii)\ \log t = o(\omega(t)), \quad (iv)\varphi : t \mapsto \omega(e^t) \text{ is convex.}$$

For a compact set $K \subset \mathbb{R}^N$ and $\lambda > 0$, we define the Banach space

$$D_\lambda(K) = \{f \in C^\infty(\mathbb{R}^N)\,|\mathrm{supp f} \subset \mathrm{K\,and} \int_{\mathbb{R}^N} |\hat{\mathrm{f}}(\mathrm{t})| \exp(\lambda\omega(|\mathrm{t}|))\mathrm{dt} < +\infty\},$$

where $\hat{f}$ denotes the Fourier transform of f.

We set $D_\omega(K) = \liminf_{\lambda\to 0} D_\lambda(K)$ endowed with the topology of the inductive limit.

Let us point out that in paper [1] the space $D_\omega(K)$is also considered with the projective limit topology. However, we do not know whether the reasonings which are presented in our paper are also true in the case of projective limit topology.

3. THE SPACE OF POLYNOMIALS FOR THE SPACE OF ω–ULTRADIFFERENTIABLE FUNCTIONS

Let K be a compact subset of $\mathbb{R}$ and let $D_\omega = D_\omega(K)$.

Let us remark that any reasoning presented in this section can be carried on when K is a compact subset of $\mathbb{R}^N$. We shall stick to the case $N = 1$ in order to simplify the notation.

By $\otimes^n D_\omega := D_\omega \otimes \ldots \otimes D_\omega$ we denote the algebraic tensor product. Let $\widehat{\otimes}^n_{\mathfrak{p}} D_\omega$ denote its completion in the projective topology. In the space $\otimes^n D_\omega$, we consider the operation of symmetrization

$$\varsigma_n : \otimes^n D_\omega \ni x_1 \otimes \ldots \otimes x_n \longmapsto x_1 \odot \ldots \odot x_n := \frac{1}{n!} \sum_{\varsigma \in \mathfrak{G}_n} x_{\varsigma(1)} \otimes \ldots \otimes x_{\varsigma(n)},$$

where $\mathfrak{G}_n$ is the group of permutations.

The operator ς_n is the continuous projection in the space $\otimes^n D_\omega$, [3], thus it can be extended onto the completion of $\otimes^n D_\omega$. This extension will also be denoted by ς_n.

We shall use the following notation: $\odot^n D_\omega := \varsigma_n(\otimes^n D_\omega)$, and $\widehat{\odot}^n_{\mathfrak{p}} D_\omega := \varsigma_n\,(\widehat{\otimes}^n_{\mathfrak{p}} D_\omega)$.

Let χ_n denote the canonical inclusion of the Cartesian product into the tensor product

$$\chi_n : \prod_{\imath=1}^n D_\omega \ni (x_1, \ldots, x_n) \longmapsto x_1 \otimes \ldots \otimes x_n \in \otimes^n D_\omega.$$

In the space $D_\omega(K^n)$ we consider the following operator

$$\varsigma_n^* : D_\omega(K^n) \ni \varphi(t) \longmapsto (\varsigma_n^* \circ \varphi)(t) := \frac{1}{n!} \sum_{\varsigma \in \mathfrak{G}_n} \varphi(t_{\varsigma(1)}, \ldots, t_{\varsigma(n)}),$$

where $t = (t_1, \ldots, t_n) \in K^n$.

The operator ς_n^* is the projection on the closed subspace of $D_\omega(K^n)$ of symmetric functions

$$S_\omega(K^n) := \mathcal{R}(\varsigma_n^*) \subset D_\omega(K^n).$$

We would like to describe the dual space for $\prod_{n\in\mathbb{N}_1} S_\omega(K^n)$. Let $S'_\omega(K^n)$ denote the dual space for $S_\omega(K^n)$ with the strong topology $\beta\,\langle S'_\omega(K^n) \mid S_\omega(K^n)\rangle$. We shall prove the following theorem (cf. [4])

Theorem 3.1. *The following topological isomorphisms*

$$\begin{array}{ccccc} S'_\omega(K^n) & \xrightarrow{\varrho} & \widehat{\odot}^n_{\mathfrak{p}} D'_\omega & \xrightarrow{\vartheta} & \mathcal{P}_n(D_\omega), \\ T_n & \xmapsto{\varrho} & \varrho(T_n) = f_n & \xmapsto{\vartheta} & F_n \end{array}$$

hold; moreover the second of them

$$\vartheta : \widehat{\odot}^n_{\mathfrak{p}} D'_\omega \ni f_n \longmapsto F_n := f_n \circ \chi_n \circ \Delta_n \in \mathcal{P}_n(D_\omega)$$

is given by the formula

$$F_n(\varphi) = \big\langle f_n \mid \otimes^n \varphi \big\rangle, \qquad \varphi \in D_\omega \tag{3.1}$$

and it is the extension of the superposition of canonical mappings

$$\begin{array}{ccccc} D_\omega & \xrightarrow{\Delta_n} & \prod_{\iota=1}^{n} D_\omega & \xrightarrow{\chi_n} & \otimes^n D_\omega, \\ \varphi & \xmapsto{\Delta_n} & {}^n\varphi & \xmapsto{\chi_n} & \otimes^n \varphi := \varphi \otimes \ldots \otimes \varphi, \end{array}$$

where $\otimes^n\varphi$ is the scalar function in n real variables,

$$\otimes^n \varphi(t) := \varphi(t_1) \cdot \ldots \cdot \varphi(t_n), \qquad t = (t_1, \ldots, t_n) \in K^n.$$

Proof: Let operators ς_n and ς'_n be adjoint to each other with respect to the dual pair $\langle \otimes^n D'_\omega \mid \otimes^n D_\omega\rangle$ given by the bilinear form

$$\langle u_1 \otimes \ldots \otimes u_n | \varphi_1 \otimes \ldots \otimes \varphi_n \rangle = \langle u_1|\varphi_1\rangle \ldots \langle u_n|\varphi_n\rangle\,. \tag{3.2}$$

Then for any $u_1, \ldots, u_n \in D'_\omega$ and $\varphi_1, \ldots, \varphi_n \in D_\omega$, the operator ς_n satisfies the equality

$$\langle u_1 \odot \ldots \odot u_n \mid \varphi_1 \otimes \ldots \otimes \varphi_n \rangle = \langle u_1 \otimes \ldots \otimes u_n \mid \varphi_1 \odot \ldots \odot \varphi_n \rangle\,,$$

$$\varsigma_n : \varphi_1 \otimes \ldots \otimes \varphi_n \longrightarrow \varphi_1 \odot \ldots \odot \varphi_n := \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{\varsigma(1)} \otimes \ldots \otimes \varphi_{\varsigma(n)}\,.$$

If the set of seminorms $\{p_i\}_{i\in\mathbb{I}}$ defines the topology in D_ω, then the set of seminorms

$$(p_{i_1} \otimes \ldots \otimes p_{i_n})(\psi) = \inf_{\psi = \sum\limits_{m\in{\mathbb{N}_1}^n} \varphi_{m_1} \otimes \ldots \otimes \varphi_{m_n} \in \otimes^n D_\omega} \; \sum_{m\in{\mathbb{N}_1}^n} p_{i_1}(\varphi_{m_1}) \ldots p_{i_n}(\varphi_{m_n})$$

defines the projective topology in $\otimes^n D_\omega$.

The following inequality holds:

$$(p_{i_1}\otimes\ldots\otimes p_{i_n})(\varsigma_n\circ\psi)\le\inf\sum_{m\in\mathbb{N}_1{}^n}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}p_{i_1}(\varphi_{m_{\varsigma(1)}})\ldots p_{i_n}(\varphi_{m_{\varsigma(n)}})=$$

$$=\inf\sum_{m\in\mathbb{N}_1{}^n}\frac{1}{n!}\sum_{\varsigma\in\mathfrak{S}_n}p_{i_{\varsigma(1)}}(\varphi_{m_1})\ldots p_{i_{\varsigma(n)}}(\varphi_{m_n})=\varsigma_n\circ(p_{i_1}\otimes\ldots\otimes p_{i_n})(\psi),$$

where $\varsigma_n\circ(p_{i_1}\otimes\ldots\otimes p_{i_n}):=\dfrac{1}{n!}\sum\limits_{\sigma\in\mathfrak{S}_n}p_{i_{\varsigma(1)}}\otimes\ldots\otimes p_{i_{\varsigma(n)}}$ is a seminorm in $\otimes^n D_\omega$, continuous in projective topology.

Hence the projection ς_n is continuous. The continuity of ς_n and the fact that the subspace $\otimes^n D_\omega$ is dense in $\widehat{\otimes}_{\mathfrak{p}}^n D_\omega$ imply that there exists the continuous extension of ς_n on the completion of the respective space $\varsigma_n:\widehat{\otimes}_{\mathfrak{p}}^n D_\omega\longrightarrow\widehat{\odot}_{\mathfrak{p}}^n D_\omega$. Hence we can represent the space $\widehat{\otimes}_{\mathfrak{p}}^n D_\omega$ as locally convex direct sum

$$\widehat{\otimes}_{\mathfrak{p}}^n D_\omega=\widehat{\odot}_{\mathfrak{p}}^n D_\omega\dotplus\mathcal{N}(\varsigma_n),\tag{3.3}$$

where $\mathcal{N}(\varsigma_n)$ denotes the kernel of ς_n.

Corollary 3.6 from [1] implies that D_ω is a (DF)-space, thence D'_ω is a (F)-space. Also D_ω is nuclear [1, Corollary 3.6]. For such spaces, the following isomorphism holds

$$(\widehat{\otimes}_{\mathfrak{p}}^n D_\omega)'_\beta\simeq\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega.\tag{3.4}$$

Isomorphism (3.4) implies that bilinear form (3.2) defines the dual pair $\left\langle\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega\mid\widehat{\otimes}_{\mathfrak{p}}^n D_\omega\right\rangle$ and that the operator ς'_n is adjoint to ς_n with respect to this duality. In particular ς'_n is continuous in strong topology.

Hence, and also from equality (3.3), we conclude that the space $\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega$ may be represented as the locally convex space

$$\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega=\widehat{\odot}_{\mathfrak{p}}^n D'_\omega\dotplus\mathcal{N}(\varsigma'_n).\tag{3.5}$$

When, in the space $\mathcal{L}^n(D_\omega,\mathbb{C})$, we consider the topology τ_β of uniform convergence on bounded absolutely convex subsets of $\prod\limits_{i=1}^n D_\omega$, then

$$\big(\mathcal{L}^n(D_\omega,\mathbb{C}),\tau_\beta\big)\simeq(\widehat{\otimes}_{\mathfrak{p}}^n D_\omega)'_\beta.\tag{3.6}$$

Then from (3.4), (3.5) and (3.6) we get that

$$\big(\mathcal{L}^n(D_\omega,\mathbb{C}),\tau_\beta\big)\simeq\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega\simeq\widehat{\odot}_{\mathfrak{p}}^n D'_\omega\dotplus\mathcal{N}(\varsigma'_n).\tag{3.7}$$

The first isomorphism in (3.7) implies, in particular, that any form $\bar{f}_n\in\mathcal{L}^n(D_\omega,\mathbb{C})$ may be represented as $\bar{f}_n=\bar{F}_n\circ\chi_n$ for some $\bar{F}_n\in\widehat{\otimes}_{\mathfrak{p}}^n D'_\omega$, and that there exists the representation of $\bar{F}_n$ in the form of absolutely convergent series

$$\bar{F}_n = \sum_{l \in \mathbb{N}_1{}^n} u_{l_1} \otimes \ldots \otimes u_{l_n} \in \widehat{\otimes}^n_{\mathfrak{p}} D'_\omega,$$

where $u_{l_\iota} \in D'_\omega$, $\iota = 1, \ldots, n$ [7, Theorem 6.4]. Hence, for any $\varphi_1, \ldots, \varphi_n \in D_\omega$, the operator ς'_n satisfies the following equalities

$$\begin{aligned}(\varsigma'_n \circ \bar{f}_n)(\varphi_1, \ldots, \varphi_n) &= \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \sum_{l \in \mathbb{N}_1{}^n} \langle u_{l_{\varsigma(1)}} \mid \varphi_1 \rangle \ldots \langle u_{l_{\varsigma(n)}} \mid \varphi_n \rangle = \\ &= \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \sum_{l \in \mathbb{N}_1{}^n} \langle u_{l_1} \mid \varphi_{\varsigma(1)} \rangle \ldots \langle u_{l_n} \mid \varphi_{\varsigma(n)} \rangle,\end{aligned}$$

which means that the composition $f_n^\varsigma := \varsigma'_n \circ \bar{f}_n$ belongs to the space $\mathcal{L}^n_\varsigma(D_\omega, \mathbb{C})$ of symmetric continuous n-linear forms on D_ω.

The second isomorphism in (3.7) implies that

$$\mathcal{R}(\varsigma'_n) = \big(\mathcal{L}^n_\varsigma(D_\omega, \mathbb{C}), \tau_\beta\big) \simeq \widehat{\odot}^n_{\mathfrak{p}} D'_\omega. \tag{3.8}$$

Now we shall prove that the following topological isomorphism holds:

$$\big(\mathcal{L}^n_\varsigma(D_\omega, \mathbb{C}), \tau_\beta\big) \simeq \mathcal{P}_n(D_\omega). \tag{3.9}$$

For any symmetric form $f_n^\varsigma \in \mathcal{L}^n_\varsigma(D_\omega, \mathbb{C})$, the polarization formula (2.1) is true. Hence, its restriction to the diagonal of the Cartesian product

$$\Delta'_n : \big(\mathcal{L}^n_\varsigma(D_\omega, \mathbb{C}), \tau_\beta\big) \ni f_n^\varsigma \longrightarrow f_n^\varsigma \circ \Delta_n \in \mathcal{P}_n(D_\omega)$$

should be isomorphism (3.9) we are looking for.

Since Δ'_n is surjective, then it is enough to prove its continuity. Any continuous seminorm on the space $\mathcal{L}^n_\varsigma(D_\omega, \mathbb{C})$ has the form

$$p_{S_1 \ldots S_n}(f_n^\varsigma) = \sup_{\iota \in \{1, \ldots, n\}} \sup_{\varphi_\iota \in S_\iota} \big| f_n^\varsigma(\varphi_1, \ldots, \varphi_n) \big|, \qquad f_n^\varsigma \in \mathcal{L}^n_\varsigma(D_\omega, \mathbb{C}),$$

where $\{S_\iota : \iota = 1, \ldots, n\}$ are bounded absolutely convex subsets of D_ω. Hence, any continuous seminorm on $\mathcal{P}_n(D_\omega)$ has the form

$$p_S(f_n^\varsigma \circ \Delta_n) = \sup_{\varphi \in S} \big| (f_n^\varsigma \circ \Delta_n)(\varphi) \big|, \qquad f_n^\varsigma \circ \Delta_n \in \mathcal{P}_n(D_\omega),$$

where S is a bounded absolutely convex subset of D_ω. Polarization formula (2.1) implies that

$$\begin{aligned}p_{S_1 \ldots S_n}(f_n^\varsigma) &\leq \frac{1}{2^n \cdot n!} \sum_{\mathfrak{e}_\iota = \pm 1} \sup_{\iota \in \{1, \ldots, n\}} \sup_{\varphi_\iota \in S_\iota} \Big| (f_n^\varsigma \circ \Delta_n)\Big(\sum_{\iota=1}^n \mathfrak{e}_\iota \varphi_\iota\Big) \Big| = \\ &= \frac{n^n}{2^n \cdot n!} \sum_{\mathfrak{e}_\iota = \pm 1} \sup_{\iota \in \{1, \ldots, n\}} \sup_{x_\iota \in S_\iota} \Big| (f_n^\varsigma \circ \Delta_n)\Big(\frac{1}{n} \sum_{\iota=1}^n \mathfrak{e}_\iota \varphi_\iota\Big) \Big| \leq \\ &\leq \frac{n^n}{n!} p_S(f_n^\varsigma \circ \Delta_n).\end{aligned}$$

Hence, Δ'_n is isomorphism (3.9) searched for.

Combining isomorphisms (3.8) and (3.9), we conclude that the mapping

$$\widehat{\odot}^n_{\mathfrak{p}} D'_\omega \ni f_n \longrightarrow F_n \in \mathcal{P}_n(D_\omega), \tag{3.10}$$

where

$$F_n(\varphi) := \langle f_n \mid \otimes^n \varphi \rangle = (f_n \circ \chi_n \circ \Delta_n)(\varphi), \quad \varphi \in D_\omega, \tag{3.11}$$

is the second isomorphism ϑ.

From [1, Theorem 8.1] we obtain the following toplogical isomorphism

$$D_\omega(K^n) \simeq \widehat{\otimes}^n_{\mathfrak{p}} D_\omega. \tag{3.12}$$

Isomorphism (3.12) implies that the set of functions of the form $\varphi(t) = \sum_{l \in \mathbb{N}_1^n} \varphi_{l_1}(t_1) \dots \varphi_{l_n}(t_n)$, where $\varphi_{l_1}, \dots, \varphi_{l_n} \in D_\omega$, is a dense subspace $\otimes^n D_\omega$ of the space $D_\omega(K^n)$. Since

$$\begin{aligned}(\varsigma_n^* \circ \varphi)(t) &= \frac{1}{n!} \sum_{l \in \mathbb{N}_1{}^n} \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{l_1}(t_{\varsigma(1)}) \dots \varphi_{l_n}(t_{\varsigma(n)}) = \\ &= \frac{1}{n!} \sum_{l \in \mathbb{N}_1{}^n} \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{l_{\varsigma(1)}}(t_1) \dots \varphi_{l_{\varsigma(n)}}(t_n) = (\varsigma_n \circ \varphi)(t),\end{aligned}$$

then the continuity of projections implies that the following algebraic equality holds:

$$S_\omega(K^n) = \widehat{\odot}^n_{\mathfrak{p}} D_\omega.$$

The topological isomorphism $S_\omega(K^n) \simeq \widehat{\odot}^n_{\mathfrak{p}} D_\omega$ is a corollary of (3.12) and the adjoint topological isomorphism

$$\varrho : \; S'_\omega(K^n) \longrightarrow \widehat{\odot}^n_{\mathfrak{p}} D'_\omega$$

is obvious. □

Let us denote

$$S' := \sum_{n \in \mathbb{N}_1} S'_\omega(K^n), \qquad S := \prod_{n \in \mathbb{N}_1} S_\omega(K^n).$$

The following equality is known (see e.g. [7])

$$\sum_{n \in \mathbb{N}_1} S'_\omega(K^n) = \Big(\prod_{n \in \mathbb{N}_1} S_\omega(K^n) \Big)'. \tag{3.13}$$

This equality in particular means that $\langle S', S \rangle$ is a dual pair according to its canonical bilinear form

$$\langle T \mid \bar{\varphi} \rangle = \sum_{n \in \mathbb{N}_1} \langle T_n \mid \varphi_n \rangle, \qquad T = \sum_{n \in \mathbb{N}_1} T_n \in S', \quad \bar{\varphi} = \prod_{n \in \mathbb{N}_1} \varphi_n \in S,$$

where $T_n \in S'_\omega(K^n)$ and $\varphi_n \in S_\omega(K^n)$. Therefore, S' may be treated as the space of ω–ultradistributions for the space S of ω–ultradifferentiable symmetric functions in infinitely many variables.

We shall prove the following

Theorem 3.2.

(i) *The locally convex space* $\sum_{n\in\mathbb{N}_1} \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$ *is a topological algebra with respect to a convolution given by the formula*

$$f * h := \sum_{n\in\mathbb{N}_1} \Big(\sum_{m=1}^{n} f_m \odot h_{n-m+1} \Big),$$

where $f = \sum_{n\in\mathbb{N}_1} f_n$, $h = \sum_{n\in\mathbb{N}_1} h_n \in \sum_{n\in\mathbb{N}_1} \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$ *and* $f_n, h_n \in \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$.

(ii) *The following topological isomorphisms*

$$\begin{array}{ccccc} \mathcal{S}' & \xrightarrow{\varrho} & \sum_{n\in\mathbb{N}_1} \widehat{\odot}_{\mathfrak{p}}^n D'_\omega & \xrightarrow{\vartheta} & \mathcal{P}(D_\omega) \\ T = \sum_{n\in\mathbb{N}_1} T_n & \xmapsto{\varrho} & f = \sum_{n\in\mathbb{N}_1} f_n & \xmapsto{\vartheta} & F = \sum_{n\in\mathbb{N}_1} F_n \end{array}$$

hold, where $f_n = \varrho(T_n)$ *and* $F_n = f_n \circ \chi_n \circ \Delta_n = \vartheta(f_n)$.

(iii) *The convolution in algebra* $\sum_{n\in\mathbb{N}_1} \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$ *is transformed by the isomorphism* ϑ *into the product of polynomials in algebra* $\mathcal{P}(D_\omega)$, *i.e.,*

$$\vartheta(f * h) = F \cdot H, \qquad F = \vartheta(f), \quad H = \vartheta(h) \in \mathcal{P}(D_\omega). \tag{3.14}$$

Remark. The inverse mapping for the superposition

$$\ell = (\vartheta \circ \varrho)^{-1} : \mathcal{P}(D_\omega) \longrightarrow \mathcal{S}'$$

may be treated as the linearisation of polynomials, since this mapping allows us to identify polynomials on D_ω with linear forms on the space S, which is predual for $\sum_{n\in\mathbb{N}_1} \mathcal{S}'_\omega(K^n)$.

Proof of Theorem 3.2. If we put

$$\varrho(T) = \sum_{n\in\mathbb{N}_1} \varrho(T_n) = \sum_{n\in\mathbb{N}_1} f_n = f, \qquad \vartheta(f) = \sum_{n\in\mathbb{N}_1} \vartheta(f_n) = \sum_{n\in\mathbb{N}_1} F_n = F,$$

then Theorem 3.1 implies that there exist the isomorphisms ϱ and ϑ.

Now we prove (i). For any $f_n \in \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$ and $h_m \in \widehat{\odot}_{\mathfrak{p}}^m D'_\omega$ there is

$$f_n \odot h_m \in (\widehat{\odot}_{\mathfrak{p}}^n D'_\omega) \odot (\widehat{\odot}_{\mathfrak{p}}^m D'_\omega) \subset \widehat{\odot}_{\mathfrak{p}}^{n+m} D'_\omega$$

and the convolution "$*$" in the direct sum $\sum_{n\in\mathbb{N}_1} \widehat{\odot}_{\mathfrak{p}}^n D'_\omega$ is well defined. Its continuity follows from the continuity of the canonical mapping in the symmetric tensor product

$$(\widehat{\odot}_{\mathfrak{p}}^n D'_\omega) \times (\widehat{\odot}_{\mathfrak{p}}^m D'_\omega) \ni (f_n, h_m) \longrightarrow f_n \odot h_m \in \widehat{\odot}_{\mathfrak{p}}^{n+m} D'_\omega.$$

From formula (3.11), we obtain

$$\begin{aligned} F_n(\varphi)\cdot H_m(\varphi) &= \langle f_n \mid \otimes^n\varphi\rangle\cdot\langle h_m \mid \otimes^m\varphi\rangle = \big\langle f_n\otimes h_m \mid \otimes^{n+m}\varphi\big\rangle = \\ &= \big\langle f_n\odot h_m \mid \otimes^{n+m}\varphi\big\rangle = (f_n\odot h_m)\circ\chi_{n+m}\circ\Delta_{n+m}(\varphi). \end{aligned}$$

Hence, $F_n\cdot H_m\in\mathcal{P}_{n+m}(D_\omega)$ and, for any polynomials $F=\sum_{n\in\mathbb{N}_1}F_n$ and $H=\sum_{n\in\mathbb{N}_1}H_n$ from the space $\mathcal{P}(D_\omega)$, we obtain

$$\begin{aligned} F(\varphi)\cdot H(\varphi) &= \sum_{n\in\mathbb{N}_1}\sum_{m=1}^{n}F_m(\varphi)\cdot H_{n-m+1}(\varphi) = \\ &= \sum_{n\in\mathbb{N}_1}\sum_{m=1}^{n}(f_m\odot h_{n-m+1})\circ\chi_{n+1}\circ\Delta_{n+1}(\varphi) = \\ &= (f\ast h)\circ\chi_{n+1}\circ\Delta_{n+1}(\varphi). \end{aligned}$$

Therefore, the mapping

$$\sum_{n\in\mathbb{N}_1}\widehat{\odot}^n_{\mathfrak{p}}D'_\omega\ni f=\sum_{n\in\mathbb{N}_1}f_n\overset{\vartheta}{\longmapsto}F=\sum_{n\in\mathbb{N}_1}f_n\circ\chi_n\circ\Delta_n\in\mathcal{P}(D_\omega)$$

transforms the convolution into the product of polynomials. □

4. SOME REMARKS AND OPEN PROBLEMS

Let us remark that if we denote by $\mathcal{V}$ a fundamental system of convex balanced neighbourhoods of 0 in D_ω, then the familly $\bigcap_{V\in\mathcal{V}}\Gamma(\odot^n\delta_V V)$ forms a fundamental system of bounded subsets in the space $\widehat{\odot}^n_{\mathfrak{p}}D_\omega$ as δ_V ranges over all collections of positive real numbers, and $\Gamma(\odot^n\delta_V V)$ denotes the closed absolutely convex balanced hull of $\odot^n\delta_V V$. If S is a bounded subset of D_ω and $V\in\mathcal{V}$, then there exists $\varepsilon_V>0$ such that $S\subset\varepsilon_V V$. Hence $\Gamma(\odot^n S)\subset\Gamma(\odot^n\varepsilon_V V)$ and $\Gamma(\odot^n S)$ is a bounded subset of $\widehat{\odot}^n_{\mathfrak{p}}D_\omega$. Therefore, the strong topology in the dual space $(\widehat{\odot}^n_{\mathfrak{p}}D_\omega)'=\widehat{\odot}^n_{\mathfrak{p}}D'_\omega$ is always stronger than the topology which is induced by the topology τ_β in the space $\mathcal{P}_n(D_\omega)$ through the isomorphism ϑ. Theorem 3.1 gives the answer to the question of when these two topologies are equal. Equality (3.1) implies that

$$\sup_{\varphi\in S}|F_n(\varphi)| = \sup_{\varphi\in\Gamma(\odot^n S)}\big|\big\langle f_n \mid \otimes^n\varphi\big\rangle\big|, \tag{4.1}$$

hence these topologies are equal if and only if the sets of the form $\Gamma(\odot^n S)$, for S bounded in D_ω, form a basis of bounded sets in the space $\widehat{\odot}^n_{\mathfrak{p}}D_\omega$, which means that the space D_ω has the $(BB)_n$ property (cf. [3]). Theorem 3.1 gives the positive answer to some version of Grothendieck's *Probléme des topologies* (cf. [3, 5]) for symmetric tensor products.

For the space D_ω Grothendieck's question has a positive answer, since D_ω is a (DF)-space. It is also the case when we consider the space G, the space of ultradifferentiable functions in the sense of Gevrey ([4]). However, the author does not know whether Theorem 3.1 holds for the space of test functions of Beurling type (see e.g. [1]) or for the space of test functions introduced by Ciorânescu and Zsidó (see [2]).

Braun, Meise and Taylor have shown in [1] that the theory of ultradistributions $\{D_\omega\}_{\omega\in W}$ (W denotes the set of all weight functions) is equivalent to some other theories of ultradistributions ([1, Proposition 8.8]); however, the author does not know whether the theories of polynomials on different spaces of ultradifferentiable functions are also equivalent.

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