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DIFFERENTIAL EQUATION OF TRANSVERSE VIBRATIONS OF A BEAM WITH A LOCAL STROKE CHANGE OF STIFFNESS

Abstract. The aim of this paper is to derive a differential equation of transverse vibrations of a beam with a local, stroke change of stiffness, and to solve it. The presented method is based on the theory of distributions.

Keywords: equation of a beam, joint point, distribution.

Mathematics Subject Classification: 74K10, 46F99.

1. INTRODUCTION

It is well known that the equation of transverse vibrations of a beam with changeable stiffness is of the form:

$$\frac{\partial^2}{\partial x^2}\left(\alpha(x)\frac{\partial^2 u}{\partial x^2}(t,x)+\mu\alpha(x)\frac{\partial^3 u}{\partial t\partial x^2}(t,x)\right)+\rho F\frac{\partial^2 u}{\partial t^2}(t,x)=f(t,x), \tag{1}$$

where μ is a constant that characterizes internal damping, ρF is mass per unit length, a summable function f represents a distribution of external forces that act on a beam of length l on a symmetry plane of the beam. Assuming that $\alpha \in \mathcal{C}^2(0,l)$ and $f \equiv 0$, we obtain the existence and uniqueness of the solution u of $\mathcal{C}^2$ class in t and $\mathcal{C}^4$ class in x of equation (1) with given boundary and initial conditions.

This paper is devoted to the case of a beam with a local, stroke change of stiffness. Let us consider a joint point at $x_0 \in (0,l)$. In this case

$$\alpha(x)=\begin{cases}\alpha_0 = EI & \text{for } x\in[0,l]\setminus\{x_0\},\\ 0, & \text{for } x=x_0,\end{cases} \tag{2}$$

(EI is a bending stiffness), and (1) can not be longer understood in a classical manner. Usually, the segment $(0,l)$ is divided into two: $(0,x_0)$, (x_0,l), and two problems, related to each other by a geometrical condition

$$\frac{\partial^3}{\partial t\partial x^2}u(t,x_0^+) = \frac{\partial^3}{\partial t\partial x^2}u(t,x_0^-) = 0 \quad \text{for } t \geq 0, \tag{3}$$

are taken into consideration, and on account of (2), a solution u is continuous as a function of x in $(0,l)$.

Here we present another method. First we consider the homogeneous equation

$$\frac{\partial^2}{\partial x^2}\left(\alpha(x)\frac{\partial^2 u}{\partial x^2} + \mu\alpha(x)\frac{\partial^3 u}{\partial x^2\partial t}\right) + \rho F\frac{\partial^2 u}{\partial t^2} = 0, \quad \rho F \text{ is a constant.} \tag{4}$$

Separating variables in (4) as follows:

$$u(t,x) = T(t)\cdot X(x) \tag{5}$$

we obtain a system of equations: for a function T

$$\ddot{T} + \mu\omega^2\dot{T} + \omega^2 T = 0,$$

and a distributional equation:

$$X^{(4)} - \lambda^4 X = -\sigma_3\delta_{x_0} - \sigma_1\delta''_{x_0}, \tag{6}$$

where δ_{x_0} denotes the Dirac distribution concentrated at a point x_0, the left-hand-side of (6) is understood as a regular distribution generated by a function $X^{(4)} - \lambda^4 X$ with $X \in \mathcal{C}^4((0,x_0)\cup(x_0,l))\cap\mathcal{C}^0(0,l)$ and such that $X''(x_0^+) = X''(x_0^-) = 0$, and $\lambda^4 = \frac{\omega^2\rho F}{EI}$, $\sigma_1 = X'(x_0^+) - X'(x_0^-)$, $\sigma_3 = X^{(3)}(x_0^+) - X^{(3)}(x_0^-)$, ω is a constant.

In consequence, we obtain a new form of (1), namely,

$$\begin{aligned}\alpha_0\frac{\partial^4 u}{\partial x^4} + \mu\alpha_0\frac{\partial^5 u}{\partial t\partial x^4} + \rho F\frac{\partial^2 u}{\partial t^2} + \frac{\alpha_0}{\omega^2}\left(\frac{\partial^2 u(t,x_0^+)}{\partial t\partial x} - \frac{\partial^2 u(t,x_0^-)}{\partial t\partial x}\right)\delta''_{x_0} + \\ + \frac{\alpha_0}{\omega^2}\left(\frac{\partial^4 u(t,x_0^+)}{\partial t\partial x^3} - \frac{\partial^4 u(t,x_0^-)}{\partial t\partial x^3}\right)\delta_{x_0} = f(t,x).\end{aligned} \tag{7}$$

Next we formulate an eigenproblem of boundary-value problem corresponding to (6) and derive orthogonality condition. Finally we consider a boundary-initial problem corresponding to (7).

The idea of an analytic description of a joint point was presented in [2]. In [3], a differential equation of transverse vibrations of a beam with a local, stroke change of stiffness was derived, based on the sequential definition of a distribution.

2. PRELIMINARIES

For the convenience of the reader, we recall some basic ideas from the theory of distributions. Let $\Omega \subset \mathbb{R}$ be an open set. We introduce the following notations, for more details see [4,5]. Let $\mathcal{D}(\Omega)$ denote the space of test functions,

$$\mathcal{D}(\Omega) := \{\varphi \in \mathcal{C}^\infty(\Omega) : \quad \operatorname{supp}\varphi := \overline{\{\varphi \neq 0\}} \text{ is compact in } \Omega\}$$

and $\mathcal{D}'(\Omega)$ the space of distributions, i.e., the space of all linear continuous functionals defined on $\mathcal{D}(\Omega)$.

A locally summable function $u : \Omega \to \mathbb{R}$ induces the functional

$$[u]: \quad \mathcal{D}(\Omega) \ni \varphi \mapsto \int_\Omega \varphi(x)u(x)dx \in \mathbb{R}.$$

Obviously, $[u] \in \mathcal{D}'(\Omega)$ and is called a regular distribution.

Let $T \in \mathcal{D}'(\Omega)$, $k \in \mathbb{N}$. The k-th derivative of the distribution T is given by the formula:

$$T^{(k)}(\varphi) := (-1)^k T(\varphi^{(k)}) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Observe that for smooth functions

$$[u]^{(k)} = [u^{(k)}] \quad \text{for all } k \in \mathbb{N}.$$

On the other hand, if u is of class $\mathcal{C}^k$ in the set $\mathbb{R} \setminus \{a\}$, then

$$[u]^{(k)} = [u^{(k)}] + \sum_{j=0}^{k-1} \sigma_{k-j-1}\delta_a^{(j)}, \tag{8}$$

where $\sigma_m = \lim_{x\to a^+} u^{(m)}(x) - \lim_{x\to a^-} u^{(m)}(x)$ is the jump of the m-th derivative of u at the point a. As usual, δ_a denotes the Dirac distribution concentrated at the point a

$$\delta_a : \mathcal{D}(\Omega) \ni \varphi \mapsto \varphi(a) \in \mathbb{R}.$$

Let $T \in \mathcal{D}'(\Omega)$. The least $N \in \mathbb{N}$ such that the restriction $T_{|\mathcal{D}_K(\Omega)}$ is p_N–continuous for each compact $K \subset \Omega$ is called the order of the distribution T. Here

$$\mathcal{D}_K(\Omega) := \{u \in \mathcal{C}^\infty(\Omega) : \operatorname{supp} \mathrm{u} \subset \mathrm{K}\},$$

$$p_N(u) := \sup_{x\in\Omega} \sup_{s\le N} |D^s u(x)|.$$

Remind that the Dirac distribution has order zero. Similarly, a regular distribution has order zero.

Let $T \in \mathcal{D}'(\Omega)$, $\psi \in \mathcal{C}^\infty(\Omega)$. The product ψT is defined as a following distribution:

$$\psi T : \qquad \mathcal{D}(\Omega) \ni \varphi \mapsto T(\varphi\psi) \in \mathbb{R}.$$

The notion of the product of a smooth function and a distribution can be extended to the case of a function of class $\mathcal{C}^m$, whenever the order of the distribution does not exceed m. In particular, for the Dirac distribution δ_a the product $\psi\delta_a$ can be defined, whenever ψ is continuous. In this case

$$\psi\delta_a = \psi(a)\delta_a. \tag{9}$$

For the same reason

$$\psi[u] = [\psi u] \tag{10}$$

as long as ψ is continuous and u is locally summable. Similarly,

$$\psi\delta_a'' = \psi''(a)\delta_a - 2\psi'(a)\delta_a' + \psi(a)\delta_a'', \tag{11}$$

for $\psi \in \mathcal{C}^2(\mathbb{R})$.

Also the product $H_a\delta_a$ makes sense, where H_a is Heaviside function:

$$H_a(x) = \begin{cases} 1 & \text{for } x > a, \\ 0 & \text{for } x < a, \\ \frac{1}{2} & \text{for } x = a, \end{cases}$$

and

$$H_a\delta_a = \frac{1}{2}\delta_a. \tag{12}$$

3. A JOINT POINT

Let $l > 0$. Let us first consider homogeneous equation (4) and separate variables as in formula (5). In a usual manner, we obtain a system of equations for functions T and X:

$$\ddot{T} + \mu\omega^2\dot{T} + \omega^2 T = 0,$$
$$\frac{d^2}{dx^2}(\alpha(x)X''(x)) - \rho F\omega^2 X(x) = 0, \tag{13}$$

with some constant ω. We consider an ideal joint point at $x_0 \in (0, l)$ thus, according to (2):

$$\alpha(x) = \begin{cases} \alpha_0 & \text{for } \ x \neq x_0, \\ 0 & \text{for } \ x = x_0, \end{cases}$$

and from geometrical condition (3) we infer that

$$X''(x_0^+) = X''(x_0^-) = 0. \tag{14}$$

We are looking for $X \in \mathcal{C}^0([0, l])$ which is of class $\mathcal{C}^4$ in the intervals $(0, x_0)$, (x_0, l) and satisfies (13) in the distributional sense,

$$\frac{d^2}{dx^2}(\alpha(x)[X]'') - \rho F\omega^2[X] = 0.$$

Observe that $\alpha = \alpha_0(H_{x_0} + \check{H}_{x_0})$, where $\check{}: x \mapsto -x$. On the other hand, from differential formula (8) there follows:

$$[X]'' = [X''] + \sigma_1\delta_{x_0}, \tag{15}$$

for $\sigma_1 = X'(x_0^+) - X'(x_0^-)$. Remembering (12), it is meaningful to consider products $\alpha(x)[X'']$, $\alpha(x)\delta_{x_0}$, and

$$\alpha(x)[X''] = [\alpha_0 X''], \quad \alpha(x)\delta_{x_0} = \alpha_0\delta_{x_0}.$$

Thus

$$\frac{d^2}{dx^2}(\alpha(x)[X]'') = \alpha_0[X^{(4)}] + \alpha_0\sigma_3\delta_{x_0} + \alpha_0\sigma_1\delta''_{x_0}. \tag{16}$$

Combining (16) and (13), we infer that

$$\alpha_0[X^{(4)}] - \omega^2\rho F[X] = -\alpha_0\sigma_3\delta_{x_0} - \alpha_0\sigma_1\delta''_{x_0}. \tag{17}$$

Therefore, putting $\alpha_0 = EJ$, $\lambda^4 = \dfrac{\omega^2\rho F}{EJ}$ we finally arrive to:

$$[X^{(4)} - \lambda^4 X] = -\sigma_3\delta_{x_0} - \sigma_1\delta''_{x_0}.$$

Remark. On account of the above computations, we obtain that (1) is equivalent to (7).
Remark. It is easy to compute that the solution of (6) is of the form

$$\begin{aligned} X(x) = {} & P\cos\lambda x + Q\sin\lambda x + R\,\mathrm{ch}\,\lambda x + S\,\mathrm{sh}\,\lambda x - \\ & - \frac{\sigma_1}{2\lambda}(\mathrm{sh}\,\lambda(x-x_0) + \sin\lambda(x-x_0))H_{x_0} - \\ & - \frac{\sigma_3}{2\lambda^3}(\mathrm{sh}\,\lambda(x-x_0) - \sin\lambda(x-x_0))H_{x_0}. \end{aligned} \tag{18}$$

4. EIGENPROBLEM OF A BOUNDARY-INITIAL PROBLEM

Let us consider equation (4) with α of form (2) and initial conditions:

$$u(0,x) = \psi_0(x), \quad \frac{\partial u}{\partial x}(0,x) = \psi_1(x). \tag{19}$$

In order to get geometrical stability of a beam, boundary conditions have one of the following forms:

$$u(t,0) = u(t,l) = 0, \quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,l) = 0, \tag{20}$$

$$u(t,0) = u(t,l) = 0, \quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,l) = 0, \tag{21}$$

$$u(t,0) = u(t,l) = 0, \quad \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial u}{\partial x}(t,l) = 0. \tag{22}$$

We also assume that

$$\frac{\partial^3 u}{\partial t\partial x^2}(t,x_0^+) = \frac{\partial^3 u}{\partial t\partial x^2}(t,x_0^-) = 0. \tag{23}$$

Separating variables according to (5), we obtain

$$\ddot{T} + \mu\omega^2\dot{T} + \omega^2 T = 0, \quad T(0) = \psi_0, \quad T'(0) = \psi_1, \tag{24}$$

and due to the result of Section 3:

$$[X^{(4)} - \lambda^4 X] = -\sigma_3 \delta_{x_0} - \sigma_1 \delta''_{x_0}, \tag{25}$$

with one of the following conditions:

$$X(0) = X(l) = 0, \quad X'(0) = X'(l) = 0, \tag{26}$$

$$X(0) = X(l) = 0, \quad X'(0) = X''(l) = 0, \tag{27}$$

$$X(0) = X(l) = 0, \quad X''(0) = X'(l) = 0, \tag{28}$$

and

$$X''(x_0^+) = X''(x_0^-) = 0. \tag{29}$$

Remembering that the solution of (25) is of form (18), taking one of (26)–(28) and (29) into account, we obtain a system of equation with the unknowns: $P, Q, R, S, \sigma_1, \sigma_3$. In the matrix notation:

$$A \cdot C^T = 0, \quad C = (P, Q, R, S, \sigma_1, \sigma_3), \tag{30}$$

where $A = A(\lambda)$ is 6×6 matrix depending on λ, x_0, l. Equation (30) has a non-zero solution iff $\det A(\lambda) = 0$. There is a countable number of λ's satisfying this. Let us set them into an increasing sequence λ_n ($\lambda_1 > 0$). Now putting λ_n instead of λ in (25) and denoting solutions of problems (24) and (25)–(29) with T_n and X_n, respectively, we obtain the eigenproblem of boundary-initial problem (4), (19)–(23). Finally, the solution of the problem considered can be represented as:

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x).$$

5. ORTHOGONALITY CONDITION

Now let X_i and X_j be solutions of (25) with λ_i and λ_j respectively ($\lambda_i > \lambda_j$). Then

$$[X_i^{(4)} - \lambda_i^4 X_i] + \sigma_{3i} \delta_{x_0} = -\sigma_{1i} \delta''_{x_0}, \quad [X_j^{(4)} - \lambda_j^4 X_j] + \sigma_{3j} \delta_{x_0} = -\sigma_{1j} \delta''_{x_0}. \tag{31}$$

Functions X_i, X_j are continuous at x_0 and distributions on the right-hand-side of equations in (31) are of order zero. Moreover, note that

$$(\sigma_{1i} X_j - \sigma_{1j} X_i)^{(k)}(x_0^+) = (\sigma_{1i} X_j - \sigma_{1j} X_i)^{(k)}(x_0^-) = (\sigma_{1i} X_j - \sigma_{1j} X_i)^{(k)}(x_0)$$

for $k = 0, 1, 2$. Thus we obtain

$$[X_i X_j^{(4)} - X_i^{(4)} X_j] - (\lambda_j^4 - \lambda_i^4)[X_i X_j] = (\sigma_{3i} X_j - \sigma_{3j} X_i)\delta_{x_0} + (\sigma_{1i} X_j - \sigma_{1j} X_i)\delta''_{x_0}.$$

Hence, for any test function $\varphi \in \mathcal{D}(0,l)$:

$$\int_{\mathbb{R}} (X_i X_j^{(4)} - X_j X_i^{(4)})(x)\varphi(x)dx - (\lambda_j^4 - \lambda_i^4)\int_{\mathbb{R}} X_i(x)X_j(x)\varphi(x)dx =$$
$$= (\sigma_{3i}X_j(x_0) - \sigma_{3j}X_i(x_0))\varphi(x_0) + (\sigma_{1i}X_j(x_0) - \sigma_{1j}X_i(x_0))\varphi''(x_0) +$$
$$+2(\sigma_{1i}X_j'(x_0^+) - \sigma_{1j}X_i'(x_0^+))\varphi'(x_0).$$

In particular,

$$\int_0^l (X_i X_j^{(4)} - X_j X_i^{(4)})(x)dx - (\lambda_j^4 - \lambda_i^4)\int_0^l X_i(x)X_j(x)dx =$$
$$= (\sigma_{3i}X_j(x_0) - \sigma_{3j}X_i(x_0)).$$

Next, integrating the first integral by parts and applying conditions (29) and one of (26)–(28), we compute

$$(\lambda_i^4 - \lambda_j^4)\int_0^l X_i(x)X_j(x)dx = 0.$$

Thus

$$\int_0^l X_i(x)X_j(x)dx = \begin{cases} 0 & \text{for } i \neq j, \\ \kappa_i & \text{for } i = j. \end{cases}$$

6. BOUNDARY-INITIAL PROBLEM

We now turn to the case of $f(x,t) \neq 0$ in (1). We are looking for the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)F_n(t) \tag{32}$$

where X_n is an eigenfunction corresponding to an eigenvalue λ_n. Then

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)X_n(x), \quad f_n(t) = \frac{1}{\kappa_n}\int_0^l f(x,t)X_n(x)dx. \tag{33}$$

Substituting (32) and (33) into (7), we can proceed as in the homogeneous case to conclude that F_n satisfies the equation

$$\ddot{F_n} + \mu\omega_n^2\dot{F_n} + \omega_n^2 = f_n.$$

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