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POROUS SETS FOR MUTUALLY NEAREST POINTS IN BANACH SPACES

Abstract. Let $\mathfrak{B}(X)$ denote the family of all nonempty closed bounded subsets of a real Banach space X, endowed with the Hausdorff metric. For $E, F \in \mathfrak{B}(X)$ we set $\lambda_{EF} = \inf\{\|z - x\| : x \in E, z \in F\}$. Let $\mathfrak{D}$ denote the closure (under the maximum distance) of the set of all $(E, F) \in \mathfrak{B}(X) \times \mathfrak{B}(X)$ such that $\lambda_{EF} > 0$. It is proved that the set of all $(E, F) \in \mathfrak{D}$ for which the minimization problem $\min_{x\in E,\ z\in F} \|x - z\|$ fails to be well posed in a σ-porous subset of $\mathfrak{D}$.

Keywords: minimization problem, well-posedness, H_ρ-topology, σ-porous set.

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1. INTRODUCTION

Let X be a real Banach space. By $\mathfrak{B}(X)$ we denote the family of all nonempty closed bounded subsets of X. For $E, F \in \mathfrak{B}(X)$ we set

$$\lambda_{EF} := \inf\{\|z - x\| : x \in E, z \in F\}.$$

We consider the minimization problem, denoted by $\min(E, F)$, of finding a pair (x_0, z_0) with $x_0 \in E, z_0 \in F$ such that $\|x_0 - z_0\| = \lambda_{EF}$. Such a pair is called a solution of the minimization problem $\min(E, F)$. Moreover, any sequence $\{(x_n, z_n)\}$ with $x_n \in E, z_n \in F$ such that $\lim_{n\to\infty} \|x_n - z_n\| = \lambda_{EF}$ is called a *minimizing* sequence for the problem $\min(E, F)$. A minimization problem is said to be *well-posed* if it has a unique solution and every minimizing sequence converges strongly to this solution.

Recall that the Hausdorff distance on the space $\mathfrak{B}(X)$ is defined by

$$h(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \|a - b\|,\ \sup_{b\in B} \inf_{a\in A} \|a - b\|\right\}, \quad A, B \in \mathfrak{B}(X).$$

It is well known that $\mathfrak{B}(X)$ endowed with the Hausdorff distance is a complete metric space.

Define $\mathfrak{C}(X) = \{A \in \mathfrak{B}(X) : A \text{ is convex}\}$. For a given set G, let $\mathfrak{C}_G(X)$ stand for the closure of the set $\{A \in \mathfrak{C}(X) : \lambda_{AG} > 0\}$. It is proved in [3] that if X is a uniformly convex Banach space, then the set of all $A \in \mathfrak{C}_G(X)$ such that the minimization problem $\min(A, G)$ is well-posed is a dense G_δ-subset of $\mathfrak{C}_G(X)$. This result has been extended to the framework of strongly convex and/or strictly convex Banach spaces in [7-12]. For further related results see [5, 6, 14-16].

Let $\mathfrak{B}(X) \times \mathfrak{B}(X)$ denote the Cartesian product endowed with the distance

$$d((A,B),(E,F)) = \max\{h(A,E), h(B,F)\} \quad \text{for} \quad A,B,E,F \in \mathcal{B}(X).$$

Let $\mathfrak{D}$ denote the closure of the set of all $(E,F) \in \mathcal{B}(X) \times \mathfrak{B}(X)$ such that $\lambda_{EF} > 0$. In this note, we will show that the set of all $(E,F) \in \mathfrak{D}$ such that the minimization problem $\min(E,F)$ is well-posed is a dense G_δ-subset of $\mathfrak{D}$. In particular, we also show that the set of all $(E,F) \in \mathfrak{D}$ such that the minimization problem $\min(E,F)$ fails to be well-posed is a σ-porous subset of $\mathfrak{D}$.

2. AUXILIARY RESULTS

For a subset A of X, $\overline{A}$ stands for the closure of A, $\operatorname{diam} A$ for the diameter of A, $\overline{\text{co}}A$ for the closed convex hull of A, and $d(x, A)$ for the distance from x to A. We use $S(x, r)$ to denote the closed ball with center x and radius r in X, in particular, S stands for $S(0,1)$.

Let $E,\ F \in \mathfrak{B}(X)$ and $\sigma > 0$. Define

$$L_{E,F}(\sigma) := \{x \in E : d(x,F) \le \lambda_{EF} + \sigma\} \tag{2.1}$$

It is clear that $L_{E,F}(\sigma_1) \subseteq L_{E,F}(\sigma_2)$ if $\sigma_1 \le \sigma_2$. The following propositions can be found in [3, 4].

Proposition 2.1. *Let $E,\ F \in \mathcal{B}(X)$. Then the problem $\min(E,F)$ is well-posed if and only if*

$$\inf_{\sigma>0} \operatorname{diam} L_{E,F}(\sigma) = 0 \quad \text{and} \quad \inf_{\sigma>0} \operatorname{diam} L_{F,E}(\sigma) = 0.$$

Proposition 2.2. *Let $A,\ B,\ E,\ F \in \mathfrak{B}(X)$ and $z \in X$. Then:*

(i) $|d(z,E) - d(z,F)| \le h(E,F)$;

(ii) $\lambda_{EF} \le d(z,E) + d(z,F)$;

(iii) $|\lambda_{AB} - \lambda_{EF}| \le 2d\big((A,B),(E,F)\big)$.

Define the function Λ on $\mathfrak{D}$ by the formula

$$\Lambda(E,F) = \inf_{\sigma>0} \operatorname{diam} L_{E,F}(\sigma), \quad \text{for} \quad (E,F) \in \mathfrak{D}.$$

Proposition 2.3. Λ *is upper semi-continuous on* $\mathfrak{D}$.

Proof. Let $(E_0, F_0) \in \mathcal{D}$. Let $\sigma > 0$ and $\delta > 0$. We will show that

$$L_{E,F}(\sigma) \subseteq L_{E_0,F_0}(\sigma + 4\delta) + \delta S \tag{2.2}$$

holds for all $(E, F) \in \mathcal{D}$ with $d\big((E, F), (E_0, F_0)\big) < \delta$. Indeed let $y \in L_{E,F}(\sigma)$. Since $h(E, E_0) < \delta$, there exists $x \in E_0$ such that $\|x - y\| < \delta$. Hence, by Proposition 2.2 and relation (2.1) there is

$$\begin{aligned} d(x, F_0) &\le d(x, F) + h(F, F_0) \le d(y, F) + \|x - y\| + h(F, F_0) \le \\ &\le d(y, F) + 2\delta \le \lambda_{EF} + \sigma + 2\delta \le \lambda_{E_0F_0} + \sigma + 4\delta, \end{aligned}$$

which shows that $x \in L_{E_0,F_0}(\sigma + 4\delta)$. Hence (2.2) holds. Let $\varepsilon > 0$. Choose $\tau > 0$ such that

$$\operatorname{diam} L_{E_0,F_0}(\tau) < \Lambda(E_0, F_0) + \frac{\varepsilon}{2}. \tag{2.3}$$

Taking $\sigma > 0$ and $\delta > 0$ such that $\sigma + 4\delta < \tau$ and $\delta < \varepsilon/4$, by (2.2) and (2.3), we obtain

$$\Lambda(E, F) \le \operatorname{diam} L_{E,F}(\sigma) \le \operatorname{diam} L_{E_0,F_0}(\sigma + 4\delta) + 2\delta < \Lambda(E_0, F_0) + \varepsilon$$

for all $(E, F) \in \mathfrak{D}$ with $d\big((E, F), (E_0, F_0)\big) < \delta$. This shows that Λ is upper semi-continuous at (E_0, F_0). □

The following lemma (see [4]) is essential in our proofs.

Lemma 2.1. *Let* $\varepsilon > 0$, $\rho > 0$ *and let* $E \in \mathfrak{C}(X)$. *Let* $\delta_0 = (\rho/2)\min\{1, \varepsilon\}$. *Then for each* $u \in X$ *with* $d(u, E) \ge \rho$ *and each* $0 < \delta \le \delta_0$, *there is*

$$\operatorname{diam} C_{E,u}(\delta) < (\operatorname{diam} E + \delta)\varepsilon,$$

where

$$C_{E,u}(\delta) = \overline{\mathrm{co}}(E \cup \{u\}) \setminus (E + (d(u, E) - \delta)S).$$

3. A GENERIC RESULT FOR MUTUALLY NEAREST POINTS

Let $\mathfrak{D}_0$ denote the set of all $(E, F) \in \mathfrak{D}$ such that the minimization problem $\min(E, F)$ is well-posed. By virtue of Proposition 2.1,

$$\mathfrak{D}_0 = \bigcap_{k \in \mathbb{N}} \mathfrak{D}_k, \tag{3.1}$$

where

$$\mathfrak{D}_k := \Big\{(E, F) \in \mathfrak{D} : \Lambda(E, F) < \frac{1}{k},\ \Lambda(F, E) < \frac{1}{k}\Big\}.$$

Theorem 3.1. $\mathfrak{D}_0$ *is a dense* G_δ *subset of* $\mathfrak{D}$.

Proof. By (3.1), it suffices to verify that each $\mathfrak{D}_k$ ($k \in \mathbb{N}$) is open and dense in $\mathcal{D}$. The openness of $\mathcal{D}_k$ is a direct consequence of Proposition 2.3. It remains to show that for every $k \in \mathbb{N}$ the set $\mathfrak{D}_k$ is dense in $\mathfrak{D}$. To this end, let $(E, F) \in \mathfrak{D}$. Without loss of generality, we may assume that $\lambda_{EF} > 0$. Let $k \in \mathbb{N}$ and $0 < r < \lambda_{EF}/4$. By Lemma 2.1, there exists $0 < \delta < r/2$ such that, for all $u \in X$ with $d(u, E) \geq r/2$ and all $v \in X$ with $d(v, F) \geq r/2$, there holds

$$\operatorname{diam} C_{E,u}(\delta) < \frac{1}{k} \quad \text{and} \quad \operatorname{diam} C_{F,v}(\delta) < \frac{1}{k}.$$

Pick $\hat{x} \in E$ and $\hat{y} \in F$ such that

$$\|\hat{x} - \hat{y}\| < \lambda_{EF} + \delta/2.$$

Note that $\|\hat{x} - \hat{y}\| \geq \lambda_{EF} \geq 4r$. Choose such two points u and v in the interval $[\hat{x}, \hat{y}]$ that $\|\hat{x} - u\| = \|\hat{y} - v\| = r$ and define

$$\widetilde{E} = \overline{co}\big(E \cup \{u\}\big), \quad \widetilde{F} = \overline{co}\big(F \cup \{v\}\big).$$

Obviously $h(\widetilde{E}, E) \leq r$, $h(\widetilde{F}, F) \leq r$ and $\lambda_{\widetilde{E}\widetilde{F}} \geq \lambda_{EF} - 2r > 0$. Hence $(\widetilde{E}, \widetilde{F}) \in \mathfrak{D}$. To complete the proof it suffices to show that $(\widetilde{E}, \widetilde{F}) \in \mathfrak{D}_k$ for every $k \in \mathbb{N}$. Note that

$$\|u - \hat{y}\| = \|\hat{x} - \hat{y}\| - \|u - \hat{x}\| \leq \lambda_{EF} + \frac{\delta}{2} - r$$

and

$$d(u, F) \;\leq\; \|u - \hat{y}\| \;\leq\; \lambda_{EF} + \frac{\delta}{2} - r.$$

From Proposition 2.2, the last inequality and the choice of δ, we conclude

$$d(u, E) \;\geq\; \lambda_{EF} - d(u, F) \geq r - \frac{\delta}{2} \;\geq\; \frac{3r}{4}. \tag{3.2}$$

On the other hand, since $u \in \widetilde{E}$ and $v \in \widetilde{F}$, then

$$\lambda_{\widetilde{E}\widetilde{F}} \;=\; \|u - v\| \leq \|\hat{x} - \hat{y}\| - 2r \;\leq\; \lambda_{EF} + \frac{\delta}{2} - 2r. \tag{3.3}$$

We claim that

$$L_{\widetilde{E}\widetilde{F}}\big(\delta/2\big) \subseteq C_{E,u}(\delta). \tag{3.4}$$

Indeed, let $y \in L_{\widetilde{E}\widetilde{F}}(\delta/2) = \overline{co}(E \cup \{u\})$. By (2.1) and (3.3), there holds

$$d(y, \widetilde{F}) \;\leq\; \lambda_{\widetilde{E}\widetilde{F}} + \frac{\delta}{2} \;\leq\; \lambda_{EF} + \delta - 2r. \tag{3.5}$$

Then, by Proposition 2.2, relation (3.5), and the inequality $d(u, E) \leq \|u - \hat{x}\| = r$, there follows

$$\begin{aligned} d(y, E) \geq \lambda_{E\widetilde{F}} - d(y, \widetilde{F}) \;&\geq\; \lambda_{E\widetilde{F}} - (\lambda_{EF} + \delta - 2r) \geq \\ &\geq \lambda_{EF} - r - (\lambda_{EF} + \delta - 2r) \;=\; r - \delta \geq d(u, E) - \delta, \end{aligned}$$

This means that (3.4) holds. From (3.4) and Lemma 2.1 it follows that

$$\Lambda(\widetilde{E}, \widetilde{F}) \leq \operatorname{diam} C_{E,u}(\delta) < \frac{1}{k}.$$

Similarly, one can show that

$$\Lambda(\widetilde{F}, \widetilde{E}) \leq \operatorname{diam} C_{F,v}(\delta) < \frac{1}{k}.$$

This means that $(\widetilde{E}, \widetilde{F}) \in \mathfrak{D}_k$, which completes the proof. □

4. A POROSITY RESULT

Definition 4.1. *A subset Y in a metric space (X, d) is said to be porous in X if there are $0 < t \leq 1$ and $r_0 > 0$ such that for every $x \in X$ and $r \in (0, r_0]$ there is a point $y \in X$ such that $S(y, tr) \subseteq S(x, r) \cap (X \setminus Y)$. A subset Y is said to be σ-porous in X if it is a countable union of sets which are porous in X.*

Note that an equivalent definition of a porous set can be obtained by replacing "for every $x \in X$" with "for every $x \in Y$" (see [1, 3]).

For $(E, F) \in \mathfrak{D}_0$, let $(u_E, u_F) \in E \times F$ denote the unique solution of the minimization problem $\min(E, F)$. Let

$$u_{\alpha,E} = (1-\alpha)u_E + \alpha u_F, \quad \text{and} \quad E_\alpha = \overline{co}\left(E \cup \{u_{\alpha,E}\}\right), \quad \alpha \in [0,1].$$

Furthermore, for $r > 0$, set

$$\mathcal{O}(F, r) = \left\{E \in \mathcal{B}(X) : \ h(E, F) < r\right\}.$$

Define

$$\widetilde{\mathfrak{D}} = \bigcap_{k \in \mathbb{N}} \ \bigcup_{(E,F) \in \mathfrak{D}_0} \ \bigcup_{0 \leq \alpha \leq 1/4} \Big(\mathcal{O}\big(E_\alpha, \gamma_{E_\alpha}(1/k)\big) \times \mathcal{O}\big(F_\alpha, \gamma_{F_\alpha}(1/k)\big)\Big),$$

where

$$\gamma_{E_\alpha}(\varepsilon) = \min\left\{d(u_{\alpha,E}, E), 1\right\}\varepsilon, \quad \gamma_{F_\alpha}(\varepsilon) = \min\left\{d(u_{\alpha,F}, F), 1\right\}\varepsilon.$$

Lemma 4.1. $\widetilde{\mathfrak{D}} \subseteq \mathfrak{D}_0$.

Proof. Let $(E, F) \in \widetilde{\mathcal{D}}$. By Proposition 2.2, we only need to show that

$$\Lambda(E, F) = \lim_{\delta \to 0+} \operatorname{diam} L_{E,F}(\delta) = 0 \quad \text{and} \quad \Lambda(F, E) = \lim_{\delta \to 0+} \operatorname{diam} L_{F,E}(\delta) = 0. \tag{4.1}$$

By the definition of $\widetilde{\mathfrak{D}}$, for each $k \in \mathbb{N}$, there exist $(E^k, F^k) \in \mathfrak{D}_0$ and $0 \leq \alpha_k \leq 1/4$ such that

$$h(E, E^k_{\alpha_k}) \leq \gamma_{E^k_{\alpha_k}}(1/k) \quad \text{and} \quad h(F, F^k_{\alpha_k}) \leq \gamma_{F^k_{\alpha_k}}(1/k). \tag{4.2}$$

Without loss of generality, we may assume that $\lambda_{E^kF^k} > 0$ and $\alpha_k > 0$ for each $k \in \mathbb{N}$. For convenience, we write

$$r_k = \lambda_{E^kF^k} \quad \text{and} \quad \delta_k = \gamma_{E^k_{\alpha_k}}(1/k) = \gamma_{F^k_{\alpha_k}}(1/k).$$

Then, it is easy to see that, for each $k \in \mathbb{N}$,

$$\delta_k \le \alpha_k r_k/k,$$

$$\lambda_{E^k_{\alpha_k}F^k_{\alpha_k}} = (1-2\alpha_k)r_k, \tag{4.3}$$

$$\lambda_{E^kF^k_{\alpha_k}} = \lambda_{F^kE^k_{\alpha_k}} = (1-\alpha_k)r_k, \tag{4.4}$$

$$d(u_{\alpha_k,E^k}, E^k) = d(u_{\alpha_k,F^k}, F^k) = \alpha_k r_k. \tag{4.5}$$

We claim that, for each $\delta > 0$,

$$L_{E^k_{\alpha_k},F^k_{\alpha_k}}(\delta/2) \subseteq C_{E^k,u_{\alpha_k,E^k}}(\delta) \quad \text{for } k \in \mathbb{N}. \tag{4.6}$$

To see this, let $k \in \mathbb{N}$, $\delta > 0$ and $y \in L_{E^k_{\alpha_k},F^k_{\alpha_k}}(\delta/2)$. Obviously, $y \in \overline{co}(E^k \cup \{u_{\alpha_k,E^k}\})$. By (2.1) and (4.3), there is

$$d(y, F^k_{\alpha_k}) \le \lambda_{E^k_{\alpha_k}F^k_{\alpha_k}} + \frac{\delta}{2} = (1-2\alpha_k)r_k + \frac{\delta}{2}. \tag{4.7}$$

Consequently, by Proposition 2.2, relation (4.4), (4.7) and (4.5) we obtain

$$\begin{aligned} d(y, E^k) &\ge \lambda_{E^kF^k_{\alpha_k}} - d(y, F^k_{\alpha_k}) \ \ge\ (1-\alpha_k)r_k - (1-2\alpha_k)r_k - \delta/2 = \\ &= \alpha_k r_k - \delta/2 \ =\ d(u_{\alpha_k,E^k}, E^k) - \delta/2 \ >\ d(u_{\alpha_k,E^k}, E^k) - \delta. \end{aligned}$$

Hence $y \in C_{E^k,u_{\alpha_k,E^k}}(\delta)$. Since

$$d((E,F),(E^k_{\alpha_k}, F^k_{\alpha_k})) < \delta_k,$$

from (2.2) it follows that

$$L_{E,F}(\delta_k) \subseteq L_{E^k_{\alpha_k},F^k_{\alpha_k}}(5\delta_k) + \delta_k S.$$

By the last inclusion and (4.6) we obtain

$$\begin{aligned} \Lambda(E,F) &\le \operatorname{diam} L_{E,F}(\delta_k) \ \le\ \operatorname{diam} L_{E^k_{\alpha_k},F^k_{\alpha_k}}(5\delta_k) + 2\delta_k \le \\ &\le \operatorname{diam} C_{E^k,u_{\alpha_k}}(10\delta_k) + 2\delta_k \end{aligned} \tag{4.8}$$

for each $k \in \mathbb{N}$. Recall that

$$d(u_{\alpha_k,E^k}, E^k) = \alpha_k r_k \quad \text{and} \quad \delta_k \le \frac{\alpha_k r_k}{k} \quad \text{for each} \quad k > 1.$$

Then using Lemma 2.1 we conclude that

$$\operatorname{diam} C_{E^k,u_{\alpha_k}}(10\delta_k) \le \frac{2}{k}(\operatorname{diam} E^k + 10\alpha_k r_k),$$

and hence, by (4.8),

$$\Lambda(E,F) \le \frac{2}{k}(\operatorname{diam}E^k + 10\alpha_k r_k) + 2\delta_k \le \frac{2}{k}(\operatorname{diam}E^k + 11\alpha_k r_k). \tag{4.9}$$

Note that

$$h(E,E^k) \le h(E,E^k_{\alpha_k}) \le \gamma_{E^k_{\alpha_k}}(1/k) \le 1.$$

Analogously $h(F,F^k) \le 1$. Thus $h(E^k,F^k) \le h(E,F)+2$. It follows that sequences $\{\operatorname{diam}E^k\}$ and $\{r_k\}$ are bounded. Hence (4.9) implies that $\Lambda(E,F)=0$. Similarly, we can verify that $\Lambda(F,E)=0$. Hence (4.1) holds and the proof of Lemma 4.1 is complete. □

Theorem 4.1. *The set $\mathfrak{D}\setminus\mathfrak{D}_0$ is σ-porous in $\mathfrak{D}$.*

Proof. For k, $l \in \mathbb{N}$, define

$$\widetilde{\mathfrak{D}}_k = \mathfrak{D} \setminus \bigcup_{(E,F)\in\mathfrak{D}_0} \bigcup_{0\le\alpha\le 1/4} \Big(\mathcal{O}\big(E_\alpha, \gamma_{E_\alpha}(1/k)\big) \times \mathcal{O}\big(F_\alpha, \gamma_{F_\alpha}(1/k)\big)\Big)$$

and

$$\widetilde{\mathcal{D}}^l_k = \Big\{(E,F)\in\widetilde{\mathcal{D}}_k : \frac{1}{l} < \lambda_{EF} < l\Big\}.$$

Observe that

$$\mathfrak{D}\setminus\mathcal{D}_0 \subseteq \mathfrak{D}\setminus\widetilde{\mathcal{D}} = \bigcup_{k\in\mathbb{N}}\bigcup_{l\in\mathbb{N}}\widetilde{\mathcal{D}}^l_k.$$

It suffices to verify that, $\widetilde{\mathfrak{D}}^l_k$ is porous in $\mathfrak{D}$ for each $k,l\in\mathbb{N}$. To this end, let $k,l\in\mathbb{N}$ be arbitrary. Define $r_0 = 1/(2l)$ and $\alpha = 1/(4k)$. Let $(E,F)\in\widetilde{\mathfrak{D}}^l_k$ and $0<r\le r_0$. Then, by Theorem 2.1, there exists $(\bar{E},\bar{F})\in\mathfrak{D}_0$ such that

$$h(E,\bar{E}) < \frac{r}{4}, \qquad h(F,\bar{F}) < \frac{r}{4}$$

and

$$\frac{1}{l} < \lambda_{\bar{E}\bar{F}} < l.$$

Set $\bar{u}_{1/2} = (u_{\bar{E}} + u_{\bar{F}})/2$. Then

$$\begin{aligned} h(\bar{E}_{1/2},E) &\ge h(\bar{E}_{1/2},\bar{E}) - h(\bar{E},E) \ge \\ &\ge \sup_{y\in\bar{E}_{1/2}} d(y,\bar{E}) - r/4 \ge \\ &\ge d(\bar{u}_{1/2},\bar{E}) - r/4 = \\ &= (1/2)\lambda_{\bar{E}\bar{F}} - r/4 \ge 3r/4. \end{aligned}$$

Similarly, one can prove that

$$h(\bar{F}_{1/2}, F) \geq 3r/4.$$

From the previous two inequalities it follows that there exist $0 < t_1, t_2 \leq 1/2$ such that $h(\bar{E}_{t_1}, E) = 3r/4$ and $h(\bar{F}_{t_2}, F) = 3r/4$, where $\bar{E}_{t_1} = \overline{co}(\bar{E} \cup u_{t_1,\bar{E}})$ and $\bar{F}_{t_2} = \overline{co}(\bar{F} \cup u_{t_2,\bar{F}})$. Observe that

$$\mathcal{O}(\bar{E}_{t_1}, \alpha r) \subseteq \mathcal{O}(E, r) \quad \text{and} \quad \mathcal{O}(\bar{F}_{t_2}, \alpha r) \subseteq \mathcal{O}(F, r). \tag{4.10}$$

Indeed, for each $A \in \mathcal{O}(\bar{E}_{t_1}, \alpha r)$

$$h(A, E) \leq h(A, \bar{E}_{t_1}) + h(\bar{E}_{t_1}, E) \leq \alpha r + 3r/4 \leq r.$$

Hence the first inequality of (4.10) is proved. The second one can be proved analogously.

Now we claim that

$$\alpha r \leq \gamma_{\bar{E}_{t_1}}(1/k) \quad \text{and} \quad \alpha r \leq \gamma_{\bar{F}_{t_2}}(1/k). \tag{4.11}$$

Indeed, note that

$$h(\bar{E}_{t_1}, \bar{E}) \geq h(\bar{E}_{t_1}, E) - h(E, \bar{E}) \geq r/2.$$

Therefore,

$$\alpha r \leq 2\alpha h(\bar{E}_{t_1}, \bar{E}) \leq h(\bar{E}_{t_1}, \bar{E})/k = d(u_{t_1,\bar{E}}, \bar{E})/k.$$

Since obviously $\alpha r \leq 1/k$, the first inequality of (4.11) is proved. The second one can be proved analogously.

From (4.11) it follows that

$$\mathcal{O}(\bar{E}_{t_1}, \alpha r) \times \mathcal{O}(\bar{F}_{t_2}, \alpha r) \subseteq \mathcal{O}(\bar{E}_{t_1}, \gamma_{\bar{E}_{t_1}}(1/k)) \times \mathcal{O}(\bar{F}_{t_2}, \gamma_{\bar{F}_{t_2}}(1/k)).$$

This implies that

$$\mathcal{O}(\bar{E}_{t_1}, \alpha r) \times \mathcal{O}(\bar{F}_{t_2}, \alpha r) \subseteq \mathfrak{D} \setminus \widetilde{\mathfrak{D}}_k^l.$$

From this last inclusion and relation (4.10) it immediately follows that the set $\widetilde{\mathfrak{D}}_k^l$ is porous in $\mathfrak{D}$. □

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