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**ON THE PERFECTNESS  
OF  $C^{\infty,s}$ -DIFFEOMORPHISM GROUPS  
ON A FOLIATED MANIFOLD**

**Abstract.** The notion of  $C^{r,s}$  and  $C^{\infty,s}$ -diffeomorphisms is introduced. It is shown that the identity component of the group of leaf preserving  $C^{\infty,s}$ -diffeomorphisms with compact supports is perfect. This result is a modification of the Mather and Epstein perfectness theorem.

**Keywords:** group of  $C^{\infty}$ -diffeomorphisms, perfectness, commutator, foliation.

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## 1. INTRODUCTION

We say that a group  $G$  is *perfect* iff  $G = [G, G]$ , where  $[G, G]$  is the commutator subgroup of  $G$ . Next,  $G$  is *simple* iff it has no normal subgroups except  $\{e\}$  and  $G$  itself. In terms of homology groups it means that  $H_1(G) = G/[G, G] = 0$ .

Let  $D_c^r(M)$  be the group of all  $C^r$ -diffeomorphisms on a manifold  $M$  isotopic to the identity through compactly supported  $C^r$ -isotopies. D. B. A. Epstein proved in [2] that for a large class of homeomorphism groups perfectness implies simplicity. So it suffices to show that the group  $D_c^r(M)$  is perfect. Especially the following result is valid.

**Theorem 1.1 (Thurston, Mather).** *Let  $M$  be an  $n$ -dimensional smooth manifold and let  $1 \leq r \leq \infty$ ,  $r \neq n + 1$ . Then the group  $D_c^r(M)$  is perfect and simple.*

For  $r = \infty$  and  $M = T^n$ , where  $T^n$  is the  $n$ -dimensional torus, Theorem 1.1 was proved by M. R. Herman [4]. Small denominator theory [1] was used in the proof. W. Thurston [11] generalized Herman's considerations to an arbitrary manifold  $M$  by homological arguments.

Next, Mather [6, 7] solved the case  $r$  finite,  $r \neq n + 1$ . His method is quite different to the one mentioned above. The rolling-up operators  $\Psi_{i,A}$  play a key role. Epstein

[3] modified this method to prove the perfectness of the group  $D_c^\infty(M)$  in a way other than Thurston's. The case  $r = n + 1$  is still open. But there are strong arguments that it may not be perfect (see [8], [9]).

Let  $(M, \mathcal{F})$  be a foliated manifold of dimension  $n$  and  $\dim \mathcal{F} = k$ . We consider the perfectness of the group  $D_c^r(M, \mathcal{F})$  of leaf preserving  $C^r$ -diffeomorphisms isotopic to the identity through compactly supported isotopies.

For  $r = \infty$ , the group was proved to be perfect by Rybicki [10] following Herman and Thurston's arguments. Moreover, for  $r < k$ , it is easily checked by using Mather's proof [7].

For  $r \geq k + 1$  the perfectness of the group  $D_c^r(M, \mathcal{F})$  follows the perfectness of the group  $D_c^r(M)$  as it was presented in [5]. In particular it means that the solution of the problem in the case  $r = n + 1$  is positive. But Mather's method does not work here.

Hence we consider another kind of smoothness. Namely we introduce a notion of  $C^{r,s}$ -mappings which are of class  $C^r$  in the tangent direction and of class  $C^s$  in the transversal direction. For the definition see section 2. Then we are able to show the perfectness of the group  $D_c^{r,s}(M, \mathcal{F})$  whenever  $r - s > k + 1$  (see [5]).

Our aim is to extend this result onto the case of  $C^{\infty,s}$ -diffeomorphisms. We will use modifications of Epstein's method [3] and remarks about  $C^{r,s}$ -mappings included in [5]. Eventually we will obtain the following

**Theorem 1.2.** *Let  $s \geq 1$  and let  $(M, \mathcal{F})$  be an  $n$ -dimensional smooth foliated manifold with  $\dim \mathcal{F} = k$ . Then the group  $D_c^{r,s}(M, \mathcal{F})$  is perfect, whenever  $r - s > k + 1$  or  $r = \infty$ .*

In Section 2 we will introduce the notion of  $C^{r,s}$ -mappings and some basic properties of such mappings. Then we will present estimations which play important role in the proof.

Section 3 contains a proof of Theorem 1.2. Mather's idea of the rolling-up operators  $\Psi_{i,A}$  is used there since they are leaf preserving mappings. The construction of  $\Psi_{i,A}$  is presented in Section 4.

## 2. DEFINITIONS AND BASIC ESTIMATES

Let  $r, s \geq 1$ ,  $1 \leq k \leq n$  and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a mapping such that  $f(x, y) = (f_1(x, y), f_2(y))$ , where  $x \in \mathbb{R}^k$ ,  $y \in \mathbb{R}^{n-k}$ .

**Definition 2.1.** *A partial derivative of order  $r$  of  $f$  is  $s$ -admissible iff it contains at most  $s$  derivatives in the direction of the last  $n - k$  coordinates. Next, we say that  $f$  is of class  $C^{r,s}$  iff it has all  $s$ -admissible partial derivatives up to order  $r$  and they are continuous. Moreover,  $f$  is of class  $C^{\infty,s}$  iff it is of class  $C^{r,s}$  for all  $r \geq 1$ .*

By  $D^{r,s}f : \mathbb{R}^n \rightarrow L^r(\mathbb{R}^n, \mathbb{R}^n)$  we denote the mapping, considered as an multidimensional matrix, which entries are  $s$ -admissible partial derivatives of order  $r$  of  $f$ , and zeros in place of derivatives which are not  $s$ -admissible. We called it the  $(r, s)$ -th derivative of  $f$ .

See that  $D^{r,s}f = D^r f$  for  $r \leq s$ , where  $D^r f$  is the standard  $r$ -th derivative of  $f$ . Hence we will write  $Df = D^{1,s}f$ .

The space  $\mathbb{R}^n$  will be treated as a foliated manifold with product foliation  $\mathcal{F} = \{\mathbb{R}^k \times \{\text{pt}\}\}$ . Then by  $C^{r,s}(n, k)$  we denote the space of all  $C^{r,s}$ -mappings  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of the form  $f(x, y) = (f_1(x, y), y)$ ,  $x \in \mathbb{R}^k$ ,  $y \in \mathbb{R}^{n-k}$ .

For  $f, g \in C^{r,s}(n, k)$ ,  $r \leq s$ , we have the standard derivative formulas

$$D(f \circ g) = (Df \circ g) \cdot Dg \tag{1}$$

and

$$D^{r,s}(f \circ g) = (D^{r,s}f \circ g) \cdot (Dg \times \dots \times Dg) + (Df \circ g) \cdot D^{r,s}g + \sum C_{i,j_1,\dots,j_i} (D^{i,s}f \circ g) \cdot (D^{j_1,s}g \times \dots \times D^{j_i,s}g), \tag{2}$$

where the sum is taken over  $1 < i < r$ ,  $j_1 + \dots + j_i = r$ ,  $j_l \geq 1$ ,  $l = 1, \dots, i$ . Here  $C_{i,j_1,\dots,j_i}$  are positive constants independent of  $f$  and  $g$ .

If  $r > s$ , formula (2) is no longer valid, but we have the following

**Proposition 2.2.** *Let  $r, s \geq 1$  and  $f, g \in C^{r,s}(n, k)$ . If an entry in the matrix  $D^{r,s}(f \circ g)$  on the left hand side of (2) is an  $s$ -admissible partial derivative then the corresponding entry on the right hand side of (2) is expressed by the formula with  $s$ -admissible partial derivatives of  $f$  and  $g$  only, and they are equal.*

With using Proposition 2.2 we obtain

**Proposition 2.3.** *Let  $f, g \in C^{r,s}(n, k)$ . Then  $f \circ g \in C^{r,s}(n, k)$ . Moreover, if  $f$  is a  $C^1$ -diffeomorphism, then  $f^{-1} \in C^{r,s}(n, k)$ .*

For the proofs see [5].

Now by  $(M, \mathcal{F})$  we denote an  $n$ -dimensional smooth foliated manifold with  $\dim \mathcal{F} = k$ . We say that  $f : (M, \mathcal{F}) \rightarrow (M, \mathcal{F})$  is a *leaf preserving mapping* if  $f(L_x) \subset L_x$  for every  $x \in M$ , where  $L_x \in \mathcal{F}$  is the leaf containing  $x$ .

**Definition 2.4.** *A leaf preserving mapping  $f : (M, \mathcal{F}) \rightarrow (M, \mathcal{F})$  is of class  $C^{r,s}$  iff for every  $x \in M$  and a chart  $(V, v)$  on  $M$  with  $f(x) \in V$  there exists a chart  $(U, u)$  on  $M$  with  $x \in U$  such that  $f(U) \subset V$  and  $v \circ f \circ u^{-1}$  is of class  $C^{r,s}$ .*

See that  $f$  is locally of the form

$$(vfu^{-1})(x, y) = ((vfu^{-1})_1(x, y), (vfu^{-1})_2(y)),$$

$x \in \mathbb{R}^k$ ,  $y \in \mathbb{R}^{n-k}$ .

The symbol  $D_c^{r,s}(M, \mathcal{F})$ ,  $r = 1, \dots, \infty$ , stands for the group of all leaf preserving  $C^1$ -diffeomorphisms of class  $C^{r,s}$  which are isotopic to the identity through compactly supported  $C^{r,s}$ -isotopies. Next,  $D_K^{r,s}(M, \mathcal{F})$  will denote the subgroup of  $D_c^{r,s}(M, \mathcal{F})$  of diffeomorphisms with supports in  $K \subset M$ .

We can prove the following fragmentation property

**Lemma 2.5.** *Let  $1 \leq r \leq \infty$ ,  $s \geq 1$ . There exists a locally finite atlas  $\varphi = \{(U_i, \varphi_i)\}_{i \in I}$  on  $(M, \mathcal{F})$  such that for every  $f \in D_c^{r,s}(M, \mathcal{F})$  there are  $g_1, \dots, g_m \in D_c^{r,s}(M, \mathcal{F})$  satisfying conditions  $f = g_1 \dots g_m$  and  $\text{supp}(g_j) \subset U_{i(j)}$ ,  $j = 1, \dots, m$ .*

Using a map  $(U_i, \varphi_i)$  we obtain  $(\varphi_i g_j \varphi_i^{-1})(x, y) = ((\varphi_i g_j \varphi_i^{-1})(x, y), y)$  for every  $j = 1, \dots, m$ . Hence we may restrict our considerations to the case  $M = \mathbb{R}^n$  with foliation  $\mathcal{F} = \{\mathbb{R}^k \times \{\text{pt}\}\}$ . From now on we will write  $D^{r,s}(n, k)$  and  $D_K^{r,s}(n, k)$  instead of  $D_c^{r,s}(M, \mathcal{F})$  and  $D_K^{r,s}(M, \mathcal{F})$ .

Let  $r \geq 0$  and  $s \geq 1$ . For  $f \in C^{r,s}(n, k)$  we define the following seminorms

$$\begin{aligned}\mu_{r,s}(f) &= \sup_{x \in \mathbb{R}^n} \|D^{r,s}(f - \text{Id})(x)\|, \\ \|f\|_{r,s} &= \sup_{x \in \mathbb{R}^n} \|D^{r,s}f(x)\|.\end{aligned}$$

Here  $C^{0,s}(n, k)$  means the space of continuous mappings acting along leaves and  $D^{0,s}f(x) = f(x)$ . Further

$$M_{r,s}(f) = \sup\{\mu_{1,s}(f), \dots, \mu_{r,s}(f)\}.$$

It is easy to see that  $\|f\|_{1,s} \leq \mu_{1,s}(f) + 1$ ,  $\mu_{1,s}(f) \leq \|f\|_{1,s} + 1$  and  $\mu_{r,s}(f) = \|f\|_{r,s}$  for  $r \geq 2$ .

For a compact set  $K \subset \mathbb{R}^n$  we put

$$R_K = \sup\{\text{dist}(x, \overline{\mathbb{R}^n \setminus K} \cap L) : x \in L, L \in \mathcal{F}\} < \infty.$$

**Lemma 2.6.** *Let  $r, s \geq 1$ ,  $1 \leq i \leq k$  and let  $K \subset \mathbb{R}^n$  be a compact set. Then there exists a constant  $C > 0$  depending on  $R_K$  such that*

$$\mu_{r,s}(f) \leq C^r \mu_{r+1,s}(f)$$

whenever  $f \in D^{r+1,s}(n, k)$  satisfies one of the following conditions:

- 1)  $D^{r,s}(f - \text{Id}) = 0$  on  $\mathbb{R}^n \setminus K$ ,
- 2)  $f$  is periodic along the  $i$ -th coordinate with period 1 (then we take  $R_K = 1$ ).

*Proof.* First, we take  $f$  satisfying (1). For  $x = (x^1, x^2) \in \mathbb{R}^n$  with  $x^1 \in \mathbb{R}^k$ ,  $x^2 \in \mathbb{R}^{n-k}$  we choose  $y^1 \in \mathbb{R}^k$  such that  $y = (y^1, x^2) \in \overline{\mathbb{R}^n \setminus K}$  and  $\|x^1 - y^1\| \leq R_K$ . Then

$$\begin{aligned}\|D^{r,s}(f - \text{Id})(x)\| &= \|D^{r,s}(f - \text{Id})(x) - D^{r,s}(f - \text{Id})(y)\| \leq \\ &\leq \left\| \int_0^1 D^{r+1,s}(f - \text{Id})(tx + (1-t)y) \cdot (x^1 - y^1, 0) dt \right\| \leq \\ &\leq \mu_{r+1,s}(f) \|x^1 - y^1\| \leq R_K \mu_{r+1,s}(f).\end{aligned}$$

Next, we assume that  $f$  is periodic along the first coordinate with period 1. Let  $\frac{\partial^r(f - \text{Id})_l}{\partial x_{i_1} \dots \partial x_{i_r}}$  be an  $r$ -th  $s$ -admissible partial derivative of  $f$  and let  $x = (x_1, x^0) \in \mathbb{R}^n$ ,  $x_1 \in \mathbb{R}$ ,  $x^0 \in \mathbb{R}^{n-1}$ . There exists  $y = (y_1, x^0) \in \mathbb{R}^n$ ,  $y_1 \in \mathbb{R}$  such that  $\|x_1 - y_1\| \leq 1$  and  $\frac{\partial^r(f - \text{Id})_l}{\partial x_{i_1} \dots \partial x_{i_r}}(y) = 0$ . As above we obtain

$$\left\| \frac{\partial^r(f - \text{Id})_l}{\partial x_{i_1} \dots \partial x_{i_r}}(x) \right\| \leq \|x - y\| \mu_{r+1,s}(f) \leq \mu_{r+1,s}(f).$$

Summing up over all  $r$ -th  $s$ -admissible partial derivatives we have

$$\begin{aligned} \|D^{r,s}(f - \text{Id})(x)\| &\leq \sum \left\| \frac{\partial^r (f - \text{Id})_l}{\partial x_{i_1} \dots \partial x_{i_r}}(x) - \frac{\partial^r (f - \text{Id})_l}{\partial x_{i_1} \dots \partial x_{i_r}}(y) \right\| \leq \\ &\leq n^{r+1} \mu_{r+1,s}(f), \end{aligned}$$

which completes the proof. □

**Definition 2.7.** A semi-admissible polynomial is a polynomial with nonnegative coefficients and with no constant term. If, in addition, it has no linear term we say that it is an admissible polynomial.

From Proposition 2.2 and Lemma 2.6 we get

**Lemma 2.8.** 1. Let  $l \geq 1$ . For every  $f_1, \dots, f_l \in C^{1,s}(n, k)$  there is

$$\mu_{1,s}(f_1 \dots f_l) \leq l \left( \sup_{1 \leq i \leq l} \mu_{1,s}(f_i) \right) \left( 1 + \sup_{1 \leq i \leq l} \mu_{1,s}(f_i) \right)^{l-1}.$$

2. Let  $r \geq 2$  and  $l \geq 1$ . There exists an admissible polynomial  $F$  depending on  $r$  and  $l$  such that

$$\mu_{r,s}(f_1 \dots f_l) \leq l \left( \sup_{1 \leq i \leq l} \mu_{r,s}(f_i) \right) \left( 1 + \sup_{1 \leq i \leq l} \mu_{1,s}(f_i) \right)^{r(l-1)} + F \left( \sup_{1 \leq i \leq l} M_{r-1,s}(f_i) \right)$$

for every  $f_1, \dots, f_l \in C^{r,s}(n, k)$ .

3. For  $f \in D^{1,s}(n, k)$  with  $\mu_{1,s}(f) \leq \frac{1}{2}$  there is

$$\mu_{1,s}(f^{-1}) \leq 2\mu_{1,s}(f).$$

4. Let  $r \geq 2$ . There exists an admissible polynomial  $F$  such that

$$\mu_{r,s}(f^{-1}) \leq \mu_{r,s}(f) \left( 1 + 2\mu_{1,s}(f) \right)^{r+1} + F(M_{r-1,s}(f))$$

for every  $f \in D^{r,s}(n, k)$  with  $\mu_{1,s}(f) \leq \frac{1}{2}$ .

### 3. THE PROOF OF MAIN THEOREM

In this section we will give the proof of Theorem 1.2. Owing to the above remarks it suffices to show that the group  $D^{\infty,s}(n, k)$  is perfect. Moreover, it follows from the condition

$$D_K^{\infty,s}(n, k) \subset [D^{\infty,s}(n, k), D^{\infty,s}(n, k)], \tag{3}$$

where  $K \subset \mathbb{R}^n$  is a compact convex set. In fact, for  $f \in D^{\infty,s}(n, k)$  there exists  $g \in D_c^\infty(\mathbb{R}^n)$  of the form  $g(x, y) = (g_1(x, y), g_2(y))$ ,  $x \in \mathbb{R}^k$ ,  $y \in \mathbb{R}^{n-k}$  such that  $\text{supp}(gfg^{-1}) \subset K$ . Then  $gfg^{-1} \in D_K^{\infty,s}(n, k)$  and  $[f] = [\text{Id}] \in H_1(D^{\infty,s}(n, k))$  by (3).

We fix  $K = [-2, 2]^n$ . For a real constant  $A \geq 1$  we put

$$\begin{aligned} K_i &= [-2, 2]^i \times [-2A, 2A]^{k-i} \times [-2, 2]^{n-k}, \\ K'_i &= [-2A, 2A]^{i-1} \times S^1 \times [-2A, 2A]^{k-i} \times [-2, 2]^{n-k}, \\ K''_i &= [-2A, 2A]^{i-1} \times \mathbb{R} \times [-2A, 2A]^{k-i} \times [-2, 2]^{n-k}, \end{aligned}$$

$i = 0, \dots, k$ . Then

$$K_k = [-2, 2]^n \subset \dots \subset K_0 = [-2A, 2A] \times [-2, 2]^{n-k}.$$

Following Mather [6] we can construct rolling-up operators  $\Psi_{i,A}$  as below

**Lemma 3.1.** *Let  $A \geq 1$ . There exist a neighbourhood  $U_A$  of  $\text{Id} \in D_K^{1,s}(n, k)$  and mappings*

$$\Psi_{i,A} : U_A \rightarrow D_{K_0}^{1,s}(n, k),$$

$i = 1, \dots, k$ , such that:

1.  $\Psi_{i,A}$  preserves the identity.
2. For every  $r \geq 1$  the mapping

$$\Psi_{i,A} : U_A \cap D_{K_{i-1}}^{r,s}(n, k) \rightarrow D_{K_i}^{r,s}(n, k)$$

is continuous with respect to  $C^{r,s}$ -topology.

3. If  $f \in U_A \cap D^{\infty,s}(n, k)$  then  $[f] = [\Psi_{i,A}(f)] \in H_1(D^{\infty,s}(n, k))$ .
4. Let  $r \geq 2$ . There exist a constant  $Q > 2^{r^2}$  depending on  $r$  and independent of  $A$ , and an admissible polynomial  $F$  depending on  $r$  and  $A$  such that

$$\mu_{r,s}(\Psi_{i,A}(f)) \leq QA\mu_{r,s}(f) + F(M_{r-1,s}(f)) \quad (4)$$

for every  $f \in U_A \cap D_{K_{i-1}}^{r,s}(n, k)$ .

5. Let  $r \geq 2$ . There exist a constant  $C > 1$  independent of  $r$  and  $A$ , and a semi-admissible polynomial  $G$  depending on  $r$  and  $A$  such that

$$\mu_{r,s}(\Psi_{i,A}(f)) \leq C^r A\mu_{r,s}(f) + G(M_{r-1,s}(f)) \quad (5)$$

for every  $f \in U_A \cap D_{K_{i-1}}^{r,s}(n, k)$ .

For the proof see Section 4.

Let us take  $A \geq 1$ . We fix  $\zeta_A \in D_c^\infty(\mathbb{R}^n)$  acting along leaves and such that  $\zeta_A = (A \cdot \text{Id}, \text{Id})$  on  $K$ . For  $f, g \in D_K^{\infty,s}(n, k)$  we denote

$$g_0 = \zeta_A f g \zeta_A^{-1} \quad \text{and} \quad g_i = \Psi_{i,A}(g_{i-1}),$$

$i = 1, \dots, k$ . From Lemma 3.1 there exists a neighbourhood  $V_A$  of  $\text{Id} \in D_K^{\infty,s}(n, k)$  such that  $g_i \in U_A$ ,  $i = 0, \dots, k$ , whenever  $f, g \in V_A$ .

By using Proposition 2.2 and Lemma 2.8 we get

$$\begin{aligned} \mu_{r,s}(g_0) &= \sup_{x \in \mathbb{R}^n} \|D^{r,s}((A \cdot \text{Id}, \text{Id})fg(\frac{1}{A} \cdot \text{Id}, \text{Id}))(x)\| \leq \\ &\leq A^{1-r+s} \sup_{x \in \mathbb{R}^n} \|D^{r,s}(fg)(\frac{1}{A} \cdot \text{Id}, \text{Id})(x)\| \leq A^{1-r+s} \mu_{r,s}(fg) \leq \\ &\leq 2A^{1-r+s} (\mu_{r,s}(f) + \mu_{r,s}(g)) (1 + \mu_{1,s}(f) + \mu_{1,s}(g))^r + \\ &\quad + F(M_{r-1,s}(f) + M_{r-1,s}(g)) \leq \\ &\leq C_1^r A^{1-r+s} (\mu_{r,s}(f) + \mu_{r,s}(g)) + F_1(M_{r-1,s}(f) + M_{r-1,s}(g)) \end{aligned} \tag{6}$$

as  $\mu_{1,s}(f)$  and  $\mu_{1,s}(g)$  are bounded. Then from (4) and (6) we obtain

$$\mu_{r,s}(g_k) \leq Q_1^k A^{1-r+s+k} (\mu_{r,s}(f) + \mu_{r,s}(g)) + F_2(M_{r-1,s}(f) + M_{r-1,s}(g)), \tag{7}$$

where  $Q_1 > 2^{r^2}$  is a constant depending on  $r$  and independent of  $A$ . Similarly, from (5) and (6) there exists a constant  $C_2 > 1$  independent of  $r$  and  $A$  such that

$$\mu_{r,s}(g_k) \leq C_2^{rk} A^{1-r+s+k} (\mu_{r,s}(f) + \mu_{r,s}(g)) + G_1(M_{r-1,s}(f) + M_{r-1,s}(g)). \tag{8}$$

We fix  $r_0 \geq s + k + 2$  and  $A_0 \geq 1$  so large that

$$Q_1^k A_0^{1-r_0+s+k} \leq \frac{1}{4} \quad \text{and} \quad C_2^{ik} A_0^{1-i+s+k} \leq \frac{1}{4}$$

for every  $i > r_0$ . It suffices to take  $A_0 \geq \max\{4Q_1^k, 4C_2^{k(s+k+2)}\}$ .

**Lemma 3.2.** *Let  $f \in D_K^{\infty,s}(n, k)$ . There exists a sequence  $\{\varepsilon_i\}_{i \geq r_0}$  of positive constants depending on  $f$  such that for every  $i \geq r_0$  there is  $\mu_{i,s}(g_k) \leq \varepsilon_i$  and  $\mu_{i,s}(f) \leq \varepsilon_i$  whenever  $\mu_{j,s}(g) \leq \varepsilon_j$ ,  $j = r_0, \dots, i - 1$ .*

*Proof.* First, let us take  $\varepsilon_{r_0} > 0$  such that for every  $f \in D_K^{\infty,s}(n, k)$  with  $\mu_{r_0,s}(f) \leq \varepsilon_{r_0}$  there is  $f \in V_A$ . Now, by taking  $\varepsilon_{r_0} > 0$  sufficiently small, from (7) we get

$$\mu_{r_0,s}(g_k) \leq \frac{1}{2} \varepsilon_{r_0} + F_2(M_{r_0-1,s}(f) + M_{r_0-1,s}(g)) \leq \varepsilon_{r_0}$$

for every  $f, g \in D_K^{\infty,s}(n, k)$  and  $\mu_{r_0,s}(f) \leq \varepsilon_{r_0}$ ,  $\mu_{r_0,s}(g) \leq \varepsilon_{r_0}$ .

Next, we will use (8). Assume that  $\mu_{j,s}(g) \leq \varepsilon_j$ ,  $j = r_0, \dots, i - 1$ . There exists a constant  $a_i > 0$  depending on  $f, i, \varepsilon_{r_0}, \dots, \varepsilon_{i-1}$  such that  $G_1(M_{i-1,s}(f) + M_{i-1,s}(g)) \leq a_i$ . We take  $\varepsilon_i = \mu_{i,s}(f) + 2a_i$ . Then from (8) there follows

$$\begin{aligned} \mu_{i,s}(g_k) &\leq C_2^{ik} A^{1-i+s+k} (\mu_{i,s}(f) + \mu_{i,s}(g)) + G_1(M_{i-1,s}(f) + M_{i-1,s}(g)) \leq \\ &\leq \frac{1}{4} (2\mu_{i,s}(f) + 2a_i) + a_i < \varepsilon_i \end{aligned}$$

for every  $g \in D_K^{\infty,s}(n, k)$  with  $\mu_{i,s}(g) \leq \varepsilon_i$ . □

We fix  $f \in D_K^{\infty,s}(n, k)$  such that  $\mu_{r_0,s}(f) \leq \varepsilon_{r_0}$ , where  $\{\varepsilon_i\}_{i \geq r_0}$  is the sequence from Lemma 3.2. Let us denote

$$L_f = \{h \in D_K^{\infty,s}(n, k) : \mu_{i,s}(h) \leq \varepsilon_i, i \geq r_0\}.$$

We will show that the mapping

$$L_f \ni g \mapsto g_k \in L_f$$

has a fixed point. It follows from the following

**Lemma 3.3.** *The set  $L_f$  equipped with  $C^{\infty,s}$ -topology has the fixed-point property, i.e. every continuous mapping  $L_f \rightarrow L_f$  has a fixed point.*

*Proof.* We take

$$L'_f = \{h \in C_K^{\infty,s}(n, k) : \|h\|_{i,s} \leq \varepsilon_i, i \geq r_0\} \subset C_K^{\infty,s}(n, k)$$

equipped with the topology induced from  $\{\|\cdot\|_{i,s}\}_{i \in \mathbb{N}}$ . Then the mapping

$$L_f \ni h \mapsto h - \text{Id} \in L'_f$$

is a homeomorphism for sufficiently small  $\varepsilon_{r_0}$ .

We define operators

$$S_i : (C_K^{\infty}(n, k), \{\|\cdot\|_{i,s}\}_{i \in \mathbb{N}}) \ni h \mapsto D^{i,s}h \in (C_K^0(\mathbb{R}^n, L^i(\mathbb{R}^n, \mathbb{R}^n)), \|\cdot\|_{\text{sup}})$$

for  $i \geq 1$ . Then we have

$$\|S_i h\|_{\text{sup}} = \sup_{x \in \mathbb{R}^n} \|D^{i,s}h(x)\| = \|h\|_{i,s}, \quad (9)$$

so  $S_i$  is continuous. Next, for every  $h \in L'_f$  we obtain

$$\|S_i h(x) - S_i h(y)\| = \|D^{i,s}h(x) - D^{i,s}h(y)\| \leq \|h\|_{i+1,s} \|x - y\| \leq \varepsilon_{i+1} \|x - y\|.$$

Hence  $S_i(L'_f)$  is equicontinuous. See also that it is bounded. For  $i \geq r_0$  it follows from (9) and for  $i \geq r_0$  from Lemma 2.6. By virtue of Ascoli-Arzelà's theorem, the set  $S_i(L'_f)$  is relatively compact in  $(C_K^0(\mathbb{R}^n, L^i(\mathbb{R}^n, \mathbb{R}^n)), \|\cdot\|_{\text{sup}})$ .

Now we can show that every sequence  $\{h_i\}_{i \in \mathbb{N}}$  in  $L'_f$  has a subsequence satisfying the Cauchy condition in  $C_K^{\infty,s}(n, k)$  with respect to  $\|\cdot\|_{j,s}$  for every  $j \in \mathbb{N}$ . It means that  $\{h_i\}_{i \in \mathbb{N}}$  is relatively compact in  $L'_f$ .

Summing up,  $L'_f$  and  $L_f$  are compact. The set  $L_f$  is a convex subset of a Fréchet space so from Schauder-Tychonoff's theorem every continuous mapping  $L_f \rightarrow L_f$  has a fixed point.  $\square$

By Lemma 3.3 there exists  $g \in L_f$  such that  $g_k = g$ . Then we get

$$[fg] = [g_0] = [g_k] = [g] \in H_1(D^{\infty,s}(n, k)).$$

Hence  $[f] = [\text{Id}] \in H_1(D^{\infty,s}(n, k))$ . But the set

$$\{f \in D_K^{\infty,s}(n, k) : \mu_{r_0,s}(f) \leq \varepsilon_{r_0}\}$$

generates the space  $D_K^{\infty,s}(n, k)$  so condition (3) is valid and the group  $D^{\infty,s}(n, k)$  is perfect.



#### 4. CONSTRUCTION OF ROLLING-UP OPERATORS

Now we will give a proof of Lemma 3.1. Notice that mappings in the construction act along leaves.

First, let  $A \geq 1$ . We choose  $\tilde{\chi}_A \in C^\infty(\mathbb{R}, [0, 1])$  such that  $\tilde{\chi}_A = 1$  on  $[-2A, 2A]$  and  $\text{supp}(\tilde{\chi}_A) = [-2A - 1, 2A + 1]$ . Then we define  $\chi_A(x) = \tilde{\chi}_A(x_1) \dots \tilde{\chi}_A(x_k)$ ,  $x \in \mathbb{R}^n$ . It is obvious that  $\chi_A \in C^\infty(\mathbb{R}^n, [0, 1])$ ,  $\chi_A = 1$  on  $[-2A, 2A]^k \times \mathbb{R}^{n-k}$  and  $\text{supp}(\chi_A) = [-2A - 1, 2A + 1]^k \times \mathbb{R}^{n-k}$ .

Let us take  $1 \leq i \leq k$ . We will use the following notions and objects:

- $S^1 \cong \mathbb{R}/\mathbb{Z}$  is a unit circle,
- $B_i = \mathbb{R}^{i-1} \times S^1 \times \mathbb{R}^{n-i}$ ,
- $\pi_i : \mathbb{R}^n \rightarrow B_i$  is the covering projection,
- $S^1$ -action on  $B_i$  given by

$$S^1 \times B_i \ni \beta \cdot (\theta_1, \dots, \theta_i, \dots, \theta_n) \mapsto (\theta_1, \dots, \beta + \theta_i, \dots, \theta_n) \in B_i,$$

- $\tau_{i,A} = \varphi_i^{\chi_A \partial_i} \in D^\infty(\mathbb{R}^n)$ ,
- $T_i = \varphi_1^{\partial_i}$  is the unit translation in the direction of the  $i$ -th coordinate.

Here  $\partial_i$  denotes the unit vector field on  $\mathbb{R}^n$  in the direction of the  $i$ -th coordinate and  $\varphi_i^X$  is the flow of a vector field  $X$ .

The covering projection  $\pi_i$  gives us a system of coordinates in a neighborhood of any point of  $B_i$  compatible with the foliation  $\mathcal{F} = \{\mathbb{R}^{i-1} \times S^1 \times \mathbb{R}^{k-i} \times \{\text{pt}\}\}$ . Therefore, the seminorms introduced in Section 2 also make sense on  $B_i$ . We define groups  $D^{r,s}(B_i, k)$  and  $D_K^{r,s}(B_i, k)$ ,  $1 \leq r \leq \infty$ , as before. Now we may introduce the group of equivariant diffeomorphisms on  $B_i$

$$G_i^{r,s} = \{f \in D^{r,s}(B_i, \mathcal{F}) : f(\beta \cdot \theta) = \beta \cdot f(\theta) \forall \theta \in B_i \forall \beta \in S^1\}.$$

The proof of Lemma 3.1 consists of construction of several mappings. First, let  $f \in D_{K_0}^{1,s}(n, k)$  with  $\mu_{1,s}(f) \leq \frac{1}{2}$ . For  $\theta \in B_i$  there exists  $x \in \mathbb{R}^n$  such that  $\pi_i(x) = \theta$  and  $x_i < -2A$ . We choose  $N \in \mathbb{N}$  such that  $((T_i f)^N(x))_i > 2A$ . Here  $x_i$  denotes the  $i$ -th coordinate of  $x$ . Then we define  $\Gamma_{i,A}(f) : B_i \rightarrow B_i$ ,

$$\Gamma_{i,A}(f)(\theta) = \pi_i((T_i f)^N(x)).$$

It is obvious that  $\Gamma_{i,A}(f)$  does not depend on the choice of  $x$  and  $N$ .

**Lemma 4.1.** *There exists a neighbourhood  $U$  of Id in  $D^{1,s}(n, k)$  such that:*

1.  $\Gamma_{i,A}$  preserves Id.
2. The mapping

$$\Gamma_{i,A} : U \cap D_{K_{i-1}}^{r,s}(n, k) \rightarrow D_{K_i}^{r,s}(B_i, k)$$

*is continuous with respect to  $C^{r,s}$ -topology.*

3. There exists an admissible polynomial  $F$  depending on  $r$  and  $A$  such that

$$\mu_{1,s}(\Gamma_{i,A}(f)) \leq 11A\mu_{1,s}(f)(1 + \mu_{1,s}(f))^{11A}$$

and

$$\mu_{r,s}(\Gamma_{i,A}(f)) \leq 11A\mu_{r,s}(f)(1 + \mu_{1,s}(f))^{11Ar} + F(M_{r-1,s}(f))$$

for every  $f \in U \cap D_{K_{i-1}}^{r,s}(n, k)$ .

*Proof.* We take  $N \in \mathbb{N}$  such that  $8A + 1 < N < 8A + 3$ . See that  $\mu_{r,s}(T_i f) = \mu_{r,s}(f)$  for every  $r \geq 1$ . Hence from Lemma 2.8 we obtain

$$\begin{aligned} \mu_{1,s}(\Gamma_{i,A}(f)) &= \mu_{1,s}((T_i f)^N) \leq N\mu_{1,s}(f)(1 + \mu_{1,s}(f))^{N-1} \leq \\ &\leq 11A\mu_{1,s}(f)(1 + \mu_{1,s}(f))^{11A}. \end{aligned}$$

Analogically, using Lemma 2.8 we get

$$\begin{aligned} \mu_{r,s}(\Gamma_{i,A}(f)) &\leq N\mu_{r,s}(f)(1 + \mu_{1,s}(f))^{r(N-1)} + F(M_{r-1,s}(f)) \leq \\ &\leq 11A\mu_{r,s}(f)(1 + \mu_{1,s}(f))^{11Ar} + F(M_{r-1,s}(f)). \end{aligned}$$

□

Now the following holds

**Lemma 4.2.** *There exists a neighbourhood  $U'_A$  of  $\text{Id} \in D^{1,s}(n, k)$  such that for every  $f, g \in U'_A \cap D_{K_0}^{r,s}(n, k)$  with  $\Gamma_{i,A}(f)\Gamma_{i,A}(g)^{-1} \in G_i^{r,s}$  the mapping  $\tau_{i,A}f$  is conjugated with  $\tau_{i,A}g$  in  $D^{r,s}(n, k)$ .*

The proof proceeds as in [6].

Let  $U_A = U'_A$  be as in Lemma 4.2. For  $f \in U_A$  we define  $h \in C^{1,s}(B_i, k)$ ,

$$h(\theta_1, \dots, \theta_i, \dots, \theta_n) = \theta_i \cdot \Gamma_{i,A}(f)(\theta_1, \dots, 0, \dots, \theta_n).$$

By shrinking  $U_A$  we may assume that  $h \in D_{K_i}^{1,s}(n, k)$  whenever  $f \in U_A \cap D_{K_{i-1}}^{1,s}(n, k)$ . It is obvious that  $h = \Gamma_{i,A}(f)$  on  $\{\theta \in B_i : \theta_i = 0\}$  and  $h \in G_i^{r,s}$  for  $f \in U_A \cap D_{K_{i-1}}^{r,s}(n, k)$ .

Let  $g = h^{-1}\Gamma_{i,A}(f) \in D_{K_i}^{1,s}(n, k)$ . Simple computation yields

$$\mu_{r,s}(h) \leq \mu_{r,s}(\Gamma_{i,A}(f)) \tag{10}$$

for every  $r \geq 1$ . We shrink  $U_A$  so that  $\mu_{1,s}(\Gamma_{i,A}(f)) \leq \frac{1}{2}$  and  $\mu_{1,s}(f) \leq \frac{1}{A}$  for every  $f \in U_A$ . Then from Lemmas 2.8, 4.1 and from (10) we obtain

$$\begin{aligned} \mu_{1,s}(g) &\leq 2(\mu_{1,s}(h^{-1}) + \mu_{1,s}(\Gamma_{i,A}(f)))(1 + \mu_{1,s}(h^{-1}) + \mu_{1,s}(\Gamma_{i,A}(f))) \leq \\ &\leq 15 \cdot 11A\mu_{1,s}(f)(1 + \mu_{1,s}(f))^{11A} \leq CA\mu_{1,s}(f), \end{aligned} \tag{11}$$

where  $C > 0$  is a constant independent of  $A$ . Here we use the fact that  $(1 + \frac{1}{A})^A$  is bounded.

Analogically

$$\begin{aligned} \mu_{r,s}(g) &\leq 2(\mu_{r,s}(h^{-1}) + \mu_{r,s}(\Gamma_{i,A}(f)))(1 + \mu_{1,s}(h^{-1}) + \mu_{1,s}(\Gamma_{i,A}(f)))^r + \\ &\quad + F(M_{r-1,s}(h^{-1}) + M_{r-1,s}(\Gamma_{i,A}(f))) \leq \\ &\leq C^r A\mu_{r,s}(f)(1 + \mu_{1,s}(f))^{11Ar} + F(M_{r-1,s}(f)) \leq \\ &\leq C^r A\mu_{r,s}(f) + F(M_{r-1,s}(f)). \end{aligned} \tag{12}$$

We can lift  $g$  to  $\tilde{g} \in D_{K_i}^{1,s}(n, k)$  such that  $g\pi_i = \pi_i\tilde{g}$  and  $\tilde{g} = \text{Id}$  on  $\{x \in \mathbb{R}^n : x_i \in \mathbb{Z}\}$ . Then  $\mu_{r,s}(\tilde{g}) = \mu_{r,s}(g)$ .

We fix  $\xi \in C^\infty(\mathbb{R}, [0, 1])$  of period 1 such that  $\xi = 0$  near  $m$  and  $\xi = 1$  near  $m + \frac{1}{2}$  for  $m \in \mathbb{Z}$ . Next, we define

$$\tilde{g}_1 = (\xi \circ \text{pr}_i) \cdot (\tilde{g} - \text{Id}) + \text{Id}, \quad \tilde{g}_2 = \tilde{g}_1^{-1}\tilde{g},$$

where  $\text{pr}_i$  is the projection onto the  $i$ -th coordinate. We may shrink  $U_A$  so that  $\tilde{g}_1, \tilde{g}_2 \in D_{K_i}^{1,s}(n, k)$ .

There exists a constant  $Q > 2^{r^2}$  depending on  $r$  and independent of  $A$  such that

$$\begin{aligned} \mu_{r,s}(\tilde{g}_1) &\leq \sup_{x \in \mathbb{R}^n} \|D^{r,s}((\xi \circ \text{pr}_i) \cdot (\tilde{g} - \text{Id}))(x)\| \leq \\ &\leq \sum_{l=0}^r \binom{r}{l} \sup_{x \in \mathbb{R}^n} \|D^{l,s}(\xi \circ \text{pr}_i)(x)\| \|D^{r-l,s}(\tilde{g} - \text{Id})(x)\| \leq \\ &\leq \mu_{r,s}(\tilde{g}) + \sum_{l=1}^{r-1} \binom{r}{l} \|\xi \circ \text{pr}_i\|_{l,s} \mu_{r-l,s}(\tilde{g}) + \|\xi \circ \text{pr}_i\|_{r,s} \mu_{0,s}(\tilde{g}) \leq \\ &\leq \mu_{r,s}(\tilde{g}) + Q M_{r-1,s}(\tilde{g}) \end{aligned}$$

since  $\mu_{0,s}(\tilde{g}) \leq \mu_{1,s}(\tilde{g})$ . Now from (11) and (12) we get two parallel estimations

$$\mu_{r,s}(\tilde{g}_1) \leq C^r A \mu_r(f) + G(M_{r-1,s}(f))$$

and

$$\mu_{r,s}(\tilde{g}_1) \leq Q^r A \mu_r(f) + F(M_{r-1,s}(f)).$$

To obtain the second one we also use Lemma 2.6. Notice that we may obtain analogical inequalities for  $\tilde{g}_2$ .

Now we take

$$E_1 = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}, \quad E_2 = \left\{x \in \mathbb{R}^n : -\frac{3}{2} \leq x_i \leq -\frac{1}{2}\right\},$$

and we define operator  $\Psi_{i,A} : U_A \rightarrow D_K^{1,s}(n, k)$  as follows

$$\Psi_{i,A}(f)(x) = \begin{cases} \tilde{g}_1(x) & \text{for } x \in E_1, \\ \tilde{g}_2(x) & \text{for } x \in E_2, \\ x & \text{for } x \in \mathbb{R}^n \setminus (E_1 \cup E_2). \end{cases}$$

There exist  $g_1, g_2 \in D_{K_i}^{1,s}(B_i, k)$  such that  $g_j\pi_i = \pi_i\tilde{g}_j$ ,  $j = 1, 2$ . Then  $\widetilde{g_1 g_2} = \tilde{g}_1 \tilde{g}_2 = \tilde{g}$  and from the definition of  $\Psi_{i,A}(f)$  we derive

$$\Gamma_{i,A}(\Psi_{i,A}(f))(\theta) = \pi_i((T_i \Psi_{i,A}(f))^N(x)) = g_1 g_2 = g.$$

Therefore

$$\Gamma_{i,A}(f)(\Gamma_{i,A}(\Psi_{i,A}(f)))^{-1} = \Gamma_{i,A}(f)g^{-1} = h \in G_i^{r,s}$$

for every  $f \in U_A \cap D_{K_{i-1}}^{r,s}(n, k)$ . By using Lemma 4.2 we obtain (3) in Lemma 3.1. See also that

$$\mu_{r,s}(\Psi_{i,A}(f)) = \max\{\mu_{r,s}(\tilde{g}_1), \mu_{r,s}(\tilde{g}_2)\}$$

which finishes the proof of Lemma 3.1.

#### REFERENCES

- [1] V.I. Arnold, *Small denominators. I, Mappings of the circumference onto itself*, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961) = Amer. Math. Soc. Translations ser 2. **46** (1965), 213–284.
- [2] D.B.A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. **22** (1970), 165–173.
- [3] D.B.A. Epstein, *Commutators of  $C^\infty$ -diffeomorphisms. Appendix to 'A curious remark concerning the geometric transfer map' by John N. Mather*, Comment. Math. Helv. **59** (1984), 111–122.
- [4] M.R. Herman, *Sur le groupe des difféomorphismes du tore*, Ann. Inst. Fourier, Grenoble **23** (1973) 2, 75–86.
- [5] J. Lech, T. Rybicki, *Groups of  $C^{r,s}$ -diffeomorphisms related to a foliation*, Geometry and Topology of Manifolds, Banach Center Publications, **76** (2007), 437–450.
- [6] J.N. Mather, *Commutators of diffeomorphisms*, Comment. Math. Helv. **49** (1974), 512–528.
- [7] J.N. Mather, *Commutators of diffeomorphisms, II*, Comment. Math. Helv. **50** (1975), 33–40.
- [8] J.N. Mather, *A curious remark concerning the geometric transfer map*, Comment. Math. Helv. **59** (1984), 86–110.
- [9] J.N. Mather, *Commutators of diffeomorphisms, III: a group which is not perfect*, Comment. Math. Helv. **60** (1985), 122–124.
- [10] T. Rybicki, *The identity component of the leaf preserving diffeomorphism group is perfect*, Monatsh. Math. **120** (1995), 289–305.
- [11] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

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