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In Memory of Andrzej Lasota

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THE MOTION PLANNING PROBLEM AND EXPONENTIAL STABILIZATION OF A HEAVY CHAIN. PART II

Abstract. This is the second part of paper [8], where a model of a heavy chain system with a punctual load (tip mass) in the form of a system of partial differential equations was interpreted as an abstract semigroup system and then analysed on a Hilbert state space.

In particular, in [8] we have formulated the problem of exponential stabilizability of a heavy chain in a given position. It was also shown that the exponential stability can be achieved by applying a stabilizer of the colocated-type. The proof used the method of Lyapunov functionals.

In the present paper, we give other two proofs of the exponential stability, which provides an additional intrinsic insight into the exponential stabilizability mechanism. The first proof makes use of some spectral properties of the system. In the second proof, we employ some relationships between exponential stability and exact observability.

Keywords: infinite–dimensional control systems, semigroups, motion planning problem, exponential stabilization, spectral methods, Riesz bases, exact observability.

Mathematics Subject Classification: Primary 93B, 47D; Secondary 35A, 34G.

1. INTRODUCTION

Let $\mathrm{H} := \mathbb{R} \oplus \mathrm{H}^1_L(0,L) \oplus \mathrm{L}^2(0,L)$, where

$$\mathrm{H}^1_L(0,L) := \left\{\Phi \in \mathrm{H}^1(0,L) : \ \Phi(L) = 0\right\}$$

is a closed subspace of the Sobolev space $\mathrm{H}^1(0,L)$. We endow H with the *energetic* scalar product, which is equivalent to the natural scalar product of H,

$$\left\langle \begin{bmatrix} v \\ \phi \\ \psi \end{bmatrix}, \begin{bmatrix} V \\ \Phi \\ \Psi \end{bmatrix} \right\rangle = \mu v V + \int_0^L g(\xi+\mu)\phi'(\xi)\Phi'(\xi)\mathrm{d}\xi + \int_0^L \psi(\xi)\Psi(\xi)\mathrm{d}\xi.$$

The semigroup model on H of the controlled open–loop heavy chain system, considered in [8], was as follows

$$\left\{\begin{array}{rcl} \dot{X}(t) & = & \mathcal{A}\left[X(t)+d\dot{u}(t)\right] \\ y(t) & = & h^*X(t)+u(t) \end{array}\right\},$$

with a linear unbounded state operator $\mathcal{A}$

$$\mathcal{A}\begin{bmatrix} v \\ \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} g\Phi'(0) \\ \Psi \\ \left[g(\cdot+\mu)\Phi'(\cdot)\right]' \end{bmatrix},$$

$$D(\mathcal{A}) = \left\{\begin{bmatrix} v \\ \Phi \\ \Psi \end{bmatrix} \in \mathrm{H}:\ \Phi\in \mathrm{H}^2(0,L),\ \Psi\in \mathrm{H}^1_L(0,L),\ \Psi(0)=v\right\},$$

a factor control vector $d \in \mathrm{H}\setminus D(\mathcal{A})$ and an observation vector $h\in D(\mathcal{A})$:

$$d = \begin{bmatrix} -1 \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix}, \qquad h = \frac{1}{g}\begin{bmatrix} 0 \\ -\ln(\cdot+\mu)+\ln(L+\mu) \\ \mathbf{0} \end{bmatrix}.$$

Recall that [9, Thorem 2.1 and its proof] $\mathcal{A}$ is invertible with a compact skew–adjoint inverse, whence $\mathcal{A}$ has a compact resolvent and is skew-adjoint too, and has a countable spectrum consisting entirely of purely imaginary single eigenvalues $\lambda_{\pm n} \sim \pm j\frac{n\pi}{\beta-\alpha}$, $n\in\mathbb{N}$. The set of corresponding normalized eigenvectors forms an orthonormal basis (ONB) of H. Consequently, $\mathcal{A}$ generates a unitary group $\{S(t)\}_{t\in\mathbb{R}}$ on H.

To stabilize the chain position, a negative feedback control law of the colocated-type has been proposed in [8]:

$$\dot{u}(t) = -kd^{\#}X - \kappa u(t), \qquad k>0, \quad \kappa>0$$

where

$$d^{\#}X = g(L+\mu)\Phi'(L), \quad D(d^{\#}) = \{X\in\mathrm{H} : \Phi' \quad \text{is continuous at } \theta = L\}.$$

Recall that

$$d\in D(d^{\#}), \quad d^{\#}d = 0, \qquad d^{\#}\big|_{D(\mathcal{A})} = -d^*\mathcal{A} = d^*\mathcal{A}^*.$$

The closed-loop feedback control system analysis requires examination of its subsystem corresponding to $\kappa = 0$ [8, Formula (4.5)]

$$\dot{X} = \mathcal{A}\left[X - kdd^{\#}X\right] := \mathcal{A}_c X$$

and

$$X \in D(\mathcal{A}_c) \iff \left[X \in D(d^{\#}),\ X - kdd^{\#}X \in D(\mathcal{A})\right].$$

From [8, Theorem 4.1] we know that the closed-loop system operator $\mathcal{A}_c$ generates a C_0-semigroup of contractions on H. Actually, this semigroup is *exponentially stable* (**EXS**) as shown in [8, Theorem 4.2] using the method of Lyapunov functionals.

This result is simple but it does not provide an explanation of the exponential stabilization mechanism. The aim of this paper is to provide two other proofs which offer an intrinsic insight into this mechanism.

The first proof makes use of some spectral methods. It consists in showing that the generalized eigenspaces of $\mathcal{A}_c$ form an unconditional (Riesz) basis in the state space H, the fact which can be derived from the criterion given in [17] and [18]. Then **EXS** follows because the spectral mapping property holds for $\mathcal{A}_c$.

Our second proof makes use of some relationships between **EXS** and the exact observability. Such relationships where more or less known in the literature; however, they where explicitly established in a clear time-domain setting in [14]. We revised the scheme therein, getting some simplifications and generalizations. Moreover, we propose to add to that scheme some frequency-domain tools in order to make them easier to apply. The scheme in [14], in a modified form, is as follows. We start with proving that under some standard conditions there is an equivalence between **EXS** and the finite-time exact observability of the closed-loop feedback system, and this equivalence even holds for a wide class of non-colocated stabilizers – Theorem 4.1. Our next result, Theorem 4.2, says that, roughly speaking the finite-time exact observability of the closed-loop feedback system follows from the finite-time exact observability of the open-loop feedback system, provided that the input-output system operator is bounded on some $L^2(0,T)$-type space. Thus, certain frequency-domain tools may readily be applied to verify the assumptions of Theorem 4.1 effectively. In particular, the Ingham inequality may be used to verify whether the finite-time exact observability of the open-loop feedback system holds, while the Paley-Wiener theory, modulo a simple trick, is useful in getting the desired boundedness of the input-output operator.

The results are illustrated with a heavy chain system, and they confirm the **EXS** established in [8] using the Lyapunov-Datko theorem.

2. SOME AUXILIARY RESULTS

Lemma 2.1. *The closed-loop state operator $\mathcal{A}_c$ is invertible with the inverse $\mathcal{A}_c^{-1} \in$ $\mathbf{L}(H)$,*

$$\mathcal{A}_c^{-1} = \mathcal{A}^{-1} - kdd^{*}. \tag{2.1}$$

The operator $\mathcal{A}_c^{-1}$ is compact.

Proof. Invertibility of $\mathcal{A}_c^{-1}$ holds iff the equation

$$\mathrm{H} \ni y = \mathcal{A}_c X = \mathcal{A}\left[X - kdd^{\#}X\right] \tag{2.2}$$

is uniquely solvable with respect to $X \in D(\mathcal{A}_c)$. To find such a solution, we apply the operator $\mathcal{A}^{-1}$ to the both sides, getting

$$X - kdd^{\#}X = \mathcal{A}^{-1}y.$$

A solution of the last equation exists only if

$$d^{\#}X = d^{\#}\mathcal{A}^{-1}y = -d^*\mathcal{A}^{-1}\mathcal{A}y = -d^*y.$$

Now $X = \mathcal{A}^{-1}y - kdd^*y$ is the unique solution of (2.2), defining the inverse operator (2.1). Since $\mathcal{A}^{-1}$ is compact, $\mathcal{A}_c^{-1}$ is compact too, as those operators differ by a self-adjoint rank one operator. □

A consequence of Lemma 2.1 is that $\mathcal{A}_c$ may only have a point spectrum (eigenvalues). To determine the latter we need the following *spectral observability/controllability*-type result.

Lemma 2.2. *Let $\{e_k\}_{k\in\mathbb{Z}}$ be the ONB of eigenvectors of $\mathcal{A}$, corresponding to its (purely imaginary) eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$. Then*

$$\forall k \in \mathbb{Z} \qquad d^{\#}e_k \neq 0. \tag{2.3}$$

Proof. For a proof by contradiction, suppose that there exists $k\in\mathbb{Z}$ such that $d^{\#}e_k=0$. Then $\Phi'(L) = 0$, where Φ is the second component of e_k. Since e_k solves the boundary-value eigenproblem for $\mathcal{A}$:

$$\left\{\begin{array}{rcl} g(\xi+\mu)\Phi''(\xi) + g\Phi'(\xi) - \lambda^2\Phi(\xi) &=& 0 \\ \Phi(L) &=& 0 \\ g\Phi'(0) &=& \lambda^2\Phi(0) \end{array}\right\} \tag{2.4}$$

with $\lambda = \lambda_k$, it also solves

$$\left\{\begin{array}{rcl} g(\xi+\mu)\Phi''(\xi) + g\Phi'(\xi) - \lambda^2\Phi(\xi) &=& 0 \\ \Phi(L) &=& 0 \\ \Phi'(L) &=& 0 \end{array}\right\}. \tag{2.5}$$

A general solution of the first equation in (2.4) or (2.5) is [8, Appendix A]

$$\Phi(\xi) = C_1 I_0\left(2\lambda\sqrt{\frac{\xi+\mu}{g}}\right) + C_2 K_0\left(2\lambda\sqrt{\frac{\xi+\mu}{g}}\right) \tag{2.6}$$

where I_n stands for the *modified n-th order Bessel function of the first kind* and K_n denotes the *modified n-th order Bessel function of the second kind*, $n \in \{0\}\cup\mathbb{N}$. Since $I_0'(z) = I_1(z)$ [9, 6.496.9] and $K_0'(z) = -K_1(z)$ [9, 6.496.18] then substituting (2.6) into the boundary conditions of (2.5), we obtain the linear homogeneous system

$$\begin{bmatrix} I_0(\lambda\beta) & K_0(\lambda\beta) \\ -I_1(\lambda\beta) & K_1(\lambda\beta) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \beta := 2\sqrt{\frac{L+\mu}{g}}.$$

Its determinant is ruled by the Wronskian of $\{I_0, K_0\}$ [9, 6.487.2],

$$W(\{I_0, K_0\}) = I_0(z)K_1(z) + I_1(z)K_0(z) = z^{-1},$$

and thus it is nonzero. Hence $e_k = 0$ which contradicts $\|e_k\| = 1$. □

Lemma 2.2 is an important result which enables us to establish some further facts.

Lemma 2.3. *All eigenvalues of $\mathcal{A}_c$, if they exist, are in $\mathbb{C}^- := \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$.*

Proof. Indeed, if $f \in D(\mathcal{A}_c)$ is a normalized eigenvector of $\mathcal{A}_c$ corresponding to an eigenvalue λ, then by (complexified) *dissipative inequality* [8, inequality (4.4)]

$$\langle \mathcal{A}_c X, X \rangle + \langle X, \mathcal{A}_c X \rangle = -2k[d^\# X]^2 \leq 0 \qquad \forall X \in D(\mathcal{A}_c) \tag{2.7}$$

there holds

$$\operatorname{Re} \lambda = -k \left| d^\# f \right|^2 \leq 0,$$

so it remains to exclude $\operatorname{Re} \lambda = 0$. This may hold iff $d^\# f = 0$. But then f satisfies

$$\lambda f = \mathcal{A}_c f = \mathcal{A}\left[f - k d d^\# f\right] = \mathcal{A} f,$$

whence the pair (λ, f) solves the eigenproblem for the open-loop state operator $\mathcal{A}$. Since all eigenvalues of $\mathcal{A}$ are single, there exists $k \in \mathbb{Z}$ such that $f = \eta e_k$, for some scaling multiplier $\eta \in \mathbb{C}$. Consequently, $d^\# e_k = 0$, which contradicts (2.3) in Lemma 2.2. □

Our next step will be to examine the *auxiliary system transfer function*, defined for $s \in \mathbb{C}$ being not an eigenvalue of $\mathcal{A}$, as

$$\hat{G}(s) := s d^\# (sI - \mathcal{A})^{-1} d = -s^2 d^* (sI - \mathcal{A})^{-1} d + s \|d\|^2 \; (= d^\# \mathcal{A}(sI - \mathcal{A})^{-1} d). \tag{2.8}$$

To find a detailed form of $\hat{G}$, we observe that, by the definition of $d^\#$ and (2.8),

$$\hat{G}(s) = s g (L + \mu) \left[\text{second component of } (sI - \mathcal{A})^{-1} d\right]' \Big|_{\theta = L}.$$

The second component of $(sI - \mathcal{A})^{-1} d$ coincides with the (unique) solution to the boundary-value problem

$$\left\{ \begin{array}{rcl} g(\xi + \mu)\Phi''(\xi) + g\Phi'(\xi) - s^2 \Phi(\xi) & \equiv & 1 \\ \Phi(L) & = & 0 \\ g\Phi'(0) - s^2 \Phi(0) & = & 1 \end{array} \right\}.$$

Its solution is [8, Appendix B]

$$\Phi(\xi) = \frac{1}{s^2}\left[-1 + \frac{K_2(s\alpha)}{\Delta(s)} I_0\left(2s\sqrt{\frac{\xi + \mu}{g}}\right) - \frac{I_2(s\alpha)}{\Delta(s)} K_0\left(2s\sqrt{\frac{\xi + \mu}{g}}\right)\right], \; \alpha := 2\sqrt{\frac{\mu}{g}}, \tag{2.9}$$

where $\Delta(s) := I_0(s\beta)K_2(s\alpha) - K_0(s\beta)I_2(s\alpha) = 0$ stands for the *open-loop system characteristic function.* Since $I_0'(z) = I_1(z)$ [9, 6.496.9] and $K_0'(z) = -K_1(z)$ [9, 6.496.18], then

$$\Phi'(L) = \frac{2}{sg\beta\Delta(s)}\left[I_1(s\beta)K_2(s\alpha) + I_2(s\alpha)K_1(s\beta)\right],$$

and consequently

$$\hat{G}(s) = \underbrace{\frac{\beta g}{2}}_{=\sqrt{g(L+\mu)}} \frac{I_1(s\beta)K_2(s\alpha) + I_2(s\alpha)K_1(s\beta)}{I_0(s\beta)K_2(s\alpha) - I_2(s\alpha)K_0(s\beta)}. \tag{2.10}$$

Recall the concept of a Herglotz-Nevanlinna function [6, Section 2 and Appendix A], [3][1)].

Definition 2.1. *A function $f : \mathbb{C} \to \mathbb{C}$ is called a* Herglotz-Nevanlinna function *if: (a) f is analytic on $\mathbb{C}\backslash\mathbb{R}$, (b) f is symmetric with respect to real axis, i.e., $\overline{f(z)} = f(\overline{z})$ for $z \in \mathbb{C} \setminus \mathbb{R}$ and (c) $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$.*

Lemma 2.4. *The function $f(z) := j\hat{G}(-jz)$ is a Herglotz-Nevanlinna function[2)]. Moreover, for large $|s|$, $s \in \mathbb{C}$, there holds*

$$\hat{G}(s) \sim \frac{\beta g}{2} \coth[s(\beta - \alpha)] = \sqrt{g(L+\mu)} \coth[s(\beta - \alpha)]. \tag{2.11}$$

Proof. Recall that $s = -jz$ ($\Leftrightarrow z = js$) maps bijectively the upper open complex half-plane onto $\mathbb{C}^+ := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. Condition (a) of Definition 2.1 holds because $\hat{G}$ is defined outside the (purely point) spectrum of $\mathcal{A}$ entirely located on $j\mathbb{R}$.

By (2.8), for $s \notin j\mathbb{R}$, there is

$$\begin{aligned}\overline{\hat{G}(-\overline{s})} &= \overline{-\overline{s}d^{\#}(-\overline{s}I - \mathcal{A})^{-1}d} = \overline{\overline{s}d^{*}\mathcal{A}(-\overline{s}I - \mathcal{A})^{-1}d} = sd^{*}\mathcal{A}^{*}(-sI - \mathcal{A}^{*})^{-1}d = \\ &= -sd^{\#}(sI - \mathcal{A})^{-1}d = -\hat{G}(s),\end{aligned}$$

whence, with $s = -jz$, we get

$$\overline{f(z)} = \overline{j\hat{G}(-jz)} = -j\overline{\hat{G}(-jz)} = j\hat{G}(-j\overline{z}) = f(\overline{z}), \qquad z \notin \mathbb{R},$$

and (b) of Definition 2.1 is satisfied.

Observe that

$$j\omega_k\langle d, e_k\rangle = \langle d, -j\omega_k e_k\rangle = \langle d, \mathcal{A}^{*}e_k\rangle = \overline{\langle \mathcal{A}^{*}e_k, d\rangle} = -\overline{d^{*}\mathcal{A}e_k} = \overline{d^{\#}e_k}.$$

1) The first paper brings a self-contained collection of basic facts on scalar Herglotz functions beautifully compiled from many monographs and illustrated with numerous examples, while the second is recommendable out of providing certain links between Herglotz functions and control theory.

2) Such a function $\hat{G}$ was traditionally called a *lossless impedance/transfer function.* This concept has been introduced in connection with Brune-Cauer-Darlington-Foster synthesis known in the circuit theory.

Hence, a consequence of the expansion

$$\hat{G}(s) = sd^{\#}(sI - \mathcal{A})^{-1}d = \sum_{k\in\mathbb{Z}} \frac{s}{s - j\omega_k}\langle d, e_k\rangle d^{\#}e_k =$$
$$= \sum_{k\in\mathbb{Z}} \frac{s}{(s - j\omega_k)j\omega_k} \left|d^{\#}e_k\right|^2, \qquad s \neq j\omega_k, \quad k \in \mathbb{Z},$$

where $\{e_k\}_{k\in\mathbb{Z}}$ denotes ONB of eigenvectors of $\mathcal{A}^*$, corresponding to its purely imaginary single eigenvalues $\{j\omega_k\}_{k\in\mathbb{Z}}$, or equivalently zeroes of $\Delta(s)$, is that for $s \in \mathbb{C}^+$:

$$\mathrm{Re}[\hat{G}(s)] = \sum_{k\in\mathbb{Z}} \mathrm{Re}\Big[\frac{s}{(s - j\omega_k)j\omega_k}\Big] \left|d^{\#}e_k\right|^2 = \mathrm{Re}\, s \sum_{k\in\mathbb{Z}} \frac{\left|d^{\#}e_k\right|^2}{\mathrm{Re}^2\, s + (\omega_k - \mathrm{Im}\, s)^2} \geq 0.$$

Since $\mathrm{Im}\, f(z) = \mathrm{Re}\, \hat{G}(s)$, then (c) of Definition 2.1 holds. In addition, by Lemma 2.1 all residua of $\hat{G}$ are positive:

$$\operatorname*{Res}_{s=j\omega_k} \hat{G}(s) = \left|d^{\#}e_k\right|^2 > 0. \tag{2.12}$$

For any $n = 0, 1, \ldots$, the following asymptotic formulae hold for large $|z|$ [9, the simplest forms of 6.641.5 and 6.641.6 with $k = 0$] or in [5, p. 63, pp. 104–105 and 440–441]:

$$I_n(z) \sim \frac{e^z + (-1)^n\, j\ \mathrm{sign}(\mathrm{Im}\, z)\ e^{-z}}{\sqrt{2z\pi}}, \qquad |\arg z| \leq \frac{3\pi}{2} \tag{2.13}$$

and

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad |\arg z| \leq \frac{3\pi}{2}. \tag{2.14}$$

Taking (2.14) and (2.13) into account in (2.10), we easily conclude that (2.11) holds true. □

An alternative proof of (2.11) is given in Appendix A. The fact (2.14) is more or less standard, while a proof of (2.13), which is a less standard result, is given in Appendix B.

The spectrum of $\mathcal{A}_c$ is now fully characterized by knowledge of $\hat{G}$.

Lemma 2.5. *All eigenvalues of $\mathcal{A}_c$ satisfy the characteristic equation*

$$1 + k\hat{G}(s) = 0.$$

Proof. By Lemma 2.3, without loss of generality, we may assume that $\mathrm{Re}\, s < 0$, which enables us to apply the resolvent of $\mathcal{A}$ to the both sides of the eigenproblem for $\mathcal{A}_c$

$$sX = \mathcal{A}_c X = \mathcal{A}\left[X - kdd^{\#}X\right],$$

getting

$$X + k\mathcal{A}(sI - \mathcal{A})^{-1}dd^{\#}X = 0.$$

This equation may have a solution if

$$d^{\#}X\left[1+kd^{\#}\mathcal{A}(sI-\mathcal{A})^{-1}d\right]=\underbrace{d^{\#}X}_{\neq 0}\left[1+k\hat{G}(s)\right]=0.$$

Here $d^{\#}X\neq 0$, for if not, then X would be an eigenvector of $\mathcal{A}$ too, which would contradict (2.3). □

An important conclusion from Lemmas 2.5 and 2.4 is that asymptotic eigenvalues of the closed-loop operator $\mathcal{A}_c$ satisfy the equation

$$1+k\frac{\beta g}{2}\coth[s(\beta-\alpha)]=1+k\frac{\beta g}{2}\frac{e^{s(\beta-\alpha)}+e^{-s(\beta-\alpha)}}{e^{s(\beta-\alpha)}-e^{-s(\beta-\alpha)}}=0\iff e^{2s(\beta-\alpha)}=\frac{2-k\beta g}{2+k\beta g},$$

whence they are:

$$s_n=\left\{\begin{array}{ll}\dfrac{1}{\beta-\alpha}\ln\sqrt{\dfrac{2-k\beta g}{2+k\beta g}}+j\dfrac{n\pi}{\beta-\alpha}, & \text{if}\quad k<\dfrac{2}{\beta g}\\ \dfrac{1}{\beta-\alpha}\ln\sqrt{\dfrac{k\beta g-2}{2+k\beta g}}+j\dfrac{n\pi+\frac{\pi}{2}}{\beta-\alpha}, & \text{if}\quad k>\dfrac{2}{\beta g}\end{array}\right\}. \tag{2.15}$$

Notice that in the case of $k=\dfrac{2}{\beta g}=\dfrac{1}{\sqrt{g(L+\mu)}}$ there are no asymptotic eigenvalues[3)].

Lemma 2.6. *For any $s\notin[\sigma(\mathcal{A})\cup\sigma(\mathcal{A}_c)]$, the resolvent of $\mathcal{A}_c$ may be represented as:*

$$(\lambda I-\mathcal{A}_c)^{-1}=(\lambda I-\mathcal{A})^{-1}-\frac{k}{1+k\hat{G}(\lambda)}\mathcal{A}(\lambda I-\mathcal{A})^{-1}dd^{\#}(\lambda I-\mathcal{A})^{-1}. \tag{2.16}$$

Proof. Applying the resolvent of $\mathcal{A}$ to the both sides of the equation

$$sX-\mathcal{A}\left[X-kdd^{\#}X\right]=Y\in\mathrm{H}$$

one gets

$$X=(sI-\mathcal{A})^{-1}Y-k\mathcal{A}(sI-\mathcal{A})^{-1}dd^{\#}X. \tag{2.17}$$

It is clear that if the last equation has a solution, it belongs to $D(d^{\#})$, whence by Lemma 2.5, a solution must satisfy

$$d^{\#}X=\frac{1}{1+k\hat{G}(s)}d^{\#}(sI-\mathcal{A})^{-1}Y.$$

Eliminating $d^{\#}X$ from (2.17), we obtain (2.16). □

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3) It was established numerically, using the *principle of argument*, that the exemplary heavy chain system described in [8] has then a finite number of eigenvalues, whence the closed system cannot be *superstable* in the sense of [2].

We are going to prove an important result characterizing the asymptotic behaviour of $(sI-\mathcal{A}_c)^{-1}$ along the negative real semi-axis. For that we need to know the adjoint of $\mathcal{A}_c$.

Lemma 2.7. *The adjoint of closed-loop feedback operator is*

$$\mathcal{A}_c^* Y := -\mathcal{A}\left[Y + kdd^{\#}Y\right], \quad Y \in D(\mathcal{A}_c^*) \iff \left[Y \in D(d^{\#}),\ Y + kdd^{\#}Y \in D(\mathcal{A})\right] \tag{2.18}$$

and it satisfies the dissipative inequality

$$\langle \mathcal{A}_c^* Y, Y\rangle + \langle Y, \mathcal{A}_c^* Y\rangle = -2k[d^{\#}Y]^2 \le 0 \qquad \forall\, Y \in D(\mathcal{A}_c^*). \tag{2.19}$$

Proof. Let $X \in D(\mathcal{A}_c)$. Then

$$\begin{aligned}\langle Y, \mathcal{A}_c X\rangle &= \left\langle Y, \mathcal{A}\left[X - kdd^{\#}X\right]\right\rangle = \\ &= \underbrace{\left\langle Y + kdd^{\#}Y, \mathcal{A}\left[X - kdd^{\#}X\right]\right\rangle}_{①} + \underbrace{kd^{\#}Y\left\langle -d, \mathcal{A}\left[X - kdd^{\#}X\right]\right\rangle}_{②},\end{aligned}$$

provided that $Y \in D(d^{\#})$. Since

$$-\left\langle \mathcal{A}\left[X - kdd^{\#}X\right], d\right\rangle = d^{\#}\left[X - kdd^{\#}X\right] = d^{\#}X$$

then $② = kd^{\#}Y d^{\#}X$. Assuming $Y + kdd^{\#}Y \in D(\mathcal{A}^*) = D(\mathcal{A})$ as $\mathcal{A}^* = -\mathcal{A}$, we get

$$\begin{aligned}① &= \left\langle -\mathcal{A}\left[Y + kdd^{\#}Y\right], X - kdd^{\#}X\right\rangle = \\ &= \left\langle -\mathcal{A}\left[Y + kdd^{\#}Y\right], X\right\rangle + \left\langle \mathcal{A}\left[Y + kdd^{\#}Y\right], d\right\rangle kd^{\#}X = \\ &= \left\langle -\mathcal{A}\left[Y + kdd^{\#}Y\right], X\right\rangle - ②.\end{aligned}$$

Hence $① + ② = \langle -\mathcal{A}\left[Y + kdd^{\#}Y\right], X\rangle$ which yields (2.18).

As regards (2.19), observe that for $Y \in D(\mathcal{A}_c^*)$ there holds

$$\begin{aligned}\langle \mathcal{A}_c^* Y, Y\rangle + \langle Y, \mathcal{A}_c^* Y\rangle &= \langle -\mathcal{A}\underbrace{\left[Y + kdd^{\#}Y\right]}_{:=w}, Y\rangle + \langle Y, -\mathcal{A}\left[Y + kdd^{\#}Y\right]\rangle = \\ &= \langle -\mathcal{A}w, w - kdd^{\#}w\rangle + \langle w - kdd^{\#}w, -\mathcal{A}w\rangle = \\ &= kd^{\#}w\langle \mathcal{A}w, d\rangle + kd^{\#}w\langle d, \mathcal{A}w\rangle = -2k[d^{\#}w]^2 = \\ &= -2k[d^{\#}Y]^2 \le 0.\end{aligned} \tag{2.20}$$

□

Theorem 2.1. $|\lambda|\left\|(\lambda I - \mathcal{A}_c)^{-1}\right\|$ *is bounded for any large negative* λ*, provided that* $k \ne \dfrac{1}{\lim\limits_{\lambda\to\infty} \hat{G}(\lambda)}$.

Proof. By (2.7)

$$\|f\|^2 = 2k\int_0^\infty \left|d^{\#}S_c(t)f\right|^2 \mathrm{d}t = 2k\|d^{\#}S_c(\cdot)f\|^2_{\mathrm{L}^2(0,\infty)}, \quad f \in D(\mathcal{A}_c).$$

Since the Laplace transform of $S_c(t)f$ is $(sI-\mathcal{A}_c)^{-1}f$, then, with $\epsilon>0$, the Schwarz inequality yields for each $f\in D(\mathcal{A}_c)$:

$$\begin{aligned}\left|\sqrt{2\epsilon}\, d^{\#}(\epsilon I-\mathcal{A}_c)^{-1}f\right| &= \left|\int_0^\infty \sqrt{2\epsilon}e^{-\epsilon t}\, d^{\#}S_c(t)f\mathrm{d}t\right| = \\ &= \left|\left\langle \sqrt{2\epsilon}e^{-\epsilon(\cdot)}, d^{\#}S_c(\cdot)f\right\rangle_{\mathrm{L}^2(0,\infty)}\right| \le \\ &\le \left\|\sqrt{2\epsilon}e^{-\epsilon(\cdot)}\right\|_{\mathrm{L}^2(0,\infty)} \|d^{\#}S_c(\cdot)f\|_{\mathrm{L}^2(0,\infty)} = \frac{1}{\sqrt{2k}}\|f\|.\end{aligned}$$

This means that the linear functional $d^{\#}(\epsilon I-\mathcal{A}_c)^{-1}$ extends to a linear bounded everywhere defined functional, whose norm satisfies

$$\left\|\sqrt{2\epsilon}\, d^{\#}(\epsilon I-\mathcal{A}_c)^{-1}\right\| \le \frac{1}{\sqrt{2k}}, \quad \epsilon>0. \tag{2.21}$$

Similarly, the linear functional $d^{\#}(\eta I-\mathcal{A}_c^*)^{-1}$ extends to a linear bounded everywhere defined functional, whose norm satisfies

$$\left\|\sqrt{2\eta}\, d^{\#}(\eta I-\mathcal{A}_c^*)^{-1}\right\| \le \frac{1}{\sqrt{2k}}, \quad \eta>0. \tag{2.22}$$

Indeed, by (2.19)

$$\|f\|^2 = 2k\int_0^\infty \left|d^{\#}S_c^*(t)f\right|^2 \mathrm{d}t = 2k\|d^{\#}S_c^*(\cdot)f\|^2_{\mathrm{L}^2(0,\infty)}, \quad f\in D(\mathcal{A}_c^*).$$

Since the Laplace transform of $S_c^*(t)f$ is $(sI-\mathcal{A}_c^*)^{-1}f$, then with $\eta>0$, the Schwarz inequality yields for each $f\in D(\mathcal{A}_c^*)$:

$$\begin{aligned}\left|\sqrt{2\eta}\, d^{\#}(\eta I-\mathcal{A}_c^*)^{-1}f\right| &= \left|\int_0^\infty \sqrt{2\eta}e^{-\eta t}\, d^{\#}S_c^*(t)f\mathrm{d}t\right| = \\ &= \left|\left\langle \sqrt{2\eta}e^{-\eta(\cdot)}, d^{\#}S_c^*(\cdot)f\right\rangle_{\mathrm{L}^2(0,\infty)}\right| \le \\ &\le \left\|\sqrt{2\eta}e^{-\eta(\cdot)}\right\|_{\mathrm{L}^2(0,\infty)} \|d^{\#}S_c^*(\cdot)f\|_{\mathrm{L}^2(0,\infty)} = \frac{1}{\sqrt{2k}}\|f\|,\end{aligned}$$

from which (2.22) follows.

Applying $d^{\#}$ to the both sides of (2.16), we get

$$d^{\#}(\lambda I-\mathcal{A}_c)^{-1} = \frac{1}{1+k\hat{G}(\lambda)}d^{\#}(\lambda I-\mathcal{A})^{-1}, \qquad \lambda\ge 0,$$

whence, by (2.21) and (2.11)[4],

$$\begin{aligned}\left\|\sqrt{2\epsilon}\, d^{\#}(\epsilon I-\mathcal{A})^{-1}\right\| &\le \left\|\sqrt{2\epsilon}\, d^{\#}(\epsilon I-\mathcal{A}_c)^{-1}\right\| \left[1+k\max_{\epsilon\ge 0}\hat{G}(\epsilon)\right] \le \\ &\le \frac{1}{\sqrt{2k}}\left[1+k\max_{\epsilon\ge 0}\hat{G}(\epsilon)\right] < \infty, \qquad \epsilon>0.\end{aligned}$$

[4] Here (2.11) implies the existence of the limit $\lim_{s\to\infty}\hat{G}(s)=\frac{\beta g}{2}$, which means that $\hat{G}(s)$ is *regular* at the point $\{\infty\}\cap\mathbb{R}$, as $\lim_{s\to\infty}\hat{G}^{(n)}(s)=0$, for any $n\in\mathbb{N}$.

From (2.16), it also follows that

$$\lambda(\lambda I-\mathcal{A}_c)^{-1}d=\lambda(\lambda I-\mathcal{A})^{-1}d-\frac{k\hat{G}(\lambda)}{1+k\hat{G}(\lambda)}\mathcal{A}(\lambda I-\mathcal{A})^{-1}d,$$

whence

$$\mathcal{A}_c(\lambda I-\mathcal{A}_c)^{-1}d=\frac{1}{1+k\hat{G}(\lambda)}\mathcal{A}(\lambda I-\mathcal{A})^{-1}d. \tag{2.23}$$

It is clear that

$$\left\|\mathcal{A}_c(\lambda I-\mathcal{A}_c)^{-1}d\right\|=\left\|d^*\mathcal{A}_c^*(\lambda I-\mathcal{A}_c^*)^{-1}\right\|. \tag{2.24}$$

If $f\in \mathrm{H}$, then $(\lambda I-\mathcal{A}_c^*)^{-1}f\in D(\mathcal{A}_c)$ which, by Lemma 2.7, implies

$$(\lambda I-\mathcal{A}_c^*)^{-1}f\in D(d^{\#}),\quad (\lambda I-\mathcal{A}_c^*)^{-1}f+kdd^{\#}(\lambda I-\mathcal{A}_c^*)^{-1}f\in D(\mathcal{A})$$

with

$$-\mathcal{A}\left[(\lambda I-\mathcal{A}_c^*)^{-1}f+kdd^{\#}(\lambda I-\mathcal{A}_c^*)^{-1}f\right]=\mathcal{A}_c^*(\lambda I-\mathcal{A}_c^*)^{-1}f.$$

Applying d^* to the both sides and taking into account that $d^{\#}\big|_{D(\mathcal{A})}=-d^*\mathcal{A}$, we get

$$\begin{aligned}d^*\mathcal{A}_c^*(\lambda I-\mathcal{A}_c^*)^{-1}f&=-d^*\mathcal{A}\left[(\lambda I-\mathcal{A}_c^*)^{-1}f+kdd^{\#}(\lambda I-\mathcal{A}_c^*)^{-1}f\right]=\\&=d^{\#}\left[(\lambda I-\mathcal{A}_c^*)^{-1}f+kdd^{\#}(\lambda I-\mathcal{A}_c^*)^{-1}f\right]=d^{\#}(\lambda I-\mathcal{A}_c^*)^{-1}f.\end{aligned}$$

Owing to this, (2.24) and (2.22), there holds:

$$\left\|\sqrt{2\eta}\mathcal{A}_c(\eta I-\mathcal{A}_c)^{-1}d\right\|=\left\|\sqrt{2\eta}d^{\#}(\eta I-\mathcal{A}_c^*)^{-1}\right\|\le\frac{1}{\sqrt{2k}},\qquad \eta>0.$$

Now first taking the adjoint of (2.23) with $\lambda=\eta>0$ (whence $\hat{G}(\eta)$ is real), then noting that on $D(\mathcal{A}_c)$: $d^{\#}=-d^*\mathcal{A}_c$, and finally applying (2.21) yields the estimate

$$\begin{aligned}\left\|\sqrt{2\eta}\,d^{\#}(\eta I-\mathcal{A}^*)^{-1}\right\|&\le\left\|\sqrt{2\eta}\,d^{\#}(\eta I-\mathcal{A}_c^*)^{-1}\right\|\left[1+k\max_{\eta\ge0}\hat{G}(\eta)\right]\le\\&\le\frac{1}{\sqrt{2k}}\left[1+k\max_{\eta\ge0}\hat{G}(\eta)\right]<\infty,\qquad \eta>0.\end{aligned}$$

The just established estimates, holding for $\epsilon\in(0,\infty)$:

$$\left\|\sqrt{4k\epsilon}\,d^{\#}(\epsilon I-\mathcal{A})^{-1}\right\|,\left\|\sqrt{4k\epsilon}\,d^{\#}(\epsilon I-\mathcal{A}^*)^{-1}\right\|\le 1+k\max_{\epsilon\ge0}\hat{G}(\epsilon), \tag{2.25}$$

will be important in examining the behaviour of $\lambda(\lambda I-\mathcal{A}_c)^{-1}$ for a large negative λ.

Firstly, observe that (2.16) remains valid for a large negative λ, provided that $1-\frac{k\beta g}{2}\ne0$. This is because $\left\|(\lambda I-\mathcal{A})^{-1}\right\|\le 1/|\lambda|$ for any real λ (even with =), the transfer function $\hat{G}$ is, thanks to $\overline{\hat{G}(-\overline{s})}=-\hat{G}(s)$, $s\notin j\mathbb{R}$, an odd function of the real argument, i.e.,

$$\hat{G}(-\lambda)=-\hat{G}(\lambda),\qquad \lambda\in\mathbb{R}, \tag{2.26}$$

and $1+k\hat{G}(\lambda)=1-k\hat{G}(-\lambda)\longrightarrow 1-\frac{k\beta g}{2}\neq 0$ as $\lambda\to-\infty$, by (2.11).

Secondly, as (2.16) is meaningful for a large negative λ, then

$$\begin{aligned}\left\|\lambda(\lambda I-\mathcal{A}_c)^{-1}\right\| &\leq \left\|\lambda(\lambda I-\mathcal{A})^{-1}\right\|+\\ &+\underbrace{\frac{k}{\left|1+k\hat{G}(\lambda)\right|}}_{❶}\underbrace{\left\|\sqrt{-\lambda}\mathcal{A}(\lambda I-\mathcal{A})^{-1}d\right\|}_{❷}\underbrace{\left\|\sqrt{-\lambda}d^{\#}(\lambda I-\mathcal{A})^{-1}\right\|}_{❸}.\end{aligned}$$

Term ❶ is bounded for a large negative λ, because, by (2.11) and (2.26):

$$\frac{k}{|1+k\hat{G}(\lambda)|}=\frac{k}{|1-k\hat{G}(-\lambda)|}\longrightarrow\frac{2k}{2-k\beta g}<\infty \text{ as } \lambda\to-\infty.$$

Next,

$$\begin{aligned}❷ &= \left\|\sqrt{-\lambda}\mathcal{A}(\lambda I-\mathcal{A})^{-1}d\right\|=\left\|\sqrt{-\lambda}d^{*}\mathcal{A}^{*}(\lambda I-\mathcal{A}^{*})^{-1}\right\|=\\ &=\left\|-\sqrt{-\lambda}d^{\#}(\lambda I+\mathcal{A})^{-1}\right\|=\left\|\sqrt{-\lambda}d^{\#}((-\lambda)I-\mathcal{A})^{-1}\right\|\end{aligned}$$

is bounded for $\lambda<0$, by (2.25) with $\epsilon=-\lambda>0$. Similarly

$$\begin{aligned}❸ &= \left\|\sqrt{-\lambda}d^{\#}(\lambda I-\mathcal{A})^{-1}\right\|=\left\|-\sqrt{-\lambda}d^{\#}(-\lambda I+\mathcal{A})^{-1}\right\|=\\ &=\left\|-\sqrt{-\lambda}d^{\#}((-\lambda)I-\mathcal{A}^{*})^{-1}\right\|=\left\|\sqrt{-\lambda}d^{\#}((-\lambda)I-\mathcal{A}^{*})^{-1}\right\|\end{aligned}$$

is bounded for $\lambda<0$, by (2.25) with $\epsilon=-\lambda>0$. □

3. SPECTRAL PROOF OF EXPONENTIAL STABILITY

Our basic tool for deducing **EXS** from spectral data will be the results of [17, Theorem 3.4.1, p. 85, Lemma 3.1.1, p. 70 and Theorem 3.5.1, p. 89 with complementary information following from Corollary 3.5.3, p. 91] abbreviated in [18, Theorem 2.3, p. 247 together with Lemma 2.5, p. 248].

Definition 3.1. *Let $G:(D(G)\subset H)\to H$ be a closed densely defined linear operator. A closed linear hull $\mathcal{E}_\lambda(G)$ of all eigenvectors and generalized eigenvectors corresponding to an eigenvalue λ of G is called a generalized eigenspace* [5] *corresponding to λ; $\dim\mathcal{E}_\lambda(G)$ is called the algebraic multiplicity of an eigenvalue λ, while*

$$\nu_\lambda:=\min\{m\in\mathbb{N}:\ (G-\lambda I)^m\mathcal{E}_\lambda(G)=\{0\}\}$$

is the index of λ.

[5] Recall that $\mathcal{E}_\lambda(G)$ equals the range of the *Riesz projector*

$$\frac{1}{2\pi j}\int_{|s-\lambda|<\varepsilon}(sI-G)^{-1}\mathrm{d}s$$

associated with λ, where ε is a sufficiently small positive number.

Theorem 3.1 (Röh). *Let $G : (D(G) \subset H) \longrightarrow H$ be a densely defined maximal dissipative operator on a Hilbert space H with the following properties:*

$$n_0 := \dim\bigl[D(G)/\{x \in D(G) : \mathrm{Re}\langle Gx, x\rangle = 0\}\bigr] < \infty, \tag{3.1}$$

the spectrum $\sigma(G)$ of G is in the open left complex half-plane, i.e.,

$$\sigma(G) \subset \mathbb{C}^-. \tag{3.2}$$

Then G has a compact resolvent. In particular, if $\sigma(G) \neq \emptyset$, then $\sigma(G)$ consists entirely of isolated eigenvalues with finite algebraic multiplicities.

Assume that, in addition, $\sigma(G) = \{s_n\}_{n\in\mathbb{N}}$, i.e., G has countably many eigenvalues[6] *$\{s_n\}_{n\in\mathbb{N}}$ which are enumerated without repetition, satisfy the condition*

$$\nu_{s_n} \le \nu < \infty, \qquad n \in \mathbb{N}, \tag{3.3}$$

where ν_{s_n} denotes the index of the eigenvalue s_n and the Carleson condition, i.e.,

$$\exists\delta > 0 : \quad \forall k \in \mathbb{N} \qquad \prod_{\substack{n=1\\ n\neq k}}^{\infty} \left|\frac{s_k - s_n}{s_k + \overline{s_n}}\right| \ge \delta. \tag{3.4}$$

Then its generalized eigenspaces $\{\mathcal{E}_{s_n}(G)\}_{n\in\mathbb{N}}$ form and unconditional (Riesz) basis in their closed linear hull, i.e., in their closed linear hull each vector x has a unique expansion $x = \sum_{n\in\mathbb{N}} x_n$, $x_n \in \mathcal{E}_{s_n}(G)$, where the series converges unconditionally in H (the union of orthonormal bases of the eigenspaces $\{\mathcal{E}_{s_n}(G)\}_{n\in\mathbb{N}}$ is then an unconditional (Riesz) vector basis of the closed linear hull of the generalized eigenspaces $\{\mathcal{E}_{s_n}(G)\}_{n\in\mathbb{N}}$).

Moreover, if conditions (3.1)–(3.4) *hold, then a necessary and sufficient condition for completeness of generalized eigenspaces $\{\mathcal{E}_{s_n}(G)\}_{n\in\mathbb{N}}$ is the existence of a sequence of points $\{z_n\}_{n\in\mathbb{N}}$ in the resolvent set of G, $\mathrm{Re}\, z_n \longrightarrow -\infty$ as $n \to \infty$, such that:*

$$\bigl\{|\,\mathrm{Re}\, z_n|\ \|(z_nI - G)^{-1}\|\bigr\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}). \tag{3.5}$$

Furthermore, if (3.5) *holds then*[7] *the semigroup $\{T(t)\}_{t\ge 0}$ of contractions, generated by G has the spectral mapping property:*

$$\forall t \ge 0 : \qquad \sigma\left[T(t)\right] = \overline{\exp\left[\sigma(G)t\right]}, \tag{3.6}$$

which, in particular, implies that $\{T(t)\}_{t\ge 0}$ is ***EXS*** *iff* [17, *(1.1.11), p.12*][8]

$$\sup_{n\in\mathbb{N}} \mathrm{Re}\, s_n < 0. \tag{3.7}$$

Remark 3.1. *A result closely related to that part of Theorem 3.1 which concerns the existence of a Riesz basis of generalized eigenspaces in their closed linear hull, but without the completeness, has been known since* [12, *Theorem 2.2*][9]*. However, the*

[6] In [17, p. 79] this assumption appears in text rather than in formulation of a theorem.

[7] In fact, G is a spectral operator [18, Corollary 2.4, p. 248]. An operator whose spectrum consists of isolated eigenvalues is called *spectral* iff its eigenspaces form an unconditional (Riesz) basis of the whole space H.

[8] Actually the semigroup $\{T(t)\}_{t\ge 0}$ possesses the so-called *spectrum determined growth* property.

[9] Or even in the monograph [7, Section 6.6] under some more restrictive assumptions.

last paper appeared in the references neither in [17] *nor in* [18]. *A result related to Theorem 3.1 has been recently established in* [19, *Theorem 2, p. 969*] *where, however, a partial completeness result was derived starting from different arguments, and the Carleson condition was replaced by the requirement that the eigenvectors associated with exponentials* $\{e^{s_n(\cdot)}\}_{n\in\mathbb{N}}$ *form a Riesz basis of* $L^2(0,T)$ *for any* $T > 0$. *The authors included neither* [12], *nor* [17], *nor* [18] *in the references.*

We should apply Theorem 3.1 to $G = \mathcal{A}_c$. $\mathcal{A}_c$ is maximally dissipative, as a generator of a semigroup of contractions. In Röh's terminology, maximally dissipative operators satisfying (3.1) are called dissipative operators of *finite-dimensional damping*, and, in particular, by (2.7) there is $n_0 = 1$, i.e., $\mathcal{A}_c$ is of *one-dimensional damping*. Indeed, her

$$\begin{aligned} n_0 &= \operatorname{codim}\left\{X \in D(\mathcal{A}_c) :\ d^{\#}X = 0\right\} = \operatorname{codim}\left\{x \in D(\mathcal{A}) :\ d^{\#}x = 0\right\} = \\ &= \operatorname{codim}\{z \in \mathrm{H} :\ d^{*}z = 0\} = 1. \end{aligned}$$

Since, by Lemma 2.3, (3.2) holds, then $\mathcal{A}_c$ has a compact resolvent, which agrees with the assertion of our Lemma 2.1, established independently of Theorem 3.1. As we already know, $\sigma(A_c) \neq \emptyset$, whence it consists entirely of isolated eigenvalues with finite algebraic multiplicities.

From (2.15) – the asymptotic formulae for eigenvalues, we deduce that if $k \neq \frac{2}{\beta g}$ then $\mathcal{A}_c$ has countably many eigenvalues located in a bounded strip parallel to $j\mathbb{R}$ and they are *uniformly separated*, i.e.

$$\inf_{n\neq m} |s_n - s_m| > 0,$$

whence, by [17, Lemma 3.3.1, p. 81] or [18, Lemma 2.1, p. 247], they satisfy the Carleson condition (3.4). It also follows from (2.15) that asymptotic eigenvalues are single, whence all eigenvalues satisfy (3.3) and the generalized eigenspaces $\mathcal{E}_{s_n}(\mathcal{A}_c)$ are asymptotically one-dimensional eigenspaces. By Theorem 3.1, the generalized eigenspaces $\{\mathcal{E}_{s_n}(\mathcal{A}_c)\}_{n\in\mathbb{N}}$ form an unconditional (Riesz) basis in their closed linear hull.

Finally, by Theorem 2.1, (3.5) is met for, e.g., $z_n := -n$, and the last assertion of Theorem 3.1 implies that the generalized eigenspaces $\{\mathcal{E}_{s_n}(\mathcal{A}_c)\}_{n\in\mathbb{N}}$ span the whole state space H, whence they form an unconditional (Riesz) basis in H. Owing to this, **EXS** for $k \neq \frac{2}{g\beta}$ may be deduced from spectral mapping property (3.6). This is because (2.15), in conjunction with $\sigma(\mathcal{A}_c) \subset \mathbb{C}^-$ implies (3.7).

4. EXACT OBSERVABILITY APPROACH TO **EXS**

4.1. GENERAL CONSIDERATIONS

The attempt to conclude **EXS** which will be discussed in this section has for years been known for certain particular systems and its origins are associated with such names as: M. Slemrod, D. Russell, A. Haraux and E. Zuazua. Nevertheless, its general scheme has most recently been consolidated and extended by Lasiecka and Triggiani in an "almost trivial" form [14, Claim 2.3, p. 1071 and Proposition 3.1, p. 1074]. A related result, specialized to a class of well-posed linear systems, can be found in Curtain and Weiss [4, Theorem 1.1 with $Q = 0$, p. 274]. Though the arguments of [14, Proposition 3.1, p. 1074] are gathered into three steps, only two of them are in fact essential. They will be formulated below as two theorems in a context more general than the closed-loop system dictated by $\mathcal{A}_c$.

Theorem 4.1. *Let $\mathcal{A}_F : (D(\mathcal{A}_F) \subset H) \longrightarrow H$ be an infinitesimal generator of a linear C_0-semigroup $\{S_F(t)\}_{t\geq 0}$ on a Hilbert space H with a scalar product $\langle\cdot,\cdot\rangle_H$ and $\mathcal{C} \in \mathbf{L}(D(\mathcal{A}_F), Y)$, where Y is a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_Y$. Assume that $\mathcal{H} \in \mathbf{L}(H)$ is a bounded self-adjoint and coercive, i.e., $\mathcal{H} = \mathcal{H}^* \geq \epsilon I$ for some $\epsilon > 0$, and is a solution to the operator Lyapunov equation*

$$\langle \mathcal{A}_F x, \mathcal{H}x\rangle_H + \langle x, \mathcal{H}\mathcal{A}_F x\rangle_H = -\|\mathcal{C}x\|_Y^2, \qquad x \in D(\mathcal{A}_F). \tag{4.1}$$

*Then $\{S_F(t)\}_{t\geq 0}$ is **EXS** iff the pair $(\mathcal{A}_F, \mathcal{C})$ is exactly observable in a finite time $T > 0$, i.e., there exist $T > 0$ and $\gamma > 0$ such that:*

$$\int_0^T \|\mathcal{C}S_F(t)x_0\|_Y^2\, dt \geq \gamma\, \|x_0\|_H^2, \qquad x_0 \in D(\mathcal{A}_F). \tag{4.2}$$

Proof. The scalar product $\langle x_1, x_2\rangle_{\mathrm{e}} := \langle x_1, \mathcal{H}x_2\rangle_{\mathrm{H}} = \langle \mathcal{H}^{1/2}x_1, \mathcal{H}^{1/2}x_2\rangle_{\mathrm{H}}$ and the original one are equivalent. Indeed,

$$\epsilon\,\|x\|_{\mathrm{H}}^2 \leq \langle x, \mathcal{H}x\rangle_{\mathrm{H}} = \left\|\mathcal{H}^{1/2}x\right\|_{\mathrm{H}}^2 = \|x\|_{\mathrm{e}}^2 \leq \|x\|_{\mathrm{H}}^2\, \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\,.$$

Under the linear transformation $z = \mathcal{H}^{1/2}x$, the semigroup operator $S_F(t)$ is similar to $\mathcal{H}^{1/2}S_F(t)\mathcal{H}^{-1/2}$. The semigroup $\{\mathcal{H}^{1/2}S_F(t)\mathcal{H}^{-1/2}\}_{t\geq 0}$ is generated by $\mathcal{H}^{1/2}\mathcal{A}_F\mathcal{H}^{-1/2}$ with domain $\mathcal{H}^{1/2}D(\mathcal{A}_F)$.

Putting $x = S_F(t)x_0$, $x_0 \in D(\mathcal{A}_F)$, in (4.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S_F(t)x_0, \mathcal{H}S_F(t)x_0\rangle_{\mathrm{H}} = \frac{\mathrm{d}}{\mathrm{d}t}\,\|S_F(t)x_0\|_{\mathrm{e}}^2 = -\,\|\mathcal{C}S_F(t)x_0\|_{\mathrm{Y}}^2\,,$$

where the last term is continuous in $t \geq 0$. Integrating both sides from 0 to t, yields

$$\|x_0\|_{\mathrm{e}}^2 - \|S_F(t)x_0\|_{\mathrm{e}}^2 = \int_0^t \|\mathcal{C}S_F(\tau)x_0\|_{\mathrm{Y}}^2\,\mathrm{d}\tau, \qquad x_0 \in D(\mathcal{A}_F), \quad t \geq 0. \tag{4.3}$$

Sufficiency. Assume that (4.2) holds. Then applying (4.3) twice at $t = T$, we obtain

$$\|S_F(T)x_0\|_{\mathrm{e}}^2 \leq \|x_0\|_{\mathrm{e}}^2 \leq \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\,\|x_0\|_{\mathrm{H}}^2 \leq \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\,\frac{1}{\gamma}\int_0^T \|\mathcal{C}S_F(t)x_0\|_{\mathrm{Y}}^2\,\mathrm{d}t =$$
$$= \frac{1}{\gamma}\,\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\left[\|x_0\|_{\mathrm{e}}^2 - \|S_F(T)x_0\|_{\mathrm{e}}^2\right], \qquad x_0 \in D(\mathcal{A}_F),$$

whence

$$\left(1 + \frac{1}{\gamma}\,\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\right)\|S_F(T)x_0\|_{\mathrm{e}}^2 \leq \frac{1}{\gamma}\,\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\,\|x_0\|_{\mathrm{e}}^2\,.$$

Consequently,

$$\forall x_0 \in D(\mathcal{A}_F) \qquad \|S_F(T)x_0\|_{\mathrm{e}}^2 \leq \frac{\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}{\gamma + \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}} \|x_0\|_{\mathrm{e}}^2$$

or, equivalently,

$$\forall x_0 \in D(\mathcal{A}_F) \qquad \left\| \left[\mathcal{H}^{1/2} S_F(T) \mathcal{H}^{-1/2}\right] \mathcal{H}^{1/2} x_0 \right\|_{\mathrm{H}}^2 \leq \frac{\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}{\gamma + \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}} \left\|\mathcal{H}^{1/2} x_0\right\|_{\mathrm{H}}^2 .$$

Since $\overline{D(\mathcal{A}_F)} = \mathrm{H}$ then, using the similarity transformation $z = \mathcal{H}^{1/2}x$, we conclude that $\{\mathcal{H}^{1/2} S_F(T) \mathcal{H}^{-1/2}\}_{t\geq 0}$ satisfies:

$$\left\|\mathcal{H}^{1/2} S_F(T) \mathcal{H}^{-1/2}\right\|_{\mathrm{H}}^2 \leq \frac{\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}{\gamma + \|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}} < 1,$$

which is known to guarantee **EXS** of $\{\mathcal{H}^{1/2} S_F(t) \mathcal{H}^{-1/2}\}_{t\geq 0}$. Hence $\{S_F(t)\}_{t\geq 0}$ is **EXS**.

Necessity. If $\{S_F(t)\}_{t\geq 0}$ is **EXS** then $\{\mathcal{H}^{1/2} S_F(t) \mathcal{H}^{-1/2}\}_{t\geq 0}$ is **EXS** too, so there exist $M \geq 1$ and $\alpha > 0$ such that

$$\begin{aligned} \|S_F(t)x_0\|_{\mathrm{e}} = \left\| \left[\mathcal{H}^{1/2} S_F(t) \mathcal{H}^{-1/2}\right] \mathcal{H}^{1/2} x_0 \right\|_{\mathrm{H}} \leq \\ \leq M e^{-\alpha t} \left\|\mathcal{H}^{1/2} x_0\right\|_{\mathrm{H}}, \qquad x_0 \in \mathrm{H}, \quad t \geq 0 \end{aligned}$$

and, by (4.3),

$$\begin{aligned} \int_0^t \|\mathcal{C} S_F(\tau) x_0\|_{\mathrm{Y}}^2 \,\mathrm{d} = \|x_0\|_{\mathrm{e}}^2 - \|S_F(t)x_0\|_{\mathrm{e}}^2 \geq \\ \geq \left(1 - M^2 e^{-2\alpha t}\right) \|x_0\|_{\mathrm{e}}^2, \qquad x_0 \in D(\mathcal{A}_F), \quad t \geq 0. \end{aligned}$$

Taking an arbitrary $T > \frac{1}{\alpha} \ln M \geq 0$ and $\gamma := (1 - M^2 e^{-2\alpha T})\epsilon > 0$, we come to (4.2). □

Remark 4.1. *This result is inspired by* [10*, Propositions 1 and 2*] *and by* [14*, Step 3 of the proof of Proposition 3.1, p. 1075*].

While using Theorem 4.1 to deduce **EXS**, one has to verify its assumptions with respect to $\mathcal{A}_F$ being the closed-loop feedback system state operator. An important fact, which has been widely discussed in [14], is that the exact observability condition (4.2) may be deduced from its counterpart for an open-loop (uncontrolled) system, provided that the *input-output open system operator* may be extended to an operator which belongs to $\mathbf{L}(\mathrm{L}^2(0,T))$ [14, Claim 2.3, p. 1071].

Passing to the details, let us consider the feedback system depicted in Figure 4.1. We shall prove the following theorem.

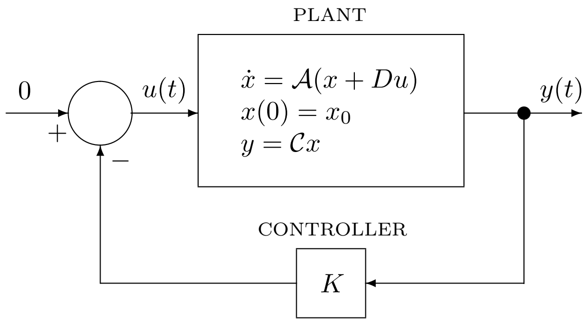

Fig. 4.1. Feedback system structure

Theorem 4.2. *Assume that the open-loop state operator* $\mathcal{A}$ *generates a linear semigroup* $\{S(t)\}_{t\geq 0}$ *on a Hilbert space* H *with a scalar product* $\langle\cdot,\cdot\rangle_H$ *and*

$$\forall s \in \mathbb{C}^+ \cup \{0\} \quad (sI - \mathcal{A})^{-1} \in \mathbf{L}(H). \tag{4.4}$$

Let $\mathcal{C} \in \mathbf{L}(D(\mathcal{A}), Y)$*, where* Y *is a Hilbert space with a scalar product* $\langle\cdot,\cdot\rangle_Y$ *and the pair* $(\mathcal{A}, \mathcal{C})$ *is exactly observable in a finite time, i.e., there exist* $T > 0$ *and* $\theta > 0$ *such that:*

$$\forall x_0 \in D(\mathcal{A}) \quad \int_0^T \|\mathcal{C}S(t)x_0\|_Y^2\, dt \geq \theta\, \|x_0\|_H^2. \tag{4.5}$$

The factor control operator $D \in \mathbf{L}(U, H)$ *satisfies* $\mathcal{R}(D) \subset D(\mathcal{C})$*,* $\mathcal{C}D \in \mathbf{L}(U, Y)$ *and* $K \in \mathbf{L}(Y, U)$*, where* U *is a Hilbert space* U *with a scalar product* $\langle\cdot,\cdot\rangle_U$*. Let the feedback system state operator*

$$\mathcal{A}_F x := \mathcal{A}\,(x - DK\mathcal{C}x)\,, \quad D(\mathcal{A}_F) \iff [x \in D(\mathcal{C}),\ (x - DK\mathcal{C}x) \in D(\mathcal{A})]$$

be such that $(\mathcal{A}_F, \mathcal{C})$ *satisfies all assumptions of Theorem 4.1 except for* (4.2)*, which is not assumed to hold. If there exists* $\lambda > 0$ *such that*

$$\hat{G}(\lambda + \cdot) \in H^\infty(\mathbb{C}^+, \mathbf{L}(U, Y)) \tag{4.6}$$

then (4.2) *holds.*

Proof. By (4.1), the semigroup $\{S_F(t)\}_{t\geq 0}$ is uniformly bounded and therefore $(sI - \mathcal{A}_F)^{-1} \in \mathbf{L}(\mathrm{H})$ for any $s \in \mathbb{C}^+$. Moreover, for any $z \in \mathrm{H}$: $(sI - \mathcal{A}_F)^{-1}z \in D(\mathcal{A}_F)$ is the unique solution of the resolvent equation

$$sx - \mathcal{A}_F x = sx - \mathcal{A}\,(x - DK\mathcal{C}x) = z.$$

By (4.4),

$$s(sI - \mathcal{A})^{-1}x - \mathcal{A}(sI - \mathcal{A})^{-1}\,(x - DK\mathcal{C}x) = x + \mathcal{A}(sI - \mathcal{A})^{-1}DK\mathcal{C}x = (sI - \mathcal{A})^{-1}z,$$

whence, in particular for $x_0 \in D(\mathcal{A})$,

$$\left\{I + \left[s\mathcal{C}(sI - \mathcal{A})^{-1}D - \mathcal{C}D\right]K\right\}\mathcal{C}(sI - \mathcal{A}_F)^{-1}x_0 \equiv \mathcal{C}(sI - \mathcal{A})^{-1}x_0 \quad \text{on} \quad \mathbb{C}^+. \tag{4.7}$$

The RHS of (4.7) is the Laplace transform of $\mathcal{C}S(\cdot)x_0 \in \mathrm{C}([0,\infty),\mathrm{Y}) \cap \mathrm{L}^2(0,T;\mathrm{Y})$. Next, $\mathcal{C}(sI-\mathcal{A}_F)^{-1}x_0$ is the Laplace transform of $\mathcal{C}S_F(\cdot)x_0 \in \mathrm{C}([0,\infty),\mathrm{Y}) \cap \mathrm{L}^2(0,\infty;\mathrm{Y})$, whilst the open system transfer function $\mathbb{C}^+ \ni s \longmapsto \hat{G}(s) \in \mathbf{L}(\mathrm{U},\mathrm{Y})$,

$$\hat{G}(s) := s\mathcal{C}(sI-\mathcal{A})^{-1}D - \mathcal{C}D = s^2 \underbrace{(\mathcal{C}\mathcal{A}^{-1})}_{\in \mathbf{L}(\mathrm{H},\mathrm{Y})}(sI-\mathcal{A})^{-1}D - s(\mathcal{C}\mathcal{A}^{-1}) - \underbrace{\mathcal{C}D}_{=\hat{G}(0)}$$

grows polynomially in $|s|$ for large $\operatorname{Re} s$. By Schwartz's theorem, it is the Laplace transform of a (Laplace transformable) distribution - the open system impulse response, $G(\cdot)$ with support in $[0,\infty)$ taking operator values in $\mathbf{L}(\mathrm{U},\mathrm{Y})$.

To get (4.2) from (4.5), we represent (4.7) in the time-domain as

$$\int_0^t \left\{ e^{-\lambda(t-\tau)} \left[I + G(t-\tau)K \right] \right\} \left[e^{-\lambda\tau}\mathcal{C}S_F(\tau)x_0 \right] \mathrm{d}\tau = e^{-\lambda t}\mathcal{C}S(t)x_0. \tag{4.8}$$

Consider now the convolution operator $\mathbb{F}_\lambda u := \left\{ e^{-\lambda(\cdot)} \left[I + G(\cdot)K \right] \right\} \star u$ induced by the LHS of (4.8). If there exists $\lambda > 0$ such that (4.6) holds, then, by a vector version of the Paley-Wiener theorem,

$$\mathbb{F}_\lambda \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y})), \qquad \|\mathbb{F}_\lambda\| = \left\| \hat{G}(\lambda+\cdot) \right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}. \tag{4.9}$$

By the *causality* of $\mathbb{F}_\lambda$ and some obvious inequalities, there follows

$$\begin{aligned} \|\mathbb{F}_\lambda u\|_{\mathrm{L}^2(0,T;\mathrm{Y})} &= \|(\mathbb{F}_\lambda u)_T\|_{\mathrm{L}^2(0,\infty;\mathrm{Y})} = \|(\mathbb{F}_\lambda u_T)_T\|_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \le \|\mathbb{F}_\lambda u_T\|_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \le \\ &\le \|\mathbb{F}_\lambda\| \, \|u_T\|_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \|\mathbb{F}_\lambda\| \, \|u\|_{\mathrm{L}^2(0,T;\mathrm{U})} \qquad \forall u \in \mathrm{L}^2(0,\infty;\mathrm{U}). \end{aligned}$$

In particular, for $u = e^{-\lambda(\cdot)}\mathcal{C}S_F(\cdot)x_0 \in \mathrm{L}^2(0,\infty;\mathrm{U})$, we obtain

$$\begin{aligned} e^{-\lambda T} \|\mathcal{C}S(\cdot)x_0\|_{\mathrm{L}^2(0,T;\mathrm{Y})} &\le \left\| e^{-\lambda(\cdot)}\mathcal{C}S(\cdot)x_0 \right\|_{\mathrm{L}^2(0,T;\mathrm{Y})} \underbrace{=}_{(4.8)} \\ &\underbrace{=}_{(4.8)} \left\| \mathbb{F}_\lambda \left[e^{-\lambda(\cdot)}\mathcal{C}S_F(\cdot)x_0 \right] \right\|_{\mathrm{L}^2(0,T;\mathrm{Y})} \le \\ &\le \|\mathbb{F}_\lambda\| \left\| e^{-\lambda(\cdot)}\mathcal{C}S_F(\cdot)x_0 \right\|_{\mathrm{L}^2(0,T;\mathrm{U})} \le \\ &\le \|\mathbb{F}_\lambda\| \, \|\mathcal{C}S_F(\cdot)x_0\|_{\mathrm{L}^2(0,T;\mathrm{U})}, \end{aligned}$$

whence

$$\theta \|\mathbb{F}_\lambda\|^{-2} e^{-2\lambda T} \|x_0\|_{\mathrm{H}}^2 \underbrace{\le}_{(4.5)} \|\mathbb{F}_\lambda\|^{-2} e^{-2\lambda T} \|\mathcal{C}S(\cdot)x_0\|_{\mathrm{L}^2(0,T;\mathrm{Y})}^2 \le \|\mathcal{C}S_F(\cdot)x_0\|_{\mathrm{L}^2(0,T;\mathrm{U})}^2,$$

i.e., (4.2) holds with the same $T > 0$ and $\gamma := \theta \|\mathbb{F}_\lambda\|^{-2} e^{-2\lambda T}$. □

The finite-time exact observability of the open-loop system, i.e., the condition (4.5), can often be deduced either from time-domain considerations or from a theorem on *Ingham's inequality* which we recall below in its original formulation [11] or [20, p. 162][10).

Theorem 4.3. *Assume that the strictly increasing sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ of real numbers satisfies the gap condition $\lambda_{n+1} - \lambda_n \geq \delta$ for all $n \in \mathbb{Z}$ and some $\delta > 0$. Then, for all $T > 2\pi/\delta$:*

$$\frac{2T}{\pi}\left(1 - \frac{4\pi^2}{T^2\delta^2}\right)\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \int_0^T \Big|\sum_{n=-\infty}^{\infty} a_n e^{jt\lambda_n}\Big|^2 dt \leq \frac{8T}{\pi}\left(1 + \frac{4\pi^2}{T^2\delta^2}\right)\sum_{n=-\infty}^{\infty} |a_n|^2 \quad (4.10)$$

for every complex sequence $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Remark 4.2. *When Theorem 4.3 is applicable, a stronger form of* (4.5) *is usually obtained:*

$$\exists T > 0,\ \Theta,\ \theta > 0 :\ \forall x_0 \in D(\mathcal{A}) \quad \Theta \|x_0\|_H^2 \geq \int_0^T \|\mathcal{C}S(t)x_0\|_Y^2\, dt \geq \theta \|x_0\|_H^2. \quad (4.11)$$

In this case, (4.7) *may be considered even for an arbitrary x_0, because the RHS of* (4.7) *is then the Laplace transform of an extended open system finite-time observability map, and $x_0 \mapsto \mathcal{C}(sI - \mathcal{A}_F)^{-1}x_0$ corresponds to the extended closed system infinite-time observability map.*

4.2. IMPLICATIONS FOR A HEAVY CHAIN SYSTEM

Theorem 4.3 enables us to get both finite-time admissibility and exact observability of the pair $(\mathcal{A}, d^\#)$ (here $\mathcal{C} = d^\#$) associated with the uncontrolled system. Recall that the open system state operator $\mathcal{A}$ has a system of eigenvectors $\{e_n\}_{n\in\mathbb{Z}}$, corresponding to its (purely imaginary) eigenvalues $\{j\omega_n\}_{n\in\mathbb{Z}}$, which forms an ONB of H. Thus, for every $x_0 \in D(\mathcal{A})$

$$\mathcal{C}S(t)x_0 = d^\# S(t)x_0 = \sum_{n=-\infty}^{\infty} e^{jt\omega_n}\langle x_0, e_n\rangle_{\mathrm{H}} d^\# e_n, \qquad t \geq 0.$$

This suggests taking $\lambda_n = \omega_n$ and $a_n = \langle x_0, e_n\rangle_{\mathrm{H}} d^\# e_n$, $n \in \mathbb{Z}$. Since $\omega_n \sim \frac{n\pi}{\beta-\alpha}$ and each of the eigenvalues $j\omega_n$ is single, then there exists $\delta > 0$ for which the gap condition of Theorem 4.3 is fulfilled.

10) See also `http://planetmath.org/encyclopedia/ProofOfInghamInequality.html` for a very similar proof of the lower bound in (4.10).

Using (2.11), we can strengthen (2.12) to get[11)]

$$0 < \left|d^{\#}e_n\right|^2 \sim \operatorname*{Res}_{s=\pm j\frac{n\pi}{\beta-\alpha}} \frac{\beta g}{2} \coth[s(\beta-\alpha)] = \frac{\beta g}{2(\beta-\alpha)}.$$

By Parseval's identity, $\{\langle x_0, e_n\rangle_{\mathrm{H}}\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and therefore $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. From Theorem 4.3 we conclude that there exists a large enough T, greater than $\frac{2\pi}{\beta-\alpha}$, for which Ingham's inequality (4.10) holds. Consequently, we get the following form of (4.11)

$$\frac{2T}{\pi}\left(1-\frac{4\pi^2}{T^2\delta^2}\right)\inf_{n\in\mathbb{Z}}\left|d^{\#}e_n\right|^2\|x_0\|_{\mathrm{H}}^2 \le \int_0^T \left|d^{\#}S(t)x_0\right|^2 \mathrm{d}t \le$$
$$\le \frac{8T}{\pi}\left(1+\frac{4\pi^2}{T^2\delta^2}\right)\sup_{n\in\mathbb{Z}}\left|d^{\#}e_n\right|^2\|x_0\|_{\mathrm{H}}^2, \qquad \forall x_0 \in D(\mathcal{A}),$$

where the first inequality establishes the finite-time admissibility of $(\mathcal{A}, d^{\#})$ which is here of lesser importance, while the second inequality gives the desired finite-time exact observability of $(\mathcal{A}, d^{\#})$.

(4.4) is clearly satisfied. The assumption of Theorem 4.1, needed for Theorem 4.2 to be useful, holds true for $\mathcal{H} = I$. To show that **EXS** follows from Theorems 4.2 and 4.1, it remains to prove that the open system finite-time exact observability implies the closed system finite-time exact observability, which is the case if (4.9) holds. But here, (4.6) is satisfied thanks to (2.11), because the Nyquist plot of $\hat{G}(\lambda+\cdot)$ asymptotically coincides with the Nyquist plot of $\frac{\beta g}{2}\coth\left[(\beta-\alpha)(\lambda+\cdot)\right]$, where the last one is an ellipse, provided that $\lambda > 0$. This is confirmed by direct plotting of those two curves, as documented by Figures 4.2, 4.3.

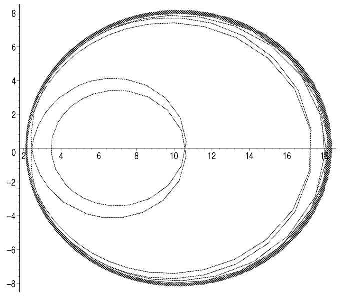

Fig. 4.2. Nyquist plot of $s \longmapsto \hat{G}(s+1)$

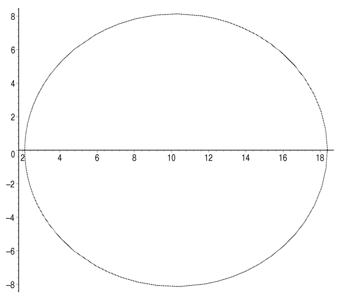

Fig. 4.3. Nyquist plot of $s \longmapsto \frac{\beta g}{2}\coth[(\beta-\alpha)(s+1)]$

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11) As expected, this agrees with the well-known *Mittag-Leffler expansion* - see, e.g. [13, Solution of Problem 5.2.5]:

$$\frac{\beta g}{2}\coth\left[(\beta-\alpha)s\right] = \frac{\beta g}{2(\beta-\alpha)}\left[\frac{1}{s}+\sum_{n\in\mathbb{Z}}\frac{s}{s^2+\left(\frac{n\pi}{\beta-\alpha}\right)^2}\right] \sim \frac{\beta g}{2(\beta-\alpha)}\sum_{n\in\mathbb{Z}}\frac{s}{s^2+\left(\frac{n\pi}{\beta-\alpha}\right)^2}.$$

5. DISCUSSION AND CONCLUSIONS

Two proofs of **EXS** of the heavy chain system, an analysis of which was initiated in [8], have been given in the present paper as alternatives to the proof given in [8, Theorem 4.2] and based on the method of Lyapunov functionals. The two new proofs are considerably more complicated than that of [8]; however, they offer an elucidation of intrinsic mechanism of exponential stabilization by the colocated feedback control law. Let us list some detailed items.

— A crucial point of the spectral method of proving **EXS** is the spectral observability/controllability results of Lemma 2.2, which asserts that (2.3) decides whether an eigenvalue of an open-loop system located on $j\mathbb{R}$ is being shifted or not, by the feedback action, to $\mathbb{C}^-$. Here (2.3) holds due to the linear independence of the Bessel functions I_0 and K_0. The method of obtaining **EXS** based on the exact observability approach required a strengthened form of (2.3), as shown is Section 4.2.

— By the result of [1] or [15], it follows from Lemmas 2.1 and 2.3 that the semigroup generated by $\mathcal{A}_c$ is strongly asymptotically stable. Both our methods show that in order to get **EXS** one has to examine certain properties of the transfer function $\hat{G}$ in details. In particular, our both new proofs involve a sharp asymptotic expression of $\hat{G}$ for large $|s|$. Obtaining such an expression is not a trivial task and we had to use an advanced asymptotic expression for the Bessel function I_n in order to derive it. The spectral method of obtaining **EXS** involves, in addition, a rather precise investigation of $(\lambda I - \mathcal{A}_c)^{-1}$ on the real axis.

— An advantage of the spectral method in relation to other approaches is that it provides additional valuable information on the feedback system properties, e.g., the existence of Riesz basis of generalized eigenspaces. Its drawback is that we did not conclude **EXS** for $k = 2/(g\beta)$, which is not the case for other proofs. Our former proof using the method of Lyapunov functionals is the simplest one but, on the other hand, its drawback is that a general rule of constructing a quadratic Lyapunov functional which yields **EXS** remains unknown. Our new proofs are more algorithmic, but they are decidedly less simple and therefore less attractive for the control engineering community. Finally, observe that the proof via the exact observability approach is mostly related to the classical control theory, though it uses less elementary mathematical apparatus, which may be difficult to understand for control engineers.

Closing our discussion, let us mention that a yet another proof of **EXS** has been drawn from the Gearhardt-Prüss-Huang criterion [16, Corollary 4, p. 853], but out of complicated (computer-aided) calculations involved, we decided to omit it.

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A. AN ALTERNATIVE PROOF OF (2.11)

Recall that each Herglotz-Nevanlinna function has the unique representation

$$f(z) = a + bz + \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] \mathrm{d}[\sigma(t)], \tag{A.1}$$

where $a = \mathrm{Re}[f(j)]$, $b = \lim_{\lambda \nearrow \infty} \dfrac{f(j\lambda)}{j\lambda} \geq 0$ and $\sigma(t)$ is a measure satisfying

$$\int_{\mathbb{R}} \frac{1}{1+t^2} \, \mathrm{d}[\sigma(t)] < \infty.$$

Substituting $-jf(js) = \hat{G}(s)$ we can convert (A.1) into a representation for $\hat{G}$:

$$\hat{G}(s) = -aj + bs - j \int_{\mathbb{R}} \left[\frac{1}{t-js} - \frac{t}{1+t^2} \right] \mathrm{d}[\sigma(t)],$$

where now $a = \mathrm{Re}[j\hat{G}(1)] = 0$, because $\hat{G}(1) \in \mathbb{R}$ and

$$b = \lim_{\lambda \nearrow \infty} \frac{j\hat{G}(\lambda)}{j\lambda} = \lim_{\lambda \nearrow \infty} \frac{\hat{G}(\lambda)}{\lambda} \underbrace{=}_{(2.8)} \lim_{\lambda \nearrow \infty} d^{\#}(\lambda I - \mathcal{A})^{-1} d = -d^{*} \lim_{\lambda \nearrow \infty} \mathcal{A}(\lambda I - \mathcal{A})^{-1} d = 0,$$

by a well-known property of a C_0-semigroup generator. Thus the *purely atomic* measure $\sigma(t) = \sum_{n \in \mathbb{Z}} \{\omega_n\}$, where $j\omega_n = \lambda_n$ is an eigenvalue of $\mathcal{A}$, enables us to recover $\hat{G}$ from its poles (spectral data). Using the asymptotic formula for eigenvalues of $\mathcal{A}$ and the corresponding purely atomic measure $\sigma_{\mathrm{as}}(t) = \sum_{n \in \mathbb{Z}} \left\{ \frac{n\pi}{\beta - \alpha} \right\}$, one recovers an asymptotic form of $\hat{G}$, which asymptotically coincides with the Mittag-Leffler expansion of $\frac{\beta g}{2} \coth[s(\beta - \alpha)]$ we have already recalled in Section 4.2,

$$\hat{G}(s) = -j \int_{\mathbb{R}} \frac{1+jst}{(t-js)(1+t^2)} \, \mathrm{d}[\sigma(t)] \sim -j \int_{\mathbb{R}} \frac{1+jst}{(t-js)(1+t^2)} \, \mathrm{d}[\sigma_{\mathrm{as}}(t)] =$$

$$= -j \sum_{n=1}^{\infty} \left[\frac{1+js\omega_n}{\omega_n - js} + \frac{1-js\omega_n}{-\omega_n - js} \right] \frac{1}{1+\omega_n^2} = \sum_{n=1}^{\infty} \frac{2s}{s^2 + \omega_n^2}.$$

B. AN ALTERNATIVE PROOF OF (2.13)

This asymptotic formula exemplifies the *Stokes phenomenon*: an analytic function may display asymptotic behaviour which varies with a part of the complex plane. In particular, for large real $X \in \mathbb{R}$, the following asymptotic relationship holds

$$J_n(X) \sim \sqrt{\frac{2}{\pi X}} \cos\left(X - \frac{n\pi}{2} - \frac{\pi}{4} \right) = a_{+}\vartheta_{+}(X) + a_{-}\vartheta_{-}(X),$$

where

$$\vartheta_{\pm}(z) := \frac{e^{\pm jz}}{\sqrt{z}}, \qquad a_{\pm} := \sqrt{\frac{1}{2\pi}} e^{\pm j\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)}.$$

The constants $a_{\pm}$ are called *Stokes multipliers*. The upper imaginary semiaxis, where ϑ_- dominates, and the lower imaginary semiaxis, where ϑ_+ dominates, are called *Stokes lines*. The real semiaxes, where both functions $\vartheta_{\pm}$ are in balance, are the so-called *anti-Stokes lines*. All those objects together create the *Stokes structure*. However, the validity of the asymptotic expression may not be analytically continued to the whole complex plane $\mathbb{C}$, as revealed by examining the *monodromy property* of J_n. Near $z = 0$, J_n has the series representation

$$J_n(z) \sim (z/2)^n \sum_{m=0}^{\infty} \frac{\left(-z^2/4\right)^m}{m!(m+n)!}$$

which implies the monodromy property of J_n,

$$J_n(ze^{2j\pi}) = e^{2jn\pi} J_n(z).$$

On comparison, the asymptotic expression yields

$$J_n(ze^{2j\pi}) = e^{-j\pi} J_n(z),$$

i.e., it does not preserve the monodromy property. By recovering the monodromy property, it is possible to get the asymptotic expression

$$J_n(z) \sim \left\{ \begin{array}{ll} a_+\vartheta_+(z) + a_-\vartheta_-(z), & \text{if} \quad z \in \mathbb{C}^+ \\ a'_+\vartheta_+(z) + a_-\vartheta_-(z), & \text{if} \quad z \in \mathbb{C}^- \end{array} \right\} \tag{B.1}$$

with $a'_+ := \sqrt{\frac{1}{2\pi}} e^{j\left(\frac{3n\pi}{2} + \frac{\pi}{4}\right)}$. Using the identity $I_n(s) = j^n J_n(js)$, i.e., by rotating the Stokes structure for J_n by angle $-\pi/2$ (clockwise), one can get (2.13) from (B.1).

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