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BEST APPROXIMATION IN CHEBYSHEV SUBSPACES OF $\mathcal{L}(l_1^n, l_1^n)$

Abstract. Chebyshev subspaces of $\mathcal{L}(l_1^n, l_1^n)$ are studied. A construction of a k-dimensional Chebyshev (not interpolating) subspace is given.

Keywords: interpolating subspace, Chebyshev subspace, strongly unique best approximation.

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1. INTRODUCTION

Let $\mathbb{K}$ be the field of real or complex numbers and let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Let $extS_{X^\star}$ denote the set of all extreme points of $S_{X^\star}$, where $S_{X^\star}$ is the unit sphere in $X^\star$.

For any $x \in X$, we put

$$E(x) = \{f \in extS_{X^\star} : f(x) = \|x\|\} \tag{1}$$

and with any $Y \subset X$ we associate the set

$$P_Y(x) = \{y \in Y : \|x - y\| = dist(x, Y)\}.$$

Note that, by the Hahn-Banach and Krein-Milman Theorems, $E(x) \neq \emptyset$.

A linear subspace $Y \subset X$ is called a Chebyshev subspace if for any $x \in X$ the set $P_Y(x)$ contains one element only.

If Y is a linear subspace of X, then the following holds

Theorem 1 ([3]). *Assume X is a normed space, $Y \subset X$ is its linear subspace, and let $x \in X \setminus Y$. Then $y_0 \in P_Y(x)$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $ref(y) \leq 0$.*

Let us recall a well-known definition

Definition 1 (see e.g. [7])**.** An element $y_0 \in Y$ is called a strongly unique best approximation for $x \in X$ if and only if there exists $r > 0$ such that for every $y \in Y$,

$$\|x - y\| \geq \|x - y_0\| + r\|y - y_0\|.$$

The largest constant r satisfying the above inequality is called the strong unicity constant. There exist two main applications of the strong unicity constant:

1. The error estimate of the Remez algorithm (see e.g. [10]).
2. The Lipschitz continuity of the best approximation mapping at x_0 (assuming that there exists a strongly unique best approximation to x_0) (see e.g. [5, 8, 9]).

The following holds true:

Theorem 2 ([14])**.** *Let $x \in X \setminus Y$ and let Y be a linear subspace of X. Then $y_0 \in Y$ is a strongly unique best approximation for x with a constant $r > 0$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $ref(y) \leq -r\|y\|$.*

In this paper, we consider $X = \mathcal{L}(l_1^n, l_1^n)$, $n > 1$ (the space of all linear and continuous operators from l_1^n to l_1^n equipped with the operator norm denoted by $\|\cdot\|_{op}$), where

$$l_1^n = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ \|x\| := \sum_{i=1}^{n} | \, x_i \, | \right\}.$$

It is known [11] that for any operator $A \in \mathcal{L}(l_1^n, l_1^n)$:

$$\|A\|_{op} = \max_{x \in extS_{l_1^n}} \|Ax\|.$$

Since (see [1])

$$extS_{l_1^n} = \{e_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni}), \ i = 1, 2, \ldots, n\},$$

then for any $A = [a_{ij}]_{i,j=1,2,\ldots,n} \in \mathcal{L}(l_1^n, l_1^n)$, we obtain

$$\|A\|_{op} = \max\left\{ \sum_{i=1}^{n} | \, a_{i1} \, |, \ldots, \sum_{i=1}^{n} | \, a_{in} \, | \right\}.$$

The aim of this paper is to show that, for any $k \leq n$, there exists a k-dimensional Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ which is not an interpolating subspace.

This result is quite different from the result obtained for the space $\mathcal{L}(l_1^n, c_0)$ (see [6]), where any finite dimensional Chebyshev subspace is an interpolating subspace. Additionally, as the space $\mathcal{L}(l_1^n, l_1^n)$ is a finite dimensional space, we get (see [13]) that the unicity of best approximation is equivalent to the strong unicity of best approximation.

2. ONE-DIMENSIONAL CHEBYSHEV SUBSPACES OF $\mathcal{L}(l_1^n, l_1^n)$

Let an operator $A \in \mathcal{L}(l_1^n, l_1^n)$ be represented by a matrix $[a_{ij}]_{i,j=1,2,\ldots,n}$. Since (see [4,12])

$$extS_{\mathcal{L}^*(l_1^n, l_1^n)} = extS_{l_\infty^n} \otimes extS_{l_1^n}, \tag{2}$$

where

$$l_\infty^n = \left\{ x = (x_1, x_2, \ldots, x_n) \ : \ \|x\| := \max_{i \in \{1,2,\ldots,n\}} | \, x_i \, | \right\}$$

we get

$$extS_{\mathcal{L}^*(l_1^n, l_1^n)} = \{x \otimes e_j \ : \ j = 1, 2, \ldots, n\},$$

where $e_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{nj})$, $j = 1, 2, \ldots, n$, and $x = (x_1, x_2, \ldots, x_n)$, $x_i \in \{-1, 1\}$, $i = 1, 2, \ldots, n$. So, for any operator $A \in \mathcal{L}(l_1^n, l_1^n)$ represented by a matrix $[a_{ij}]_{i,j=1,2,\ldots,n}$, there is

$$(x \otimes e_j)(A) = \sum_{i=1}^{n} x_i a_{ij}.$$

Since $E(A) \neq \emptyset$, then by (1) and (2), we get that there exist $x \in l_\infty^n$ and $j \in \{1, 2, \ldots, n\}$ such that

$$\|A\|_{op} = (x \otimes e_j)(A).$$

Let us recall [2] that a k-dimensional subspace $\mathcal{V}$ of the normed space X is called an interpolating subspace if and only if for any linearly independent $f_1, f_2, \ldots, f_k \in extS_{X^\star}$ and for any $v \in \mathcal{V}$, the following holds: if $f_i(v) = 0$, $i = 1, 2, \ldots, k$, then $v = 0$. It is known [2] that any finite dimensional interpolating subspace is a finite dimensional Chebyshev subspace.

Theorem 3. *Let $\mathcal{V} \subset \mathcal{L}(l_1^n, l_1^n)$ be a k-dimensional $(k < n^2)$ subspace such that $\mathcal{V} = lin\{V_1, V_2, \ldots, V_k\}$, $V_m \in \mathcal{L}(l_1^n, l_1^n)$, $m = 1, 2, \ldots, k$ and $V_1, V_2, \ldots, V_k$ are linearly independent. For $m \in \{1, 2, \ldots, k\}$, let the operator V_m be represented by the matrix $[(v_m)_{ij}]_{i,j=1,2,\ldots,n}$. Then $\mathcal{V}$ is an interpolating subspace if and only if*

$$\begin{vmatrix} (x^{j_1} \otimes e_{j_1})(V_1) & . \quad . & (x^{j_1} \otimes e_{j_1})(V_k) \\ . & . \quad . & . \\ . & . \quad . & . \\ (x^{j_k} \otimes e_{j_k})(V_1) & . \quad . & (x^{j_k} \otimes e_{j_k})(V_k) \end{vmatrix} \neq 0,$$

where $(x^{j_l} \otimes e_{j_l}), (x^{j_r} \otimes e_{j_r}) \in extS_{\mathcal{L}^*(l_1^n, l_1^n)}$ are linearly independent for $l \neq r$, $l, r \in \{1, 2, \ldots, k\}$.

Proof. This is a consequence of (2), the definition of a k-dimensional interpolating subspace and the theory of linear equations. □

Example 1. *Let $V \in \mathcal{L}(l_1^n, l_1^n)$ be represented by a matrix $[v_{ij}]_{i,j=1,2,\ldots,n}$, where $v_{1j} = j$, $v_{ij} = 0$, $i = 2, \ldots, n$, $j = 1, 2, \ldots, n$. Then $\mathcal{V} = lin\{V\}$ is a one-dimensional interpolating subspace.*

Theorem 4. *Let $\mathcal{V} = lin\{V\}$, $V \in \mathcal{L}(l_1^n, l_1^n)$, $n > 1$, $V \neq 0$, $V = [v_{ij}]_{i,j=1,2,\dots,n}$. $\mathcal{V}$ is a Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ if and only if $\mathcal{V}$ is an interpolating subspace.*

Proof. Let us assume that $\mathcal{V}$ is not an interpolating subspace. Hence, there exists $f = x^{j_0} \otimes e_{j_0}$, $j_0 \in \{1, 2, \dots, n\}$, $x^{j_0} = ((x^{j_0})_1, (x^{j_0})_2, \dots, (x^{j_0})_n)$, $(x^{j_0})_i \in \{-1, 1\}$, $i = 1, 2, \dots, n$, such that $f(V) = 0$. Let us define $A = [a_{ij}]_{i,j=1,2,\dots,n}$ as follows:

$$a_{ij_0} = -(x^{j_0})_i, \quad a_{ij} = 0,\ j \neq j_0,\ j \in \{1, 2, \dots, n\},\ i = 1, 2, \dots, n.$$

Note that $\|A\| = n$. Let us consider an operator $A - \alpha V$, where $\alpha \in \mathbb{R}$. For small enough α we get $\|A - \alpha V\| = \|A\|$. The proof is complete. □

3. k-DIMENSIONAL CHEBYSHEV SUBSPACES OF $\mathcal{L}(l_1^n, l_1^n)$

Theorem 5. *Let $\mathcal{V} = lin\{V_1, V_2, \dots, V_k\} \subset \mathcal{L}(l_1^n, l_1^n)$, $k < n^2$, $V_m \in \mathcal{L}(l_1^n, l_1^n)$ (where V_m are linearly independent for $m = 1, 2, \dots, k$) be a k-dimensional subspace of $\mathcal{L}(l_1^n, l_1^n)$. Let V_m, $m \in \{1, 2, \dots, k\}$ be represented by a matrix $[(v_m)_{ij}]_{i,j=1,2,\dots,n}$. If $\mathcal{V}$ is a Chebyshev subspace, then vectors $w_1, w_2, \dots, w_h$ where*

$$\begin{aligned} w_1 &= (f_1(V_1), \dots, f_1(V_k)), \\ w_2 &= (f_2(V_1), \dots, f_2(V_k)), \\ &\dots \\ w_h &= (f_h(V_1), \dots, f_h(V_k)) \end{aligned} \tag{3}$$

are linearly independent for any $f_1, \dots, f_h \in extS_{\mathcal{L}^*(l_1^n, l_1^n)}$ such that $f_m = x^{j_m} \otimes e_{j_m}$, $m = 1, 2, \dots, h$, $j_m \neq j_r$ for $m \neq r$, where $h = k$ if $dim\mathcal{V} = k \leq n$, $h = n$ if $n < dim\mathcal{V} = k < n^2$.

Proof. Let us assume that (3) does not hold. From this assumption there follows that there exist $f_1, \dots, f_h \in extS_{\mathcal{L}^*(l_1^n, l_1^n)}$ such that $f_m = x^{j_m} \otimes e_{j_m}$, $m = 1, 2, \dots, h$, where $j_m \neq j_r$ for $m \neq r$ and $w_1, w_2, \dots, w_h$ are linearly dependent. Hence, there exists $l \in \{1, 2, \dots, h\}$ and there exist $\gamma_p \in \mathbb{R}$, $p \in \{1, 2, \dots, h\}$, such that

$$(f_l(V_1), \dots, f_l(V_k)) = \sum_{p \in \{1,2,\dots,h\},\ p \neq l} \gamma_p (f_p(V_1), \dots, f_p(V_k)). \tag{4}$$

From (4) we obtain:

$$f_l(V_m) = \sum_{p \in \{1,2,\dots,h\},\ p \neq l} \gamma_p f_p(V_m),\ m = 1, 2, \dots, k.$$

We shall construct an operator $A \in \mathcal{L}(l_1^n, l_1^n)$ which has more than one best approximation in $\mathcal{V}$. Let $[a_{ij}]_{i,j=1,2,\dots,n}$ be a matrix representation for A. If in (4), $\gamma_p < 0$ for some $p \in \{1, 2, \dots, h\}$, $p \neq l$, we put

$$a_{ij_p} = (x^{j_p})_i, \quad i = 1, 2, \dots, n.$$

If in (4), $\gamma_p > 0$ for some $p \in \{1, 2, \ldots, h\},\ p \neq l$, we put

$$a_{ij_p} = -(x^{j_p})_i, \quad i = 1, 2, \ldots, n.$$

Additionally, we put

$$a_{ij_l} = (x^{j_l})_i, \quad i = 1, 2, \ldots, n.$$

If $j \neq j_p,\ j \in \{1, 2, \ldots, n\}$, we put $a_{ij} = 0,\ i = 1, 2, \ldots, n$. Let us consider the operator

$$A(\alpha_1, \alpha_2, \ldots, \alpha_k) := A - (\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_k V_k), \quad \text{where} \quad \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}.$$

For $(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0)$ there is $\|A(0, 0, \ldots, 0)\| = \|A\|$. As we have assumed that (3) does not hold, we conclude that there exists $(\alpha_1, \alpha_2, \ldots, \alpha_k) \neq (0, 0, \ldots, 0)$ such that

$$\|A(\alpha_1, \alpha_2, \ldots, \alpha_k)\| = \|A\|.$$

For α_i small enough for $i = 1, 2, \ldots, k$, the norm of the operator $A(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is equal to the largest of the following values:

$$\|A\| - [\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)],$$

for some $p \in \{1, 2, \ldots, h\},\ p \neq l$ for which $\gamma_p < 0$;

$$\|A\| + [\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)],$$

for some $p \in \{1, 2, \ldots, h\},\ p \neq l$ for which $\gamma_p > 0$; or

$$\begin{aligned} &\|A\| - [\alpha_1 f_l(V_1) + \alpha_2 f_l(V_2) + \ldots + \alpha_k f_l(V_k)] = \\ &= \|A\| - \Big[\sum_{p \in \{1,2,\ldots,h\},\ p \neq l} \gamma_p(\alpha_1 f_p(V_1) + \ldots + \alpha_k f_p(V_k)) \Big]. \end{aligned} \tag{5}$$

From the above, if for some $\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0$ we want to obtain

$$\|A(\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0)\| < \|A\|,$$

we need the inequality

$$[\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)] > 0$$

to hold for $p \in \{1, 2, \ldots, h\},\ p \neq l$ for which $\gamma_p < 0$, and

$$[\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)] < 0$$

for $p \in \{1, 2, \ldots, h\},\ p \neq l$ for which $\gamma_p > 0$. But then, by (5), we get

$$\|A(\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0)\| > \|A\|.$$

□

Note that (3) is satisfied for any k-dimensional interpolating subspace. But the condition presented in Theorem 5 is not sufficient for a k-dimensional subspace ($k \geq 2$) to be Chebyshev.

Example 2. *Let $\mathcal{V} = lin\{V_1, V_2\}$, where*

$$V_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that (3) *is satisfied for $\mathcal{V} = lin\{V_1, V_2\}$. Let*

$$A = \begin{bmatrix} 0 & 0 \\ 100 & 0 \end{bmatrix}.$$

Then $\|A\| = 100$. Let

$$A(\alpha_1, \alpha_2) := A - (\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} -\alpha_1 - 3\alpha_2 & -2\alpha_1 - \alpha_2 \\ 100 & 0 \end{bmatrix}.$$

Hence, for $(\alpha_1, \alpha_2) = (0, 0)$, we get

$$\|A(\alpha_1, \alpha_2)\| = \|A\| = 100 = \inf_{\alpha_1, \alpha_2 \in \mathbb{R}} \|A(\alpha_1, \alpha_2)\|.$$

But for $(\alpha_1, \alpha_2) = (3, -1)$ we get $\|A(\alpha_1, \alpha_2)\| = \|A\| = 100$.

Now we shall construct a k-dimensional Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ which is not an interpolating subspace.

Theorem 6. *Let $V_1, V_2, \ldots, V_k \in \mathcal{L}(l_1^n, l_1^n)$, $k \leq n$, $n > 1$ be linearly independent and let V_m, $m \in \{1, 2, \ldots, k\}$ be represented by a matrix*

$$V_m = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ v_{m1} & v_{m2} & . & . & . & v_{mn} \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \end{bmatrix},$$

where $v_{mj} \neq 0$ for any $j = 1, 2, \ldots, n$, $m = 1, 2, \ldots, k$. $\mathcal{V}(m_1, \ldots, m_r) := lin\{V_{m_1}, \ldots, V_{m_r}\}$, $m_1, \ldots, m_r \in \{1, 2, \ldots, k\}$, $m_p \neq m_q$, $p \neq q$ is an r-dimensional Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ for any $1 \leq r \leq k$ if and only if

$$\begin{gathered} \text{for all} \quad 1 \leq r \leq k, \quad 1 \leq j_1 < j_2 < \ldots < j_r \leq n, \\ 1 \leq i_1 < i_2 < \ldots < i_r \leq k, \quad x^1, x^2, \ldots, x^r \in \{-1, 1\}^r \quad \text{there is} \\ det[(x^m)_l v_{i_m j_l}]_{m=1,2,\ldots,r,\ l=1,2,\ldots,r} \neq 0. \end{gathered} \tag{6}$$

Proof. Let us assume that (6) holds. If $r = 1$, then $\mathcal{V}(m_1) = lin\{V_{m_1}\}$, $m_1 \in \{1, 2, \ldots, k\}$ is a Chebyshev subspace, because it is an interpolating subspace. Let us now assume that for $1 < r < k$ the space

$$\mathcal{V}_r := \mathcal{V}(m_1, \ldots, m_r) = lin\{V_{m_1}, \ldots, V_{m_r}\},$$
$$m_1, \ldots, m_r \in \{1, 2, \ldots, k\},\ m_p \neq m_q,\ p \neq q$$

is a Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ and let

$$\mathcal{V}_{r+1} := \mathcal{V}(m_1, \ldots, m_r, m_{r+1}) = lin\{V_{m_1}, \ldots, V_{m_r}, V_{m_{r+1}}\},$$
$$m_1, \ldots, m_r \in \{1, 2, \ldots, k\},\ m_{r+1} \in \{1, 2, \ldots, k\} \setminus \{m_1, \ldots, m_r\}$$

be not a Chebyshev subspace. From this we conclude that there exists an operator $A \in \mathcal{L}(l_1^n, l_1^n)$ such that $\sharp\, \mathcal{P}_{\mathcal{V}_{r+1}}(A) > 1$. We can assume that $0, W \in \mathcal{P}_{\mathcal{V}_{r+1}}(A)$, where $W \neq 0$. Let $\mathcal{U} := \{j \in \{1, 2, \ldots, n\} :\ \|A \circ e_j^T\| = \|A\|\}$, where $e_j = (\delta_{1j}, \ldots, \delta_{nj})$. For any $j \in \mathcal{U}$, we put

$$E_j := \{x = (x_1, x_2, \ldots, x_n)\ :\ x_i \in \{-1, 1\},\ i = 1, 2, \ldots, n\ :\ (x \circ A)_j = \|A\|\}.$$

Since $0, W \in \mathcal{P}_{\mathcal{V}_{r+1}}(A)$, we conclude that for $j \in \mathcal{U}$ and $x \in E_j$ the following holds

$$(x \otimes e_j)(W) \geq 0. \tag{7}$$

Let

$$\mathcal{U}_1 := \{j \in \mathcal{U}\ :\exists\ x \in E_j\ :\ (x \otimes e_j)(W) = 0\}.$$

Since $0 \in \mathcal{P}_{\mathcal{V}_{r+1}}(A)$, then $\mathcal{U}_1 \neq \emptyset$. Now we shall show that

$$\forall\ j \in \mathcal{U}_1\ \ \exists!\ x \in E_j\ :\ (x \otimes e_j)(W) = 0. \tag{8}$$

Let us assume that (8) does not hold. Let $x \neq y,\ x, y \in E_j$ be such that

$$(x \otimes e_j)(W) = 0, \quad (y \otimes e_j)(W) = 0.$$

Without loss of generality, we may assume that

$$x_i = y_i, \quad i = 1, 2, \ldots, p,\ p < r + 1, \quad x_i = -y_i, \quad i = p + 1, p + 2, \ldots, r + 1.$$

Then we get

$$\sum_{i=1}^{p} x_i(w)_{ij} = 0, \quad \sum_{i=p+1}^{r+1} x_i(w)_{ij} = 0. \tag{9}$$

Since $x_i = -y_i, \quad i = p + 1, p + 2, \ldots, r + 1$, we obtain $a_{ij} = 0,\ i = p + 1, \ldots, r + 1$. By (7) there follows:

$$\sum_{i=p+1}^{r} x_i(w)_{ij} - x_{r+1}(w)_{r+1j} \geq 0,$$
$$\sum_{i=p+1}^{r} -x_i(w)_{ij} + x_{r+1}(w)_{r+1j} \geq 0,$$

and then

$$\sum_{i=p+1}^{r} x_i(w)_{ij} = x_{r+1}(w)_{r+1j}.$$

Applying (9) we conclude that $x_{r+1}(w)_{r+1j} = 0$ and then $(w)_{r+1j} = 0$. Hence $W \in \mathcal{V}_r := lin\{V_{m_1}, \ldots, V_{m_r}\}$. Additionally, $0 \in \mathcal{V}_r := lin\{V_{m_1}, \ldots, V_{m_r}\}$. But $\mathcal{V}_r$ is a Chebyshev subspace and hence (8) is proved. We shall show that there exists $\alpha_0 > 0$ such that for any $0 < \alpha \leq \alpha_0$ the following holds:

$$E(A - \alpha W) = \{x \otimes e_j \ : \ j \in \mathcal{U}_1, \ (x \otimes e_j)(W) = 0, \ (x \otimes e_j)(A) = \|A\|\}. \tag{10}$$

Let $f \notin E(A)$. Then there exist $\alpha_0 > 0, \ b > 0$ such that for any $0 < \alpha \leq \alpha_0$ the following holds:

$$f(A - \alpha W) \leq b < \|A\| \leq \|A - \alpha W\|.$$

Let $f \in E(A - \alpha W), \ f(A) = \|A\|$. If $f(W) > 0$, we get

$$\|A - \alpha W\| = f(A - \alpha W) = \|A\| - \alpha f(W) < \|A\|.$$

From the above we conclude that if $f \in E(A - \alpha W)$, then $f \in E(A), \ f(W) = 0$. Since

$$\|A - \alpha W\| = \|A\| = dist(A, \mathcal{V}_{r+1}),$$

(10) is proved. Since $\alpha W \in \mathcal{P}_{\mathcal{V}_{r+1}}(A)$ we obtain (see [13]):

$$\exists \ 1 \leq q \leq r+2, \ \ \exists \ \lambda_1, \ldots, \lambda_q > 0, \ \sum_{i=1}^{q} \lambda_i = 1,$$

such that

$$\sum_{i=1}^{q} \lambda_i (x^{j_i} \otimes e_{j_i})|_{\mathcal{V}_{r+1}} = 0, \tag{11}$$

and $(x^{j_i} \otimes e_{j_i})(A - \alpha W) = \|A - \alpha W\|$. By (8) we get $j_i \neq j_l, \ \ i \neq l, \ \ i, l \in \{1, 2, \ldots, q\}$. Let us take the least q such that $1 \leq q \leq r+2$ and (11) is satisfied. If $q = r+2$, then (see [15]) we get that αW is a strongly unique best approximation for A in $\mathcal{V}_{r+1}$. If $1 \leq q \leq r+1$, we have a contradiction with (6).

Let us assume that $\mathcal{V}_r = lin\{V_{m_1}, \ldots, V_{m_r}\}$, $m_1, \ldots, m_r \in \{1, 2, \ldots, k\}$, $m_p \neq m_q$, $p \neq q$ is a Chebyshev subspace of $\mathcal{L}(l_1^n, l_1^n)$ for any $1 \leq r \leq k$ and let (6) does not hold. Hence, there exist

$$1 \leq r \leq k, \ i_1, \ldots, i_r, \ j_1, \ldots, j_r \in \{1, 2, \ldots, n\}, \ x^1, x^2, \ldots, x^r \in \{-1, 1\}^n$$

such that

$$det[(x^m)_l v_{i_m j_l}]_{m=1,2,\ldots,r, \ l=1,2,\ldots,r} = 0.$$

From this we conclude that there exist

$$\lambda_1, \ldots, \lambda_r \in \mathbb{R}, \ \sum_{l=1}^{r} \mid \lambda_l \mid > 0$$

such that
$$\sum_{l=1}^{r} \lambda_l (x^{j_l} \otimes e_{j_l})|_{\mathcal{V}_r} = 0. \tag{12}$$

Without loss of generality, we may assume that $\lambda_l > 0,\ l = 1, 2, \ldots, r$. Let us now define an operator $B = [b_{ij}]_{i,j=1,2,\ldots,n}$ as follows:

$$b_{ij_l} = sgn(x^{j_l})_i, \quad b_{ij} = 0,\ j \neq j_l,\ l \in \{1, 2, \ldots, r\},\ i \in \{1, 2, \ldots, n\}.$$

Then $(x^{j_l} \otimes e_{j_l})(B) = \|B\|,\ l = 1, 2, \ldots, r$. By (12), there follows that $0 \in \mathcal{P}_{\mathcal{V}_r}(B)$ and
$$dim\ span\{x^{j_l} \otimes e_{j_l} | \mathcal{V}_r\} < r,$$
where $dim \mathcal{V}_r = r$. It means that there exists $V \in \mathcal{V}_r \setminus \{0\}$ such that
$$(x^{j_l} \otimes e_{j_l})(V) = 0,\ l = 1, 2, \ldots, r.$$

Note that if $f \notin E(B)$, then there exist $\alpha_0 > 0,\ b > 0$ such that for any $\alpha \in (0, \alpha_0\rangle$:
$$f(B - \alpha V) < \|B - \alpha V\|.$$

From the above we conclude that $\|B - \alpha V\| = \|B\|$. □

Corollary 1. *Let* $\mathcal{V} \subset \mathcal{L}(l_1^n, l_1^n)$ *be a k-dimensional subspace from Theorem 6. Any operator* $A \in \mathcal{L}(l_1^n, l_1^n)$ *has a unique best approximation in* $\mathcal{V}$ *if and only if A has a strongly unique best approximation in* $\mathcal{V}$.

Proof. It is a consequence of [13] and the fact that $\mathcal{L}(l_1^n, l_1^n)$ is a finite dimensional space. □

Example 3. *We shall construct a k-dimensional Chebyshev subspace* $\mathcal{V} \subset \mathcal{L}(l_1^n, l_1^n)$, $k \leq n$. *The construction is as follows. Let* $0 < t_1 < t_2 < \ldots < t_{k-1}$. *We put* $V_m = [(v_m)_{ij}]_{i,j=1,2,\ldots,n}$, $m = 1, 2, \ldots, k-1$ *as follows:*

$$(v_m)_{mj} = t_m^j,\ j = 1, 2, \ldots, n,$$
$$(v_m)_{ij} = 0, \quad i \neq m,\ j = 1, 2, \ldots, n.$$

Let us assume that the subspace $\mathcal{V}_{k-1} := lin\{V_1, V_2, \ldots, V_{k-1}\}$ *satisfies formula* (6) *for any* $1 \leq r \leq k-1$. *We shall construct an operator* $V_k \in \mathcal{L}(l_1^n, l_1^n)$ *such that* $\mathcal{V}_k := lin\{V_1, V_2, \ldots, V_{k-1}, V_k\}$ *satisfies formula* (6) *for any* $1 \leq r \leq k$, *which means that* $\mathcal{V}_k := lin\{V_1, V_2, \ldots, V_{k-1}, V_k\}$ *is a Chebyshev subspace of* $\mathcal{L}(l_1^n, l_1^n)$. *We are looking for such* $x \in \mathbb{R}$ *that, for any* $r \in \{1, 2, \ldots, k\}$, *there holds:*

$$W(x, y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1}) := \begin{vmatrix} y_1^1 t_{m_1}^{j_1} & \cdot\ \cdot\ \cdot & y_1^r t_{m_1}^{j_r} \\ \cdot & \cdot\ \cdot\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot & \\ y_{r-1}^1 t_{m_{r-1}}^{j_1} & \cdot\ \cdot\ \cdot & y_{r-1}^r t_{m_{r-1}}^{j_r} \\ y_r^1 x^{j_1} & \cdot\ \cdot\ \cdot & y_r^r x^{j_r} \end{vmatrix} \neq 0, \tag{13}$$

for any $j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, n\}$, $y^1, \ldots, y^r \in \{-1, 1\}^r$, $m_1, m_2, \ldots, m_{r-1} \in \{1, 2, \ldots, k-1\}$.

By the assumption, $W(x, y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1})$ *is not identically equal to zero. Hence, the set of roots of* $W(x, y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1})$ *is finite for arbitrary fixed* $y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1}$. *Hence, the set of roots of* $W(x, y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1})$, $y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots m_{r-1}$ *is countable. But* $\mathbb{R}$ *is not countable, so there exists* $x \in \mathbb{R}$ *which satisfies* (13).

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