

Bogusław Bożek, Wiesław Solak, Zbigniew Szydełko

ON SOME QUADRATURE RULES WITH GREGORY END CORRECTIONS

Abstract. How can one compute the sum of an infinite series $s := a_1 + a_2 + \dots$? If the series converges fast, i.e., if the term a_n tends to 0 fast, then we can use the known bounds on this convergence to estimate the desired sum by a finite sum $a_1 + a_2 + \dots + a_n$. However, the series often converges slowly. This is the case, e.g., for the series $a_n = n^{-t}$ that defines the Riemann zeta-function. In such cases, to compute s with a reasonable accuracy, we need unrealistically large values n , and thus, a large amount of computation.

Usually, the n -th term of the series can be obtained by applying a smooth function $f(x)$ to the value n : $a_n = f(n)$. In such situations, we can get more accurate estimates if instead of using the upper bounds on the remainder infinite sum $R = f(n+1) + f(n+2) + \dots$, we approximate this remainder by the corresponding integral I of $f(x)$ (from $x = n+1$ to infinity), and find good bounds on the difference $I - R$.

First, we derive sixth order quadrature formulas for functions whose 6th derivative is either always positive or always negative and then we use these quadrature formulas to get good bounds on $I - R$, and thus good approximations for the sum s of the infinite series. Several examples (including the Riemann zeta-function) show the efficiency of this new method. This paper continues the results from [3] and [2].

Keywords: numerical integration, quadrature formulas, summation of series.

Mathematics Subject Classification: 65D30, 65D32, 65G99, 65B10.

1. SOME QUADRATURE RULES WITH GREGORY END CORRECTIONS

We present one-parameter end corrections for elementary quadrature formula and we examine a property of this quadrature for the special values of the parameter. This paper continues the results from [3].

1.1. INTRODUCTION

One can compute the approximate value of the integral

$$I(f) = \int_a^b f(t) dt$$

by applying the quadrature formula in the form

$$Q(f) = \sum_{i=0}^n a_i f(t_i),$$

where quadrature nodes t_i belong to the interval $[a - c, b + c]$, $c \geq 0$. The quadrature coefficients $\{a_i\}$ satisfy the equation

$$\sum_{i=0}^n a_i = b - a.$$

If some nodes depend on β , i.e. $t_i = t_i(\beta)$ for $i \in A \subset \{0, 1, \dots, n\}$, then we call this the quadrature formula with parameter. The value

$$EQ(f) = I(f) - Q(f)$$

is called the (global) quadrature error.

One of the methods to compute the error EQ is the method that comes from Peano. First we determine the quadrature range s and next we compute the Peano kernel defined as follows

$$K_s(x) = EQ(p(t)), \quad (1.1)$$

where

$$p(t) = \frac{(t-x)_+^{s-1}}{(s-1)!} \quad (1.2)$$

$$a_+ = \max\{a, 0\}, \quad x - \text{parametr.}$$

Peano's theory (see [1]) says, that for the function $f \in C^{(s)}([a - c, b + c])$ we have

$$EQ(f) = \int_{a-c}^{b+c} K_s(x) f^{(s)}(x) dx. \quad (1.3)$$

If $K_s(x)$ is of constant sign, then from (1.3) we obtain a useful formula

$$EQ(f) = f^{(s)}(\xi) \int_{a-c}^{b+c} K_s(x) dx, \quad \xi \in [a - c, b + c]. \quad (1.4)$$

A quadrature formula obtained by adding some correction terms to the trapezoidal rule is called the Gregory type. One of the examples of such quadrature can be written as follows

$$Q_{n+5}^\beta(f) := T_{n+1}(f) + G_n(f, \beta), \quad (1.5)$$

where

$$\begin{aligned} G_n(f, \beta) &= \frac{h}{24\beta} (-3(f_0 + f_n) + 4(f_\beta + f_{n-\beta}) - (f_{2\beta} + f_{n-2\beta})), \\ f_t &:= f(a + th), \quad h = \frac{b-a}{n}, \\ T_{n+1}(f) &= \frac{h}{2}(f_0 + f_n) + h \sum_{i=1}^{n-1} f_i \end{aligned}$$

is the trapezoidal rule, and β is a parameter.

The polynomial

$$v_n(\beta) = \frac{EQ(t^4)}{\frac{1}{30}h^5} = 30\beta^3 - 20n\beta^2 + n \quad (1.6)$$

is called the characteristic polynomial of the quadrature Q_{n+5}^β . It is easy to verify that the quadrature (1.5) is of the fourth order if β is not a root of the characteristic polynomial v_n , and of the sixth order if β is a root of this polynomial.

In the paper [3] the properties of the quadrature Q_{n+5}^β for β from the interval $(0, \frac{1}{2}]$ are examined. The Peano kernel $K_4(x)$ is non-positive for $\beta \in [0.31, 0.5]$ and in this case the error of the quadrature formula for $f \in C^{(4)}[a, b]$ (c is equal zero) can be written in the form

$$EQ_{n+5}^\beta(f) = \frac{h^5}{720} v_n(\beta) f^{(4)}(\xi) \quad (1.7)$$

with some $\xi \in [a, b]$.

In this paper we investigate the properties of (1.5) for the roots of the characteristic polynomial v_n .

1.2. AN ANALYSIS OF GREGORY TYPE QUADRATURE FORMULAE

The roots of the characteristic polynomial $v_n(\beta)$ are

$$\begin{aligned} \alpha_n &= \frac{2}{9}n \left(1 + 2 \cos\left(\frac{\varphi_n + 2\pi}{3}\right) \right), \\ \beta_n &= \frac{2}{9}n \left(1 + 2 \cos\left(\frac{\varphi_n + 4\pi}{3}\right) \right), \\ \gamma_n &= \frac{2}{9}n \left(1 + 2 \cos\left(\frac{\varphi_n}{3}\right) \right), \end{aligned}$$

where

$$\varphi_n \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \varphi_n = \arccos\left(1 - \frac{243}{160n^2}\right).$$

It easy to verify, that

$$\begin{aligned}\lim_{n \rightarrow \infty} \varphi_n &= 0, \\ \lim_{n \rightarrow \infty} \alpha_n &= -\frac{\sqrt{5}}{10}, \\ \lim_{n \rightarrow \infty} \beta_n &= \frac{\sqrt{5}}{10}, \\ \lim_{n \rightarrow \infty} \gamma_n &= \infty,\end{aligned}$$

moreover the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are decreasing, and $\alpha_n < 0$, $\beta_n > 0$ for $n = 1, 2, \dots$

Theorem 1.1. *The quadrature (1.5) with $\beta = \alpha_n$ is of the sixth order, and the error estimation for any function $f \in C^{(6)}[a + 2\alpha_n h, b - 2\alpha_n h]$ can be expressed by*

$$EQ_{n+5}^{\alpha_n}(f) = \frac{nh^7}{4320} \left(\frac{5\alpha_n^2 + 11}{15} n\alpha_n - \left(\frac{1}{7} + \alpha_n^2 \right) \right) f^{(6)}(\eta) \quad (1.8)$$

for some $\eta \in [a + 2\alpha_n h, b - 2\alpha_n h]$.

Proof. It is clear that the support of the Peano kernel $K_6^{\alpha_n}(x)$ is the interval $[a + 2\alpha_n h, b - 2\alpha_n h]$. Taking advantage of the formula (1.4) it suffices to show, that the Peano kernel is negative in the interval $(a + 2\alpha_n h, b - 2\alpha_n h)$.

Directly from the definition, we can write the Peano kernel $K_6^{\alpha_n}(x)$ in the form:

$$K_6^{\alpha_n}(x) = \begin{cases} \phi_1\left(\frac{x-b}{h}\right) & \text{for } x \in [b - \alpha_n h, b - 2\alpha_n h], \\ \phi_2\left(\frac{x-b}{h}\right) & \text{for } x \in [b, b - \alpha_n h], \\ \phi_3^j\left(\frac{b-x}{h} - j\right) & \text{for } x \in [b - (j+1)h, b - jh], \\ & j = 0, 1, \dots, n-1, \\ \phi_2\left(\frac{a-x}{h}\right) & \text{for } x \in [a + \alpha_n h, a], \\ \phi_1\left(\frac{a-x}{h}\right) & \text{for } x \in (a + 2\alpha_n h, a + \alpha_n h), \end{cases} \quad (1.9)$$

where

$$\begin{aligned}\phi_1(t) &= \frac{-h^6}{720 \cdot 4\alpha_n} (t + 2\alpha_n)^5 \quad \text{for } -\alpha_n \leq t < -2\alpha_n, \\ \phi_2(t) &= \frac{-h^6}{720 \cdot 4\alpha_n} \left((t + 2\alpha_n)^5 - 4(t + \alpha_n)^5 \right) \quad \text{for } 0 \leq t \leq -\alpha_n, \\ \phi_3^j(t) &= \frac{h^6}{720} \left((t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2) - 7\alpha_n^4 + \frac{1}{2}(1 - 20\alpha_n^2)(t + j)(t - (n - j)) \right) \\ &\quad \text{for } 0 \leq t \leq 1, \quad j = 0, 1, \dots, n-1.\end{aligned}$$

We will check that ϕ_1 , ϕ_2 , ϕ_3^j is negative in suitable intervals.

Because of $t < -2\alpha_n$, we have $(t + 2\alpha_n)^5 < 0$. Take into consideration the fact that $\alpha_n < 0$, we get $\phi_1(t) < 0$ in the interval $[-\alpha_n, -2\alpha_n)$, and moreover $\phi(-2\alpha_n) = 0$.

Next, we observe that $\phi_2(t) < 0$ if and only if $(\sqrt[5]{4} - 1)t > (2 - \sqrt[5]{4})\alpha_n$. This inequality is evidently true as $\alpha_n < 0$ and $t \geq 0$.

Let us first define the auxiliary functions

$$f(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2,$$

$$g^j(t) = \frac{1}{2}(1 - 20\alpha_n^2)(t + j)(t - (n - j)) \quad (j = 0, 1, \dots, n - 1).$$

A simple computation shows, that $f(t) < 0$ on $(0, 1)$ and $f(0) = f(1) = 0$. For $j \in \{0, 1, \dots, n - 1\}$ we have $-j \leq 0$, $n - j \geq 1$, so $-j \leq 0 < 1 \leq n - j$ and these imply the inclusions $[0, 1] \subset [-j, n - j]$. On the interval $[0, 1]$ the parabola $(t + j)(t - (n - j))$ is non-positive (it is negative in $(-j, n - j)$). Since $1 - 20\alpha_n^2 > 0$, we see that $g^j(t) \leq 0$ on $[0, 1]$. From the above we have $\phi_3^j(t) < 0$ on $[0, 1]$ because of

$$\phi_3^j(t) = \frac{h^6}{720}(f(t) + g^j(t) - 7\alpha_n^4).$$

This finishes the proof of the fact that the Peano kernel is negative. Integrating the Peano kernel over $[a + 2\alpha_n h, b - 2\alpha_n h]$ we have (1.8), which agrees with the formula (1.4). \square

Theorem 1.2. *The quadrature (1.5) with $\beta = \beta_n$ is of the sixth order, and the error estimation for any function $f \in C^{(6)}[a, b]$ can be expressed by*

$$EQ_{n+5}^{\beta_n}(f) = \frac{nh^7}{4320} \left(\frac{5\beta_n^2 + 11}{15}n\beta_n - \left(\frac{1}{7} + \beta_n^2\right) \right) f^{(6)}(\xi) \quad (1.10)$$

with some $\xi \in [a, b]$.

Proof. Directly from the definition, we can write the Peano kernel $K_6^{\beta_n}(x)$ in the form:

$$K_6^{\beta_n}(x) = \begin{cases} \psi_1\left(\frac{x-a}{h}\right) & \text{for } x \in [a, a + \beta_n h], \\ \psi_2\left(\frac{x-a}{h}\right) & \text{for } x \in [a + \beta_n h, a + 2\beta_n h], \\ \psi_3\left(\frac{x-a}{h}\right) & \text{for } x \in [a + 2\beta_n h, a + h], \\ \psi_4^j\left(\frac{b-x}{h} - j\right) & \text{for } x \in [b - (j + 1)h, b - jh], \\ & j = 1, 2, \dots, n - 2, \\ \psi_3\left(\frac{b-x}{h}\right) & \text{for } x \in [b - h, b - 2\beta_n h], \\ \psi_2\left(\frac{b-x}{h}\right) & \text{for } x \in [b - 2\beta_n h, b - \beta_n h], \\ \psi_1\left(\frac{b-x}{h}\right) & \text{for } x \in [b - \beta_n h, b], \end{cases} \quad (1.11)$$

where

$$\begin{aligned} \psi_1(t) &= \frac{h^6}{720} t^5 \left(t + 3 \left(\frac{1}{4\beta_n} - 1 \right) \right) \quad \text{for } 0 \leq t \leq \beta_n, \\ \psi_2(t) &= \frac{h^6}{720} \left(t^6 - \left(3 + \frac{1}{4\beta_n} \right) t^5 + 5t^4 - 10\beta_n t^3 + 10\beta_n^2 t^2 - 5\beta_n^3 t + \beta_n^4 \right) \\ &\quad \text{for } \beta_n \leq t \leq 2\beta_n, \\ \psi_3(t) &= \frac{h^6}{720} \left(t^6 - 3t^5 + \frac{5}{2}t^4 - 10\beta_n^2 t^2 + 15\beta_n^3 t - 7\beta_n^4 \right) \quad \text{for } 2\beta_n \leq t \leq 1, \\ \psi_4(t) &= \frac{h^6}{720} \left(\left(t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 \right) - 7\beta_n^4 + \frac{1}{2} (1 - 20\beta_n^2)(t + j)(t - (n - j)) \right) \\ &\quad \text{for } 0 \leq t \leq 1, \quad j = 1, 2, \dots, n - 2. \end{aligned}$$

We can now proceed analogously to the proof of the previous theorem. We prove that the kernel $K_6^{\beta_n}(x)$ is non-negative and from (1.4) after the integration of the Peano kernel we have (1.10). \square

Figure 1 illustrates the graphs of Peano kernels $K_6^{\alpha_n}, K_6^{\beta_n}$ for $[a, b] = [0, 1]$ and $n = 16$.

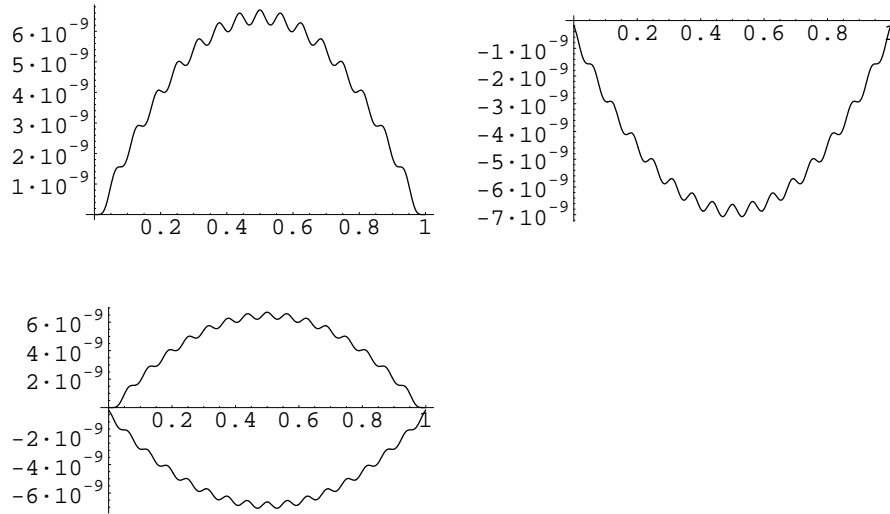


Fig. 1. The kernels $K_6^{\alpha_{16}}, K_6^{\beta_{16}}$, and the both kernels in one figure

Theorem 1.3. *If the function f is of the class $C^6[a + 2\alpha_n h, b - 2\alpha_n h]$ and $f^{(6)}$ is positive in this interval, then*

$$Q_{n+5}^{\beta_n}(f) < I(f) < Q_{n+5}^{\alpha_n}(f). \tag{1.12}$$

If $f^{(6)}$ is negative, then

$$Q_{n+5}^{\alpha_n}(f) < I(f) < Q_{n+5}^{\beta_n}(f). \tag{1.13}$$

Proof. It is easy to see, that

$$\frac{5\alpha_n^2 + 11}{15}n\alpha_n - \left(\frac{1}{7} + \alpha_n^2\right) < 0$$

and

$$\frac{5\beta_n^2 + 11}{15}n\beta_n - \left(\frac{1}{7} + \beta_n^2\right) > 0$$

for all $n \geq 2$. These inequalities, the estimations (1.8), (1.10), and the sign of the derivative $f^{(6)}$ imply the inequalities (1.12), (1.13) of the definition of the error of the quadrature. \square

Example 1. Consider the integral

$$I(f) = \int_0^{\frac{\pi}{4}} f(x) dx,$$

where $f(x) := \sqrt{\cos x}$. We can see that

$$I(f) = \sqrt{\frac{2}{\pi}} \left(\Gamma\left(\frac{3}{4}\right)\right)^2 \approx 0.74430307.$$

The derivative $f^{(6)}$ is given by

$$f^{(6)}(x) = -\frac{19}{8}\sqrt{\cos x} - \frac{289 \sin^2 x}{16 \cos^{3/2} x} - \frac{975 \sin^4 x}{32 \cos^{7/2} x} - \frac{945 \sin^6 x}{64 \cos^{11/2} x}$$

therefore $f^{(6)}(x) < 0$ for all $x \in [0, \frac{\pi}{4}]$. For example we compute:

$$\begin{aligned} Q_{25}^{\alpha_{20}}(f) &= 0.74372122 < I(f) < 0.74466093 = Q_{25}^{\beta_{20}}(f), \\ Q_{35}^{\alpha_{30}}(f) &= 0.74404307 < I(f) < 0.74446467 = Q_{35}^{\beta_{30}}(f). \end{aligned}$$

Example 2. Consider the integral

$$I(f) = \int_1^2 f(x) dx,$$

where $f(x) := \frac{e^x}{x}$. The derivative $f^{(6)}$ is given by

$$f^{(6)}(x) = \left(\frac{720}{x^7} - \frac{720}{x^6} + \frac{360}{x^5} - \frac{120}{x^4} + \frac{30}{x^3} - \frac{6}{x^2} + \frac{1}{x}\right)e^x;$$

therefore, $f^{(6)}(x) > 0$, for all $x \in [1, 2]$. For example we compute:

$$\begin{aligned} Q_{25}^{\beta_{20}}(f) &= 3.056553592 < I(f) < 3.063275128 = Q_{25}^{\alpha_{20}}(f), \\ Q_{35}^{\beta_{30}}(f) &= 3.057961330 < I(f) < 3.060972732 = Q_{35}^{\alpha_{30}}(f). \end{aligned}$$

Remark 1.4. Comparing the quadrature formulas $Q_{n+5}^{\alpha_n}(f)$, $Q_{n+5}^{\beta_n}(f)$ with Gauss sixth order quadrature formula

$$Q_{3n}^G(f) := \frac{h}{18} \sum_{j=1}^n \left(5f_{j-\frac{1}{2}-\frac{\sqrt{5}}{10}} + 8f_{j-\frac{1}{2}} + 5f_{j-\frac{1}{2}+\frac{\sqrt{5}}{10}} \right)$$

we can see that the quadrature $Q_{3n}^G(f)$ has $3n$ function calls whereas the quadratures $Q_{n+5}^{\alpha_n}(f)$, $Q_{n+5}^{\beta_n}(f)$ have $n+5$ function calls. Evidently, $n+5 < 3n$ for $n > 2$.

2. SERIES ESTIMATION VIA BOUNDARY CORRECTIONS WITH PARAMETERS

The sum of a series

$$s := \sum_{n=1}^{\infty} a_n \tag{2.1}$$

can be approximated by a finite sum $\sum_{n=1}^N a_n$. The error of this estimation can be represented as the sum of the series $\sum_{n=N+1}^{\infty} a_n$.

Therefore, if we have a method of estimating the sum of an infinite series, then this method will enable us to estimate the error of the N -term approximation. One way to estimate the sum of the series is to take into consideration the fact that a series can be viewed as an integral over an infinite domain

$$I(f) = \int_1^{\infty} f(x) dx \tag{2.2}$$

for some function $f : [1, \infty) \rightarrow \mathbb{R}$ for which $f(n) = a_n$ for all n . Therefore, if for a given series, we know an explicitly integrable function $f(x)$ with this property, then we can take the value $I(f)$ of the integral as an estimate for s .

Theorem 2.1. *We assume that the function f is such that:*

- (1) f is either positive and decreasing, or negative and increasing,
- (2) $\int_1^{\infty} f(x) dx$ is convergent,
- (3) $f \in C^6([1 - \frac{2\sqrt{5}}{5}, \infty))$,
- (4) $f^{(6)}$ is either positive or negative on $[1 - \frac{2\sqrt{5}}{10}, \infty)$. Under this assumptions, if $f^{(6)} > 0$ then

$$\begin{aligned} \sum_{i=1}^{n-1} f(i) + \frac{f(n)}{2} + \int_n^{\infty} f(x) dx + P_n(-\sqrt{5}, f) &< s < \\ &< \sum_{i=1}^{n-1} f(i) + \frac{f(n)}{2} + \int_n^{\infty} f(x) dx + P_n(\sqrt{5}, f), \end{aligned} \tag{2.3}$$

where

$$P_n(t, f) := -\frac{t}{12} \left(-3f(n) + 4f\left(n + \frac{t}{10}\right) - f\left(n + \frac{t}{5}\right) \right).$$

If $f^{(6)} < 0$, then we get a similar inequality, but with the right-hand side instead of the left-hand side, and vice versa.

Proof. Let us rewrite the inequality (1.12) in an equivalent form

$$\int_a^{a+nh} f(x) dx - G_n(f, \alpha_n) < T_{n+1}(f) < \int_a^{a+nh} f(x) dx - G_n(f, \beta_n). \quad (2.4)$$

Bearing in mind the assumptions we can apply the Theorem 1.3 for the function f with $a = m$, $h = 1$, $n \geq 4$. In our situation we have

$$\begin{aligned} T_{n+1}(f) &= \sum_{i=m}^{m+n-1} a_i - \frac{1}{2}a_m + \frac{1}{2}a_{m+n}, \\ \int_a^{a+nh} f(x) dx &= \int_m^{m+n} f(x) dx, \\ G_n(f, \zeta) &= \frac{1}{24\zeta} \left(-3(f(m) + f(m+n)) + \right. \\ &\quad \left. + 4(f(m+\zeta) + f(m+n-\zeta)) - (f(m+2\zeta) + f(m+n-2\zeta)) \right). \end{aligned}$$

Passing with n to ∞ in the inequality (2.4) we obtain

$$\int_m^{\infty} f(x) dx + P_m(-\sqrt{5}, f) \leq \sum_{i=m}^{\infty} a_i - \frac{1}{2}a_m \leq \int_m^{\infty} f(x) dx + P_m(\sqrt{5}, f). \quad (2.5)$$

We complete the proof by adding the term

$$\frac{1}{2}a_m + \sum_{i=1}^{m-1} a_i$$

to the both sides of the inequality (2.5). \square

Theorem 2.1 generalizes results from [2].

Example 3. Let us estimate the value of the Riemann function

$$\zeta(t) = \sum_{i=1}^{\infty} \left(\frac{1}{i} \right)^t$$

for $t \in (1, +\infty)$. In this case

$$f(x) = \frac{1}{x^t}, \quad f^{(6)}(x) = \frac{t(t+1)(t+2)(t+3)(t+4)(t+5)}{x^{t+6}} > 0$$

and

$$\int_n^{+\infty} f(x) dx = \frac{1}{t-1} n^{1-t}.$$

In Figure 2 we show the algebraic difference between the upper and lower estimates by the formula (2.3) for $n = 10$ and $n = 15$.

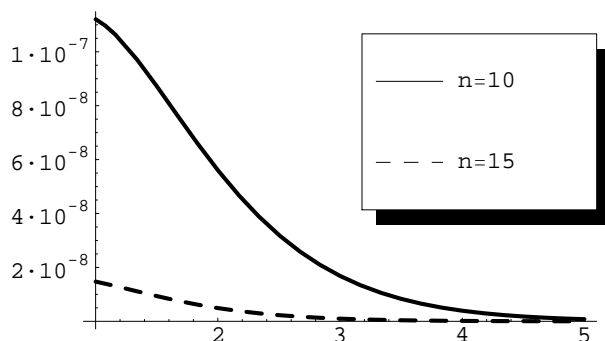


Fig. 2. The difference between the upper and lower estimates by the formula (2.3)

For instance for $n = 15$ we have

$$1.644\,934\,064\,14 \leq \zeta(2) = \frac{\pi^2}{6} \approx 1.644\,934\,066\,84 \leq 1.644\,934\,069\,06$$

and

$$1.082\,323\,233\,62 \leq \zeta(4) = \frac{\pi^4}{90} \approx 1.082\,323\,233\,71 \leq 1.082\,323\,233\,77.$$

Example 4. Let us estimate the sum of the series

$$q = \sum_{i=1}^{\infty} \frac{(-1)^i \ln i}{i}.$$

To apply our method we must represent the sum of this series in the desired form. This can be done by combining together the neighboring terms of opposite sign, for instance

$$q = \sum_{i=1}^{\infty} \frac{(-1)^i \ln i}{i} = s_c + s$$

where

$$s_c = \frac{\ln 2}{2} + \sum_{i=1}^5 \left(\frac{\ln(2i+2)}{2i+2} - \frac{\ln(2i+1)}{2i+1} \right) \approx 0.260\,832\,628\,568\,947\,6,$$

$$s = \sum_{i=1}^{\infty} \left(\frac{\ln(2i+12)}{2i+12} - \frac{\ln(2i+11)}{2i+11} \right).$$

In this case

$$f(x) = \frac{\ln(2x+12)}{2x+12} - \frac{\ln(2x+11)}{2x+11},$$

f is decreasing on interval $[1, +\infty)$,

$$\begin{aligned} f^{(6)}(x) &= 112\,896 \left(\frac{1}{(2x+11)^7} - \frac{1}{(2x+12)^7} \right) + \\ &+ 46\,080 \left(\frac{\ln(2x+12)}{(2x+12)^7} - \frac{\ln(2x+11)}{(2x+11)^7} \right) < 0 \end{aligned}$$

and

$$\int_n^{+\infty} f(x) dx = \frac{1}{4} (\ln^2(2n+11) - \ln^2(2n+12)).$$

Applying the estimate (2.3) with $n = 20$ we get

$$-0.100\,963\,724\,864\,2 \leq s \leq -0.100\,963\,724\,784\,6$$

and in consequence

$$0.159\,868\,903\,704\,6 \leq q \leq 0.159\,868\,903\,784\,2.$$

Both estimations in Theorem 2.1 differ from each other by the term $P_n(\cdot, f)$. The arithmetic mean of the upper and the lower estimates is a good approximation of the sum of the series $\sum_{i=1}^{\infty} f(i)$ so we have

$$s := \sum_{i=1}^{\infty} f(i) \approx s_n := \sum_{i=1}^{n-1} f(i) + \frac{f(n)}{2} + \int_n^{\infty} f(x) dx + S_n(f), \quad (2.6)$$

where

$$S_n(f) = \frac{\sqrt{5}}{24} \left(4 \left(f\left(n - \frac{\sqrt{5}}{10}\right) - f\left(n + \frac{\sqrt{5}}{10}\right) \right) - \left(f\left(n - \frac{\sqrt{5}}{5}\right) - f\left(n + \frac{\sqrt{5}}{5}\right) \right) \right).$$

Example 5. We consider the series

$$s = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{2i-1}.$$

We can write this series in the form

$$s = \sum_{j=1}^{\infty} \left(\frac{1}{4j-3} - \frac{1}{4j-1} \right) = \frac{\pi}{4}.$$

In this case

$$f(x) = \frac{1}{4x-3} - \frac{1}{4x-1},$$

$$f^{(6)}(x) = 2\,949\,120 \left(\frac{1}{(4x-3)^7} - \frac{1}{(4x-1)^7} \right) > 0$$

and

$$\int_n^\infty f(x) dx = \frac{\log(4n-1) - \log(4n-3)}{4}.$$

In the table (tab. 1) below we give some exemplary values of s_n by using the formula (2.6).

Table 1. Exemplary values of s_n

	s_n	$s - s_n$
$n = 10$	0.785 398 162 658 706 361 34	$7.387\,42 \cdot 10^{-10}$
$n = 40$	0.785 398 163 397 413 894 17	$3.441\,54 \cdot 10^{-14}$

REFERENCES

- [1] D. Kincaid, W. Cheney, *Numerical Analysis, Mathematics of Scientific Computing*, 3rd ed., The University of Texas at Austin, Brooks/Cole–Thomson Learning, 2002.
- [2] W. Solak, *A remark on power series estimation via boundary corections with parameter*, *Opuscula Mathematica* **19** (1999), 75–80.
- [3] W. Solak, Z. Szydełko, *Quadrature rules with Gregory–Laplace end corrections*, *Journal of Computational and Applied Mathematics* **36** (1991), 251–253.

Bogusław Bożek
bozek@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Wiesław Solak
solak@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Zbigniew Szydełko

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Received: July 3, 2008.

Revised: April 8, 2009.

Accepted: April 8, 2009.