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ON AN EVOLUTION INCLUSION IN NON-SEPARABLE BANACH SPACES

Abstract. We consider a Cauchy problem for a class of nonconvex evolution inclusions in non-separable Banach spaces under Filippov-type assumptions.We prove the existence of solutions.

Keywords: Lusin measurable multifunctions, selection, mild solution.

Mathematics Subject Classification: 34A60.

1. INTRODUCTION

In this paper we study differential inclusions of the form

$$x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map, Lipschitzean with respect to the second variable, X is a Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t,\tau), t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X, $D = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : D \to \mathbf{R}$ is continuous and $x_0 \in X$.

The existence and qualitative properties of mild solutions of problem (1.1) have been obtained in [1,2–7,13] etc.. Most of the existence results mentioned above are obtained using fixed point techniques. In [9] it is shown that Filippov's ideas ([11]) can suitably be adapted in order to prove the existence of solutions to problem (1.1). All these approaches are have proved successful the Banach space X separable.

De Blasi and Pianigiani ([10]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space X. Even if Filippov's ideas are still present, the approach in [10] is fundamental different: it consists in the construction of the measurable selections of the multifunction.

This construction does not use classical selection theorems such as Kuratowski and Ryll-Nardzewski's ([12]) or Bressan and Colombo's ([8]).

The aim of this paper is to obtain an existence result for problem (1.1) similar to the one in [10]. We will prove the existence of solutions for problem (1.1) in an arbitrary space X under Filippov-type assumptions on F.

The paper is organized as follows: in Section 2 we present the notations, definitions and preliminary results to be used in the sequel, and in Section 3 we prove the main result.

2. PRELIMINARIES

Consider X, an arbitrary real Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of X endowed with the Hausdorff pseudometric

$$\mathrm{d}_H(A,B) = \max\{\mathrm{d}^*(A,B), \mathrm{d}^*(B,A)\}, \quad \mathrm{d}^*(A,B) = \sup_{a\in A} \mathrm{d}(a,B),$$

where $\mathrm{d}(x,A) = \inf_{a\in A} |x-a|$, $A \subset X, x \in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesgue) measurable subsets of R and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of A.

Let X be a Banach space and Y be a metric space. An open (resp., closed) ball in Y with center y and radius r is denoted by $B_Y(y,r)$ (resp., $\overline{B}_Y(y,r)$. In what follows, $B = B_X(0,1)$.

A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H-continuous at $y_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_Y(y_0, r)$ there is $\mathrm{d}_H(F(y), F(y_0)) \le \varepsilon$. F is called d_H-continuous if it is so at each point $y_0 \in Y$.

Let $A \in \mathcal{L}$, with $\mu(A) < \infty$. A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$, with $\mu(A\backslash K_\varepsilon) < \varepsilon$ such that F restricted to K_ε is d_H-continuous.

It is clear that if $F, G : A \to \mathcal{P}(X)$ and $f : A \to X$ are Lusin measurable, then so are F restricted to B ($B \subset A$ measurable), $F + G$ and $t \to \mathrm{d}(f(t), F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is Lusin measurable, too.

Let I stand for the interval $[0,T]$, $T > 0$.

In what follows, $\{A(t); t \in I\}$ is the infinitesimal generator of a strongly continuous evolution system $G(t,s)$, $0 \le s \le t \le T$.

Recall that a family of bounded linear operators $G(t,s)$ on X, $0 \le s \le t \le T$ depending on two parameters is said to be a strongly continuous evolution system if the following conditions hold: $G(s,s) = I$, $G(t,r)G(r,s) = G(t,s)$ for $0 \le s \le r \le t \le T$ and $(t,s) \to G(t,s)$ is strongly continuous for $0 \le s \le t \le T$, i.e, $\lim_{t\to s, t>s} G(t,s)x = x$ for all $x \in X$.

In what follows, we are concerned with the evolution inclusion

$$x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{2.1}$$

where $F : I \times X \to \mathcal{P}(X)$ is a set-valued map, X is a Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t,\tau), t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X, $D = \{(t,s) \in I \times I; t \geq s\}$, $K(.,.) : D \to \mathbf{R}$ is continuous and $x_0 \in X$.

A continuous mapping $x(.) \in C(I,X)$ is called a *mild solution* of problem (2.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = G(t,0)x_0 + \int_0^t G(t,\tau)\int_0^\tau K(\tau,s)f(s)dsd\tau, \quad t \in I. \tag{2.3}$$

In this case, we shall call $(x(.), f(.))$ a *trajectory-selection pair* of (2.1).

We note that condition (2.3) can be rewritten as

$$x(t) = G(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I, \tag{2.4}$$

where $U(t,s) = \int_s^t G(t,\tau)K(\tau,s)d\tau$.

In what follows, we assume the following hypotheses.

Hypothesis 2.1. (i) *$\{A(t); t \in I\}$ is the infinitesimal generator of the strongly continuous evolution system $G(t,s)$, $0 \leq s \leq t \leq T$.*

(ii) *$F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and, for any $x \in X$, $F(.,x)$ is Lusin measurable on I.*

(iii) *There exists $l(.) \in L^1(I,(0,\infty))$ such that for each $t \in I$:*

$$d_H(F(t,x_1),F(t,x_2)) \leq l(t)|x_1 - x_2|, \quad \forall\, x_1,x_2 \in X.$$

(iv) *There exists $q(.) \in L^1(I,(0,\infty))$ such that for each $t \in I$:*

$$F(t,0) \subset q(t)B.$$

(v) *$D = \{(t,s) \in I \times I; t \geq s\}$, $K(.,.) : D \to \mathbf{R}$ is continuous.*

Set $n(t) = \int_0^t l(u)\mathrm{d}u$, $t \in I$, $M := \sup_{t,s\in I}|G(t,s)|$ and $M_0 := \sup_{(t,s)\in D}|K(t,s)|$ and note that $|U(t,s)| \leq MM_0(t-s) \leq MM_0T$.

The technical results summarized in the following lemma are essential in the proof of our result. For the proof, we refer the reader to [10].

Lemma 2.2 ([10] i)). *Let $F_i : I \to \mathcal{P}(X)$, i=1,2, be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, i=1,2 be such that*

$$H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I.$$

Then the multifunction $H : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

ii) Assume that Hypothesis 2.1 is satisfied. Then for any continuous $x(.) : I \to X$, $u(.) : I \to X$ measurable and any $\varepsilon > 0$ there is:

a) *the multifunction $t \to F(t, x(t))$ is Lusin measurable on I,*
b) *the multifunction $G : I \to \mathcal{P}(X)$ defined by*

$$G(t) := (F(t, x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t, x(t))) + \varepsilon)$$

has a Lusin measurable selection $g : I \to X$.

3. THE MAIN RESULT

We are now ready to prove our main result.

Theorem 3.1. *We assume that Hypothesis 2.1 is satisfied. Then, for every $x_0 \in X$, Cauchy problem* (1.1) *has a mild solution $x(.) \in C(I, X)$.*

Proof. Let us first note that if $z(.) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t, z(t)) + B$ is Bochner integrable on I. More precisely, for any $t \in I$, there holds

$$|u(t)| \leq d_H(F(t, z(t)) + B, 0) \leq d_H(F(t, z(t)), F(t, 0)) + \\ + d_H(F(t, 0), 0) + 1 \leq l(t)|z(t)| + q(t) + 1.$$

Let $0 < \varepsilon < 1, \quad \varepsilon_n = \frac{\varepsilon}{2^{n+2}}$.

Consider $f_0(.) : I \to X$, an arbitrary Lusin measurable, Bochner integrable function, and define

$$x_0(t) = G(t, 0)x_0 + \int_0^t U(t, s)f_0(s)ds, \quad t \in I.$$

Since $x_0(.)$ is continuous, by Lemma 2.2 ii) there exists a Lusin measurable function $f_1(.) : I \to X$ which, for $t \in I$, satisfies

$$f_1(t) \in (F(t, x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), d(f_0(t), F(t, x_0(t))) + \varepsilon_1).$$

Obviously, $f_1(.)$ is Bochner integrable on I. Define $x_1(.) : I \to X$ by

$$x_1(t) = G(t, 0)x_0 + \int_0^t U(t, s)f_1(s)ds, \quad t \in I.$$

By induction, we construct a sequence $x_n : I \to X$, $n \geq 2$ given by

$$x_n(t) = G(t,0)x_0 + \int_0^t U(t,s)f_n(s)ds, \quad t \in I, \tag{3.1}$$

where $f_n(.) : I \to X$ is a Lusin measurable function which, for $t \in I$, satisfies:

$$f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n). \tag{3.2}$$

At the same time, as we saw at the begining of the proof, $f_n(.)$ is also Bochner integrable.

From (3.2), for $n \geq 2$ and $t \in I$, we obtain

$$|f_n(t) - f_{n-1}(t)| \leq d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \leq d(f_{n-1}(t), F(t, x_{n-2}(t)))+$$
$$+d_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \leq \varepsilon_{n-1} + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n.$$

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$, for $n \geq 2$, we deduce that

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)|x_{n-1}(t) - x_{n-2}(t)|. \tag{3.3}$$

Denote $p_0(t) := d(f_0(t), F(t, x_0(t))), t \in I$. We next prove by recurrence, that for $n \geq 2$ and $t \in I$:

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| \leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{(MM_0T)^{k+1}(n(t) - n(u))^k}{k!} du + \\ + \varepsilon_0 \int_0^t \frac{(MM_0T)^n(n(t) - n(u))^{n-1}}{(n-1)!} du + \int_0^t \frac{(MM_0T)^n(n(t) - n(u))^{n-1}}{(n-1)!} p_0(u)du. \end{aligned} \tag{3.4}$$

We start with $n = 2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, there is

$$|x_2(t) - x_1(t)| \leq \int_0^t |U(t,s)| \cdot |f_2(s) - f_1(s)|ds \leq \int_0^t MM_0T[\varepsilon_0 + l(s)|x_1(s) - x_0(s)|]ds \leq$$
$$\leq \varepsilon_0 MM_0Tt + \int_0^t [MM_0Tl(s) \int_0^s |U(s,r)|.|f_1(r) - f_0(r)|dr]ds \leq$$
$$\leq \varepsilon_0 MM_0Tt + \int_0^t [(MM_0T)^2 l(s) \int_0^s (p_0(u) + \varepsilon_1)du]ds \leq$$
$$\leq \varepsilon_0 MM_0Tt + \int_0^t [(MM_0T)^2(p_0(u) + \varepsilon_1) \int_u^t l(s)ds]du =$$
$$= \varepsilon_0 MM_0Tt + \int_0^t (MM_0T)^2(n(t) - n(s))[p_0(s) + \varepsilon_0]ds,$$

i.e, (3.4) is verified for $n = 2$.

Using again (3.3) and (3.4), we conclude:

$$|x_{n+1}(t)-x_n(t)|\leq \int\limits_0^t |U(t,s)|\cdot|f_{n+1}(s)-f_n(s)|ds\leq$$

$$\leq \int\limits_0^t MM_0T[\varepsilon_{n-1}+l(s)|x_n(s)-x_{n-1}(s)|]ds\leq\leq \varepsilon_{n-1}MM_0Tt+$$

$$+\int\limits_0^t l(s)[\sum_{k=0}^{n-2}\int\limits_0^s \varepsilon_{n-2-k}\frac{(MM_0T)^{k+2}(n(s)-n(u))^k}{k!}du+$$

$$+\int\limits_0^s \frac{(MM_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}(p_0(u)+\varepsilon_0)du]ds=$$

$$=\varepsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int\limits_0^t[\int\limits_0^s \frac{(MM_0T)^{k+2}(n(s)-n(u))^k}{k!}l(s)du]ds+$$

$$+\int\limits_0^t l(s)(\int\limits_0^s \frac{(MM_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}l(s)[p_0(u)+\varepsilon_0]du)ds=$$

$$=\varepsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int\limits_0^t(\int\limits_u^t \frac{(MM_0T)^{k+2}(n(s)-n(u))^k}{k!}l(s)ds)du+$$

$$+\int\limits_0^t(\int\limits_u^t \frac{(MM_0T)^{n+1}(n(s)-n(u))^{n-1}}{(n-1)!}l(s)ds)[p_0(u)+\varepsilon_0]du=$$

$$=\varepsilon_{n-1}MM_0Tt+\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int\limits_0^t \frac{(MM_0T)^{k+2}(n(s)-n(u))^{k+1}}{(k+1)!}du+$$

$$+\int\limits_0^t \frac{(MM_0T)^{n+1}(n(s)-n(u))^n}{n!}[p_0(u)+\varepsilon_0]du=$$

$$=\sum_{k=0}^{n-1}\varepsilon_{n-1-k}\int\limits_0^t \frac{(MM_0T)^{k+1}(n(s)-n(u))^k}{k!}du+$$

$$+\int\limits_0^t \frac{(MM_0T)^{n+1}(n(s)-n(u))^n}{n!}[p_0(u)+\varepsilon_0]du,$$

and statement (3.4) it is true for $n+1$.

From (3.4) it follows that for $n \geq 2$ and $t \in I$:

$$|x_n(t) - x_{n-1}(t)| \leq a_n, \tag{3.5}$$

where

$$a_n = \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{(MM_0T)^{k+1} n(T)^k}{k!} + \frac{(MM_0T)^n n(T)^{n-1}}{(n-1)!} [\int_0^1 p_0(u)du + \varepsilon_0],$$

Obviously, the series whose n-th term is a_n converges. So, from (3.5) we infer that $x_n(.)$ converges to a continuous function, $x(.) : I \to X$, uniformly on I.

On the other hand, in view of (3.3) there is

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, n \geq 3$$

which implies that the sequence $f_n(.)$ converges to a Lusin measurable function $f(.) : I \to X$.

Since $x_n(.)$ is bounded and

$$|f_n(t)| \leq l(t)|x_{n-1}(t)| + q(t) + 1,$$

we infer that $f(.)$ is also Bochner integrable.

Passing with $n \to \infty$ in (3.1) and using the Lebesgue dominated convergence theorem, we obtain

$$x(t) = G(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I.$$

On the other hand, from (3.2) we get

$$f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, n \geq 1$$

and letting $n \to \infty$ we obtain

$$f(t) \in F(t, x(t)), \quad t \in I,$$

which completes the proof.

Remark 3.2. If $A(t) \equiv A$ and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t); t \geq 0\}$ from X to X, then problem (1.1) reduces to the problem

$$x'(t) \in Ax(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{3.6}$$

well known ([1,2–7,13] etc.) as an integrodifferential inclusion.

Obviously, a result similar to that of Theorem 3.1 may be obtained for problem (3.6).

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